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LINEAR ORDERS WITH FINITELY MANY DESCENDING CUTS

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Abstract. We show that if $\mathcal{L}$ is a low$_n$ linear order with only finitely many descending cuts, then $\mathcal{L}$ has a computable copy. We also show that neither hypothesis can be weakened: there is a linear order of intermediate degree with a single descending cut and a low$_3$ linear order with infinitely many descending cuts in order type ω, neither of which is computable.

1. Introduction

Computable structures are one of the main objects of study within effective mathematics. The question of which mathematical structures can be represented computably is of main interest in effective mathematics and computability theory. A related important question is how much information can be encoded in the isomorphism type of a certain structure. The low$_n$ Conjecture is a well-known conjecture in effective algebra that, roughly speaking, states if the amount of information encoded in the isomorphism type of a Boolean algebra is small, then there is no information in it at all. In this case, the notion of being low$_n$ formalizes the idea of having little information content.

Definition 1.1. A set $X \subseteq \omega$ is low$_n$ if $X^{(n)} \equiv_T \emptyset^{(n)}$, where $A^{(n)}$ is the nth Turing jump of the set A.

Question 1.2. Does every low$_n$ Boolean algebra have a computable copy?

This question has been solved affirmatively up to level low$_4$ by Knight and Stob and recent work by Harris and Montalbán demonstrates that an important new obstacle appears at level five. (See [DJ94, Thu95, KS00, HMb, HMa] for partial results towards resolving it).

The *degree spectrum* of a structure is the set of Turing degrees that can code a copy if it; it is intended to measure the amount of information that can be encoded in the isomorphism type of the structure. It is a common theme in effective mathematics to attempt to understand the properties of the degree spectrums of mathematical structures. The question above falls inside this theme, but another motivation for posing this question is that an affirmative answer would say that the information content of the isomorphism type of a Boolean algebra has to be encoded in a rather uncommon way. However, we will show in this paper that the class of Boolean algebras is not the only example of this unusual behavior (if indeed it is an example), as this phenomena already occurs in the class of linear orders with only finitely many descending cuts.

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Definition 1.3. A *cut* of a linear order $\mathcal{L}$ is a partition $(\mathcal{I}, \mathcal{J})$ of $\mathcal{L}$ where $\mathcal{I}$ is an initial segment of $\mathcal{L}$ and $\mathcal{J}$ is an end segment of $\mathcal{L}$. We will usually only write $\mathcal{I}$ instead $(\mathcal{I}, \mathcal{J})$ to denote a cut.

A *descending cut* in a linear order $\mathcal{L}$ is a cut $(\mathcal{I}, \mathcal{J})$ such that $\mathcal{J}$ has no least element.

Theorem 1.4. *Let $\mathcal{L}$ be a linear order with only finitely many descending cuts. If $\mathcal{L}$ has a low$_n$ computable presentation for some n, then it has a computable presentation.*

Linear orders in general exhibit very different behavior in that for every Turing degree $\mathbf{a} > \mathbf{0}$, there is an $\mathbf{a}$-computable linear order with no computable copy (see [AK00, Theorem 9.15] for this result by Knight that culminates a sequence of results by others). Furthermore, we cannot even extend Theorem 1.4 to linear orders with countably many descending cuts in order type ω; we show there is a low$_3$ linear order with this property but with no computable presentation (Theorem 3.5). We also show there is a linear order $\mathcal{L}$ with only one descending cut which has a presentation of *intermediate* degree but no computable presentation (Corollary 3.4).

We prove Theorem 1.4 in Section 2 and show that the hypotheses cannot be stregthened in Section 3.

2. The Main Result

In preparation for proving Theorem 1.4, we analyze which classical linear orders have only finitely many descending cuts. Linear orders of the following form will be ubiquitous.

Definition 2.1. If $\Gamma = \{\gamma_i\}_{i \in \omega}$ is any nondecreasing sequence of countable ordinals, denote by $\mathcal{L}_\Gamma$ the linear order

$$\mathcal{L}_\Gamma := \cdots + \omega^{\gamma_n} + \cdots + \omega^{\gamma_1} + \omega^{\gamma_0}.$$

The importance of the $\mathcal{L}_\Gamma$ is that they can be used to characterize the linear orders with only finitely many descending cuts.

Lemma 2.2. *If $\mathcal{L}$ is a countable linear order with no least element and such that every end segment is well-ordered, then there is a nondecreasing sequence of countable ordinals $\Gamma = \{\gamma_i\}_{i \in \omega}$ and an ordinal β such that $\mathcal{L} = \mathcal{L}_\Gamma + \beta$.*

Proof. Since $\mathcal{L}$ has no least element, we can decompose $\mathcal{L}$ into an infinite ω^* sum

$$\mathcal{L} = \sum_{i \in \omega^*} \mathcal{L}_i = ... + \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0,$$

where each $\mathcal{L}_i$ is an ordinal and ω^* is the order of the negative integers. As $\mathcal{L}_i$ is an ordinal, it can be written as $\omega^{\beta_{i,k}} + \cdots + \omega^{\beta_{i,1}} + \omega^{\beta_{i,0}}$ with $\beta_{i,0} \leq \beta_{i,1} \leq \cdots \leq \beta_{i,k}$. Thus by splitting each $\mathcal{L}_i$ as necessary, we may assume each $\mathcal{L}_i$ is of the form ω^{β_i}. Let $\gamma = \limsup_{i \in \omega} \beta_i$. We consider separately when there are only finitely many ks with $\beta_k = \gamma$ from when there are infinitely many.

First, suppose there are infinitely many ks with $\beta_k = \gamma$. Let k_0 be such that $\beta_{k_0} = \gamma$ and $\forall k \geq k_0$ $(\beta_k \leq \gamma)$; let $\{k_0 < k_1 < k_2 < ...\}$ be the set of all k greater than or equal to k_0 such that $\beta_k = \gamma$. By properties of ordinal addition, we have that for each i, $\omega^{\beta_{k_{i+1}-1}} + \omega^{\beta_{k_{i+1}-2}} + \cdots + \omega^{\beta_{k_i}} = \omega^{\gamma}$. Therefore

$$\mathcal{L} = \cdots + \omega^{\gamma} + \omega^{\gamma} + \omega^{\gamma} + \left(\sum_{j=k_0-1}^{j=0} \omega^{\beta_j} \right),$$

as desired.

Second, suppose there are only finitely many ks with $\beta_k = \gamma$. Let k_0 be such that $\forall k \geq k_0$ $(\beta_k < \gamma)$. We define a sequence $k_0 < k_1 < \ldots$ as follows. Let k_i be the least $k > k_{i-1}$ such that $\beta_k \geq \beta_{k_{i-1}}$. Let $\gamma_i = \beta_{k_i}$. Note that $\{\gamma_i\}_{i\in\omega}$ is a nondecreasing sequence. Again, by properties of ordinal addition, we have that for each i, $\omega^{\beta_{k_{i+1}-1}} + \omega^{\beta_{k_{i+1}-2}} + \cdots + \omega^{\beta_{k_i}} = \omega^{\gamma_i}$. Therefore

$$\mathcal{L} = \cdots + \omega^{\gamma_2} + \omega^{\gamma_1} + \omega^{\gamma_0} + \left(\sum_{j=k_0-1}^{j=0} \omega^{\beta_j} \right),$$

as desired. □

Lemma 2.3. *If $\mathcal{L}$ has only finitely many descending cuts, then $\mathcal{L}$ is of the form*

$$\mathcal{L} = \alpha + \mathcal{L}_{\Gamma_1} + \alpha_1 + \mathcal{L}_{\Gamma_2} + \alpha_2 + \cdots + \mathcal{L}_{\Gamma_n} + \alpha_n$$

for some (possibly 0) ordinals α and α_j and linear orders $\mathcal{L}_{\Gamma_j}$ for $1 \leq j \leq n$.

Proof. Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be the initial segments of $\mathcal{L}$ that define the finitely many descending cuts; for notational convenience, let $\mathcal{I}_0 = \emptyset$ and $\mathcal{I}_{n+1} = \mathcal{L}$. Then $\mathcal{L} = \mathcal{L}_0 + \cdots + \mathcal{L}_n$, with $\mathcal{L}_j = \mathcal{I}_{j+1} \backslash \mathcal{I}_j$.

As $\mathcal{L}_0 = \mathcal{I}_1 \backslash \mathcal{I}_0$ cannot contain a descending cut, it must be an ordinal which we denote by α. It thus suffices to argue that for each $j \geq 1$, the linear order $\mathcal{L}_j$ is of the form $\mathcal{L}_{\Gamma_j} + \alpha_j$ for some nondecreasing sequence of countable ordinals Γ_j and ordinal α_j. Each $\mathcal{L}_j$ for $j \geq 1$ has no minimal element, else $\mathcal{I}_j$ would not define a descending cut. On the other hand, any proper end segment of $\mathcal{L}_j$ for any $j \geq 1$ must be well-ordered as a consequence of the hypothesis that the $\mathcal{I}_j$ define all the descending cuts. It follows from Lemma 2.2 that each $\mathcal{L}_j$ for $j \geq 1$ is of the form $\mathcal{L}_{\Gamma_j} + \alpha_j$ for some nondecreasing sequence of countable ordinals Γ_j and ordinal α_j. □

Having characterized classically the linear orders with finitely many descending cuts, we turn to showing that $\alpha + \mathcal{L}_\Gamma$ is computably presentable if it is low$_n$ for some n. In doing so, we will move from $\mathcal{L}_\Gamma$ to condensations of $\mathcal{L}_\Gamma$.

Definition 2.4. Let $\mathcal{L}$ be any linear order. Write $x \equiv_n y$ if the order type of the interval $[x, y]$ is an ordinal less than ω^n.

For a linear order $\mathcal{L}$, let $\mathcal{L}/\equiv_n$ be the linear order obtained from $\mathcal{L}$ under the equivalence relation $\equiv_n$.

We remark that *condensations* (also known as *splittings*) have been studied in more general contexts (see [Ros82] for a survey). In our context (and in many others), the passage from $\mathcal{L}$ to $\mathcal{L}/\equiv_n$ is fairly effective.

Proposition 2.5. *Uniformly in an index for a computable linear order $\mathcal{L}_\Gamma$, there is an index for a Δ^0_{2n+1} presentation of the linear order $\mathcal{L}_\Gamma/\equiv_n$.*

Proof. The oracle $\emptyset^{(2n)}$ suffices to determine whether the order type of an interval $[x, y]$ is an ordinal less than ω^n (see [AK00, Proposition 7.2]). □

Moving the opposite direction is also effective.

Proposition 2.6 ([Wat84]). *Uniformly in a Δ^0_{2n+1} index for the atomic diagram of a linear order $\mathcal{L}$ with distinguished least element, there is a Δ^0_1 index for the atomic diagram of the linear order $\omega^n \cdot \mathcal{L}$.*

Lemma 2.7. *If β is an ordinal and $\Gamma = \{\gamma_i\}_{i\in\omega}$ is a nondecreasing sequence of ordinals so that $\omega^\beta + \mathcal{L}_\Gamma$ is low$_n$, then $\omega^\beta + \mathcal{L}_\Gamma$ is computable.*

Proof. Fix a low$_n$ set A such that $\omega^\beta + \mathcal{L}_\Gamma$ is A-computable. From Proposition 2.5 relativized, the linear order $(\omega^\beta + \mathcal{L}_\Gamma)/\equiv_n$ is $A^{(2n)}$-computable. Note that we may assume $\sup\{\gamma_i : i \in \omega\}$ is infinite. For if it is finite, then Γ contains only finitely much information (namely, how many of the γ_i equal k for each $k \leq \sup\{\gamma_i : i \in \omega\}$), and thus $\omega^\beta + \mathcal{L}_\Gamma$ is computable.

Denote by $\mathcal{L}'$ the linear order $(\omega^\beta + \mathcal{L}_\Gamma)/\equiv_n$ with least point removed if $\beta < n$ and greatest point removed if $\gamma_0 < n$. Then $\mathcal{L}'$ is also $A^{(2n)}$-computable, and thus $\emptyset^{(2n)}$-computable as $A^{(2n)} \equiv_T \emptyset^{(2n)}$ since A was low$_n$. So by Proposition 2.6, the linear order $\omega^n \cdot \mathcal{L}'$ is computable.

Let N be maximal with respect to the property that $\gamma_i < n$ for all $i < N$. Then

$$\omega^\beta + \mathcal{L}_\Gamma = \omega^n \cdot \mathcal{L}' + \omega^{\gamma_{N-1}} + \cdots + \omega^{\gamma_0}.$$

if $\beta \geq n$ and

$$\omega^\beta + \mathcal{L}_\Gamma = \omega^\beta + \omega^n \cdot \mathcal{L}' + \omega^{\gamma_{N-1}} + \cdots + \omega^{\gamma_0}$$

if $\beta < n$. In either case, we see that $\omega^\beta + \mathcal{L}_\Gamma$ is computable as a consequence of $\omega^n \cdot \mathcal{L}'$ being computable. □

Proof of Theorem 1.4. Let $\mathcal{L}$ be a low$_n$ linear order with only finitely many descending cuts. Without loss of generality, we may assume that $\mathcal{L}$ has a least element. By Lemma 2.3, it is of the form $\alpha + \sum_{1\leq j\leq n}(\mathcal{L}_{\Gamma_j} + \alpha_j)$ for some ordinals α and α_j and linear orders $\mathcal{L}_{\Gamma_j}$ for $1 \leq j \leq n$. We rewrite $\mathcal{L}$ as $\sum_{j<n}(\alpha'_j + \omega^{\beta'_j} + \mathcal{L}_{\Gamma'_j}) + \alpha'_n$. Note that this is possible by considering an end segment of either $\mathcal{L}_{\Gamma_j}$ or α_j as $\omega^{\beta'_j}$.

Each of the $\omega^{\beta'_j} + \mathcal{L}_{\Gamma'_j}$ is also low$_n$ because they are intervals of $\mathcal{L}$ with endpoints in $\mathcal{L} \cup \{-\infty, +\infty\}$. Thus by Lemma 2.7, each of the $\omega^{\beta'_j} + \mathcal{L}_{\Gamma'_j}$ is computable. It follows that $\mathcal{L}$ is computable, being a finite sum of computable linear orders. □

3. OTHER RESULTS

With Theorem 1.4 established, it is natural to ask if the result can be extended. We consider two questions:

1. Can the hypothesis that the linear order is low$_n$ be weakened?
2. Can the hypothesis that the linear order has only finitely many descending cuts be weakened (say, by imposing a restriction on the order type of the descending cuts)?

The first question is answered by the following results of [HKM] and [KM].

Definition 3.1. Fix a set $X \subseteq \omega$. Define $\Sigma^0_{(2n+2)}(X)$ to be the class of all sets $S \subseteq \omega$ for which there is a Turing functional Φ_e such that $\Phi_e^{X^{(2n+1)}}(n)\downarrow$ if and only if $n \in S$.

The set X is *low for Σ-Feiner* if $\Sigma^0_{(2n+2)}(X) \subseteq \Sigma^0_{(2n+2)}(\emptyset)$.

The set X has *intermediate Turing degree* if, for every n, it satisfies $0^{(n)} <_T X^{(n)} <_T 0^{(n+1)}$.

Theorem 3.2 (Hirschfeldt, Kach and Montalbán [HKM])**.** *There is a set X of intermediate Turing degree that is not low for Σ-Feiner.*

Theorem 3.3 (Kach and Miller [KM])**.** *If $A \subseteq \omega$ is any set, then $A \in \Sigma^0_{(2n+2)}(X)$ if and only if the linear order $\mathcal{L}_{\omega,A} := \omega^\omega + (\cdots + \omega^n \cdot A(n) + \cdots + \omega \cdot A(1) + 1 \cdot A(0))$ is X-computable.*

Corollary 3.4. *There is a linear order with exactly one descending cut having a presentation of intermediate Turing degree but no computable presentation.*

Proof. By Theorem 3.2, fix a set X of intermediate Turing degree that is not low for Σ-Feiner. Let A witness this, i.e., let A belong to $\Sigma^0_{(2n+2)}(X)$ but not $\Sigma^0_{(2n+2)}(\emptyset)$. Then $\mathcal{L}_{\omega,A}$ is X-computable but not computable by Theorem 3.3. ☐

The second question is answered by the following example.

Theorem 3.5. *There is a low$_3$ linear order with descending cuts in order type ω having no computable presentation.*

Proof. Fix $A \subseteq \omega$. Let $\mathcal{L}_j$ be $\omega + 1$ if $j \in A''$ and ω otherwise. We show that

$$\mathcal{L}_A = \sum_{j\in\omega} (\omega^* + \mathcal{L}_j + 2j)$$

is A-computable. It is sufficient to note that the orders $\mathcal{L}_j$ are A-computable uniformly in j, which follows from the fact that $j \in A''$ is $\Sigma^0_2(A)$ (see [AK00, Proposition 18.2]).

Now assume that $\mathcal{L}_A$ is a computable linear order. We prove that $A'' \leq_T \emptyset''$. Note that $\emptyset''$ is powerful enough to decide the following predicates, for $x, y \in \mathcal{L}_A$:

1. The point x has no predecessor, i.e., $(\forall z)(\exists w)[z < x \implies z < w < x]$.
2. The point x has no successor, i.e., $(\forall z)(\exists w)[x < z \implies x < w < z]$.
3. The points $x < y$ are adjacent, i.e., $(\forall z)[\neg(x < z < y)]$.

Thus to determine if $j \in A''$, we can use $\emptyset''$ to search for a maximal block of size $2j$ or $2j+1$ in $\mathcal{L}_A$. The unique such block must eventually be found and $j \in A''$ if and only if it has size $2j+1$. So if $\mathcal{L}_A$ is a computable, then $A'' \leq_T \emptyset''$.

Let $A \subseteq \omega$ be a low$_3$ set that is not also low$_2$. Then the linear order $\mathcal{L}_A$ is A-computable but not computable, as required. ☐

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