APPLICATIONS OF MIN-MAX METHODS TO GEOMETRY

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Abstract. We survey and sketch some of the recent progress made regarding Yau's Conjecture and the existence of minimal hypersurfaces.

1. Introduction

Let (M^{n+1}, g) be an $(n+1)$-dimensional closed Riemannian manifold. We assume, for convenience, that (M, g) is isometrically embedded in some Euclidean space $\mathbb{R}^J$.

A closed embedded hypersurface $\Sigma \subset M$ is called a *minimal hypersurface* if it is a critical point for the area functional, meaning that for every ambient vector field X in M we have

$$\frac{d}{dt}\mathrm{vol}(\phi_t(\Sigma))_{t=0} = 0,$$

where $\{\phi_t\}_{t\in\mathbb{R}}$ is a one-parameter family of diffemorphisms generated by the vector field X. From the first variation formula we know that

$$\frac{d}{dt}\mathrm{vol}(\phi_t(\Sigma))_{t=0} = -\int_\Sigma \langle H, X\rangle d\Sigma,$$

where H is the mean curvature vector of Σ, and so minimal hypersurfaces are those which have $H = 0$.

The simplest example of a minimal surface in $\mathbb{R}^3$ is given by plane and in the unit 3-sphere $S^3 \subset \mathbb{R}^4$ simple examples can be given by equators (intersection of a hyperplane in $\mathbb{R}^4$ with S^3). Many more examples exist in $\mathbb{R}^3$ (like the catenoid, helicoid, or Schwarz P surface) and Lawson [16] showed in the 70's that S^3 has closed orientable minimal surfaces of arbitrary genus.

One of the most fundamental questions question one can ask regarding closed minimal hypersurfaces is whether they exist and this was answered in the early 80's through the combined work of Almgren-Pitts [3, 27] and Schoen-Simon.

Theorem 1.1. *Every closed Riemannian manifold (M^{n+1}, g) has a closed minimal hypersurface that is smooth and embedded outside a set of Hausdorff dimension less than or equal to $n - 7$.*

Around the same time Yau [35] made the following conjecture:

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Yau's Conjecture 1.2. *Every closed Riemannian three-manifold contains infinitely many smooth, closed minimal surfaces.*

The purpose of these notes is to present some of the recent progress made regarding Yau's Conjecture and the existence of minimal hypersurface. For the sake of brevity, we will not do an exhaustive account of the historical developments (which means we will not mention the long list of beautiful results regarding existence of geodesics), nor will we cover all the recent developments in neighboring areas (such as free boundary minimal surfaces or the Allen-Cahn regularization). We focus mainly in providing the background needed in order to prove some of the recent developments.

Around the time the conjecture was made, the combined work of Almgren-Pitts [3, 27] Schoen-Simon [28] showed the following result:

Theorem 1.3. *Every closed Riemannian manifold (M^{n+1}, g) has a closed minimal hypersurface that is smooth and embedded outside a set of Hausdorff dimension less than or equal to $n-7$.*

Not much progress was done regarding Yau's conjecture until we showed [20] (see also [20, Remark 1.6]) the following result:

Theorem 1.4. *Every closed Riemannian manifold (M^{n+1}, g) with positive Ricci curvature has infinitely many distinct minimal hypersurfaces that are smooth and embedded outside a set of Hausdorff dimension less than or equal to $n-7$.*

Recently, jointly with Irie [14], we showed a denseness result that implies Yau's conjecture in the generic case.

Denseness Theorem 1.5. *Let M^{n+1} be a closed manifold of dimension $(n+1)$, with $3 \leq (n+1) \leq 7$.*

For a C^∞-generic Riemannian metric g on M, the union of all closed, smooth, embedded minimal hypersurfaces is dense.

Later, jointly with Song [24], we showed the existence of a sequence of closed embedded minimal hypersurfaces that becomes equidistributed.

Equidistribution Theorem 1.6. *Let M^{n+1} be a closed manifold of dimension $(n+1)$, with $3 \leq (n+1) \leq 7$.*

For a C^∞-generic Riemannian metric g on M, there exists a sequence $\{\Sigma_j\}_{j\in\mathbb{N}}$ of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M: for any $f \in C^0(M)$ we have

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^q \mathrm{vol}_g(\Sigma_j)} \sum_{j=1}^q \int_{\Sigma_j} f\, d\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M f dV.$$

Actually, the equidistribution proven in [24] is slightly more general because the test functions are allowed to be symmetric 2-tensors.

Shortly after these results were proven, two serious contributions to the field were made: Firstly, Song [31] settled Yau's conjecture by showing the following result.

Theorem 1.7. *Every closed Riemannian manifold (M^{n+1}, g) with $3 \leq (n+1) \leq 7$ has infinitely many distinct closed, smooth, embedded minimal hypersurfaces.*

Secondly, X. Zhou [38] used a novel regularization of the area functional (developed by him and Zhu in [39]) to prove the Multiplicity One Conjecture proposed by the authors in [23] (see also [22]). Before we describe a consequence of his work we need to introduce some more concepts.

Consider Σ a closed minimal hypersurface of M and let $N\Sigma$, $\Gamma(N\Sigma)$ denote, respectively, the normal bundle of Σ and the space of sections of $N\Sigma$. The second variation of Σ is a quadratic form on $\Gamma(N\Sigma)$ defined by

$$\begin{aligned}\delta^2\Sigma(X, X) &:= \frac{d^2}{dt^2}\mathrm{vol}(\phi_t(\Sigma))_{|t=0} \\ &= \int_\Sigma |\nabla^\perp X|^2 - \mathrm{Ric}(X, X) - |A|^2|X|^2 d\Sigma,\end{aligned}$$

where $X \in \Gamma(N\Sigma)$, $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the one-parameter family of diffeomorphisms generated by X (after being extended to vector field on M), $\nabla^\perp$ is the natural connection on $N\Sigma$, and $|A|^2$ is the norm of the second fundamental form. Elements in the kernel of $\delta^2\Sigma$ are called *Jacobi vector fields.*

White [33] (see also [34]) proved a Bumpy Metrics Theorem which says that almost every metric (in the Baire category sense) is bumpy, i.e., every minimal hypersurface has no non-trivial Jacobi vector fields.

The Morse index of Σ is the largest possible dimension of a vector subspace $P \subset \Gamma(N\Sigma)$ so that $\delta^2\Sigma$ is a negative quadratic form when restricted to P. Intuitively speaking, the Morse index of Σ (denoted by index(Σ)) is the number of linearly independent deformations that strictly decrease the volume of Σ. For instance, on the unit 3-sphere $S^3 \subset \mathbb{R}^4$, the Morse index of an equator (intersection of a hyperplane in $\mathbb{R}^4$ with S^3) is one because normal deformations decrease the area and volume preserving deformations never decrease the area. One can find an ellipsoid in $\mathbb{R}^4$ so that the intersection of the ellipsoid with each hyperplane $\{x_i = 0\}$ is a minimal sphere with Morse index i, $i = 1, \ldots, 4$. On $\mathbb{RP}^3$ with the round metric, the Morse index of a equatorial $\mathbb{RP}^2$ is zero because it is area-minimizing in its homotopy class.

Combining Zhou's solution to the Multiplicity One Conjecture with the Morse index bounds proven by the authors in [23] and with the Weyl Law for the Volume Spectrum proven by Liokumovich and the authors in [17] we have

Theorem 1.8. *Assume (M^{n+1}, g) is a closed Riemannian manifold, $3 \leq (n+1) \leq 7$, with a bumpy metric.*

For each $k \in \mathbb{N}$ there is an embedded, two-sided, multiplicity one, minimal hypersurface Σ_k with

$$\mathrm{vol}(\Sigma_k) \simeq a(n)\mathrm{vol}(M)^{\frac{n}{n+1}} k^{\frac{1}{n+1}} \quad \textit{and} \quad \mathrm{index}(\Sigma_k) = k,$$

where $a(n)$ is a universal constant.

Organization: In Section 2 we introduce the basic concepts and describe the main results of Min-max Theory for minimal hypersurfaces. In particular, in Section 2.3 we explain how Theorem 1.1 follows from Min-max Theory. Section 3 is dedicated to the concept of volume spectrum and to the proof of the Weyl Law for the volume spectrum. In Section 3.3 we explain how Theorem 3.10 follows from the solution to the Multiplicity One Conjecture, lower bounds for Morse index, and Weyl Law for the Volume Spectrum. In Section 4.1 we prove the Denseness Theorem 1.5 and the proof of the Equidistribution Theorem 1.6 is sketched in Section 4.2.

2. Min-max Theory

2.1. **Basic notions in Geometric Measure Theory.** The following definitions are taken from [30] and they correspond to extensions of the concept of a submanifold. In a nutshell, we will be working mostly with the space of mod 2 codimension one cycles $\mathcal{Z}_k(M;\mathbb{Z}_2)$ which can thought of as the space of all closed hypersurfaces in M. The reader comfortable with these concepts can skip this section.

A set $S\subset\mathbb{R}^J$ is *countable k-rectifiable* if $S\subset S_0\cup_{j\in\mathbb{N}}S_j$, where $\mathcal{H}^k(S_0)=0$ and S_j, $j\in\mathbb{N}$, is an embedded k-dimensional C^1-submanifold. We assume in addition that the set $S\subset\mathbb{R}^J$ is $\mathcal{H}^k$-measurable and $\mathcal{H}^k(S\cap K)<+\infty$ for every compact set $K\subset\mathbb{R}^J$. In this case, k-rectifiable sets are characterized by the property that they have a well defined k-dimensional tangent plane T_xS for $\mathcal{H}^k$-a.e. $x\in S$ (see [30, Theorem 11.6]). Let S_* denote the subset of S for which the k-dimensional tangent plane is well defined.

The Grassmanian of k-planes in $\mathbb{R}^J$ is denoted by $G_k(\mathbb{R}^J)$. There is a natural projection π from $G_k(\mathbb{R}^J)$ onto $\mathbb{R}^J$. A *rectifiable k-varifold* V is a Radon measure on $G_k(\mathbb{R}^J)$ so that for every measurable set $A\subset G_k(\mathbb{R}^J)$

$$V(A)=\int_{S\cap\pi(TS\cap A)}\theta(x)d\mathcal{H}^k$$

where S is a countable k-rectifiable set, θ is a positive locally $\mathcal{H}^k$-integrable function on S, and $TS=\{(x,T_xS): x\in S_*\}$. There is a natural Radon measure $||V||$ on $\mathbb{R}^J$ defined as $||V||(A)=V(\pi^{-1}(A))$ for every measurable set $A\subset\mathbb{R}^J$. We say that $||V||(\mathbb{R}^J)$ is the *mass* of V and is the analogue of k-dimensional volume

We denote by $\mathcal{V}_n(M)$ the closure, in the weak topology, of the space of rectifiable k-varifolds in $\mathbb{R}^J$ with support contained in M. When the function θ is $\mathbb{N}$-valued, V is called an *integer k-varifold*.

Denote by $\mathcal{D}^k(\mathbb{R}^J)$ the set of smooth k-forms of $\mathbb{R}^J$ with compact support. Given an element $\omega\in\mathcal{D}^k(\mathbb{R}^J)$ we define $|\omega|=\sup_{x\in\mathbb{R}^J}\{\langle\omega(x),\omega(x)\rangle^{1/2}\}$.

A *k-current* T is a continuous linear functional on $\mathcal{D}^k(\mathbb{R}^J)$. Its boundary ∂T is a $k-1$-current that is defined as $\partial T(\phi)=T(d\phi)$, $\phi\in\mathcal{D}^{k-1}(\mathbb{R}^J)$. Naturally, $\partial^2T=0$. We will assume that every k-current has compact support. The restriction of a current T to an open set U is denoted by $T\llcorner U$.

Following [30, Section 27], we say that T is an *integer multiplicity k-current* (or simply integer multiplicity current) if it can be expressed as

$$T(\phi) = \int_S \langle \phi(x), \tau(x) \rangle \theta(x) d\mathcal{H}^k, \quad \phi \in \mathcal{D}^k(\mathbb{R}^J),$$

where S is a $\mathcal{H}^k$-measurable countable k-rectifiable set, θ is a $\mathcal{H}^k$-integrable $\mathbb{N}$-valued function, and τ is a k-form so that for all $x \in S_*$, $\tau(x)$ is a volume form for T_xS. In particular, $\tau(x)$ chooses an orientation for T_xS. The *mass* of an integer multiplicity k-current T is defined as

$$\mathbf{M}(T) = \sup\{T(\phi) : \phi \in \mathcal{D}^k(\mathbb{R}^L), |\phi| \leq 1\}$$

The space of integral currents with finite mass corresponds to the space of rectifiable currents defined in [8, 4.1.24] (see [8, Theorem 4.1.28]).

The space of k-currents T such that both T and ∂T are integer multiplicity currents with finite mass and support contained in M is denoted by $\mathbf{I}_k(M)$. This space is called the space of *integral k-currents.* The space of *k-cycles* is defined as those elements $T \in \mathbf{I}_k(M)$ so that $T = \partial Q$ for some $Q \in \mathbf{I}_{k+1}(M)$ and is denoted at $\mathcal{Z}_k(M)$. Note that in our notation $\mathcal{Z}_k(M)$ stands for the connected component containing zero of the set of integral currents with no boundary (thus differing slightly from the notation in [30] or [8]).

Given $T \in \mathbf{I}_k(M)$ there is a natural varifold $|T|$ associated to it and we denote its Radon measure by $||T||$. We have $||T||(M) = \mathbf{M}(T)$. The following varifolds appear naturally in the context of min-max theory.

Definition 2.1. We say an integer n-varifold V is a *smooth embedded minimal cycle* if there is a disjoint collection $\{\Sigma_1, \dots, \Sigma_N\}$ of closed, smooth, embedded, minimal hypersurfaces in M and a set of integers $\{m_1, \dots, m_N\} \subset \mathbb{N}$, such that

$$V = m_1|\Sigma_1| + \dots + m_N|\Sigma_N|.$$

The spaces above come with several relevant metrics. Given $T_1, T_2 \in \mathbf{I}_k(M)$, the *flat metric* is defined by

$$\mathcal{F}(T_1, T_2) = \inf\{\mathbf{M}(Q)+\mathbf{M}(R) : T_1-T_2 = Q+\partial R, P \in \mathbf{I}_k(M), Q \in \mathbf{I}_{k+1}(M)\}$$

and induces the so called *flat topology* on $\mathbf{I}_k(M)$. We also use $\mathcal{F}(T) = \mathcal{F}(T, 0)$ and one has that

$$\mathcal{F}(T) \leq \mathbf{M}(T) \quad \text{ for all } T \in \mathbf{I}_k(M).$$

The $\mathbf{F}$-*metric* on $\mathcal{V}_k(M)$ is defined in the book of Pitts [27, page 66] as:

$$\mathbf{F}(V, W) = \sup\{V(f) - W(f) : f \in C_c(G_k(\mathbb{R}^L)),$$
$$|f| \leq 1, \mathrm{Lip}(f) \leq 1\},$$

for $V, W \in \mathcal{V}_k(M)$ and induces the varifold weak topology on

$$\mathcal{V}_n(M) \cap \{V : ||V||(M) \leq c\}$$

for any $c > 0$.

Finally, the $\mathbf{F}$-*metric* on $\mathbf{I}_k(M)$ is defined by

$$\mathbf{F}(S,T) = \mathcal{F}(S-T) + \mathbf{F}(|S|,|T|).$$

We have $\mathbf{F}(|S|,|T|) \le \mathbf{M}(S-T)$ and hence $\mathbf{F}(S,T) \le 2\mathbf{M}(S-T)$ for any $S,T \in \mathbf{I}_l(M)$.

We assume that $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ have the topology induced by the flat metric. Informally, $T,S \in \mathcal{Z}_k(M)$ being very close to each other in the flat metric means that $T-S$ is the boundary of $Q \in \mathbf{I}_{k+1}(M)$ with very small mass. When endowed with the topology of the $\mathbf{F}$-metric these spaces will be denoted by $\mathbf{I}_k(M;\mathbf{F})$ and $\mathcal{Z}_n(M;\mathbf{F})$, respectively.

The Federer-Fleming Compactness Theorem [8, 4.2.17] states that the set

$$\{T \in \mathcal{Z}_k(M) : \mathbf{M}(T) \le C\}, \quad C > 0$$

is compact in the flat topology.

An important fact in the theory is that, while the mass is continuous in the varifold topology, it is only lower semicontinuous in the flat topology. The loss of mass in the limit is illustrated with the following standard example: let

$$Q_i = \{(x,y) \in \mathbb{R}^2 :\, 0 \le x \le 1, 0 \le y \le i^{-1}\}.$$

Then ∂Q_i tends to zero in the flat topology, but $\mathbf{M}(\partial Q_i)$ tends to 2. In this example $|\partial Q_i|$ tends to $2([0,1] \times \{0\})$ in the varifold topology.

For our purposes, we are interested in the space of mod 2 integral k-currents or mod 2 k-cycles that we denote by $\mathbf{I}_k(M;\mathbb{Z}_2)$ and $\mathcal{Z}_k(M;\mathbb{Z}_2)$, respectively. This space is defined via an equivalence relation, where we say that $T \equiv S$ if $T - S = 2Q$, T,S,Q being in $\mathbf{I}_k(M)$, and they were first introduced by Ziemer [36]. All the concepts we mentioned for $\mathbf{I}_k(M)$ and $\mathcal{Z}_k(M)$ can be extended to $\mathbf{I}_k(M;\mathbb{Z}_2)$ and $\mathcal{Z}_k(M;\mathbb{Z}_2)$ as well (see [36] or [9]). The Constancy Theorem [30, Theorem 26.27] says that if $T \in \mathcal{I}_{n+1}(M;\mathbb{Z}_2)$ has $\partial T = 0$, then either $T = M$ or $T = 0$.

The Isoperimetric Inequality of Federer-Fleming (adapted to mod 2 integral currents in [36, Corollary 4.7]) gives constants a_M, b_M so that for every $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ with $\mathbf{M}(T) \le a_M$ there is Ω in $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ such that

$$\partial\Omega = T \quad \text{and} \quad \mathbf{M}(\Omega) \le b_M \mathbf{M}(T)^{\frac{n+1}{n}}. \tag{2.1}$$

When combined with the Constancy Theorem we obtain the following lemma:

Lemma 2.2. *There is ε so that for every $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ with $\mathcal{F}(T) < \varepsilon$ there is a unique $S \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $\mathcal{F}(T) = \mathbf{M}(S)$.*

Proof. Choose Q and R so that $T = Q + \partial R$ and $\mathbf{M}(Q) + \mathbf{M}(R) \le \varepsilon$. Assuming $\varepsilon \le \min\{a_M, b_M^{-n}, \mathrm{vol}(M)/3\}$ we have from the Isoperimetric Inequality the existence of Ω with $\partial\Omega = Q$ and $\mathbf{M}(\Omega) \le \mathbf{M}(Q)$. As a result, setting $S = \Omega + R$, we have $T = \partial S$ and

$$\mathbf{M}(S) \le \mathbf{M}(\Omega) + \mathbf{M}(R) \le \mathbf{M}(Q) + \mathbf{M}(R) \le \mathrm{vol}(M)/3.$$

From the Constancy Theorem we have that if $S' \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is such that $\partial S' = T$ and $\mathbf{M}(S') \le \mathrm{vol}(M)/3$ then $S = S'$. This implies the lemma. $\square$

2.2. **Space of Cycles.** The basic principle of min-max theory is to use the homotopy classes of $\mathcal{Z}_n(M;\mathbb{Z}_2)$ to produce minimal hypersurfaces and so it is important that we understand the topology of $\mathcal{Z}_n(M;\mathbb{Z}_2)$. There is a map from $\mathbb{RP}^\infty$ to $\mathcal{Z}_n(M;\mathbb{Z}_2)$ that we now describe.

Let $f : M \to \mathbb{R}$ be a Morse function, with $f(M) = [0,1]$, and consider the map $\hat{\Phi} : \mathbb{RP}^\infty \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ given by

$$\hat{\Phi}([a_0 : a_1 : \cdots : a_k : 0 : \cdots]) = \partial\{x \in M : a_0 + a_1 f(x) + \cdots + a_k f(x)^k \leq 0\}.$$

The map is well defined because we are considering mod 2 cycles. In [20] (Claim 5.6), we proved the map $\hat{\Phi}$ is continuous in the flat topology.

Theorem 2.3. *The map $\hat{\Phi}$ is a weak homotopy equivalence.*

Almgren computed in [2] the homotopy groups of $\mathcal{Z}_k(M;\mathbb{Z}_2)$ for all $0 \leq k \leq n+1$ but the proof is more complicated than the argument we present (see [23, Section 5]).

Proof. Consider the continuous map

$$\partial : \mathbf{I}_{n+1}(M;\mathbb{Z}_2) \to \mathcal{Z}_n(M;\mathbb{Z}_2).$$

From the Constancy Theorem we know that $\partial U = \partial V$ implies that $U = V$ or $U = M - V$, which means that the map is 2 to 1.

Claim 1: $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ *is contractible.*

We define $H : [0,1] \times \mathbf{I}_{n+1}(M;\mathbb{Z}_2) \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ by

$$H(t,U) = U \llcorner \{f \leq t\}.$$

The map H is continuous, $H(1,U) = U$ and $H(0,U) = 0$ for every $U \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$. This proves the claim.

From the definition of $\mathcal{Z}_n(M;\mathbb{Z}_2)$ we have that the map ∂ is surjective and so it follows from the previous claim that $\mathcal{Z}_n(M;\mathbb{Z}_2)$ is path-connected.

Claim 2: $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ *is a covering space.*

We need to find an open cover $\{B_T\}_{T\in\mathcal{Z}_n(M;\mathbb{Z}_2)}$ of $\mathcal{Z}_n(M;\mathbb{Z}_2)$ such that each $\partial^{-1}(B_T)$ is a disjoint union of open sets in $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$, each of which is mapped by ∂ homeomorphically onto B_T.

Choose $\varepsilon \leq \mathrm{vol}(M)/3$ given by Lemma 2.2 and for every $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ consider the open set

$$B_T = \{R \in \mathcal{Z}_n(M;\mathbb{Z}_2) : \mathcal{F}(T,R) < \varepsilon\}.$$

The family $\{B_T\}_{T\in\mathcal{Z}_n(M;\mathbb{Z}_2)}$ forms an open cover. With $\partial^{-1}T = \{U_1, U_2\}$ set

$$C_i = \{V \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2) : \mathcal{F}(U_i, V) < \varepsilon\}, \quad i = 1,2.$$

Note that $\mathcal{F}(U_1,U_2) = \mathbf{M}(U_1 - U_2) = \mathrm{vol}(M)$ and so C_1 and C_2 are disjoint. The reader can check that $C_1 \cup C_2 \subset \partial^{-1}(B_T)$. To check the reverse inclusion suppose that $R \in B_T$. From Lemma 2.2 we have the existence of $W \in$

$\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ so that $\partial W = R - T$ and $\mathbf{M}(W) < \varepsilon$. Thus $V_i = W + U_i \in C_i$, $\partial V_i = R$, $i = 1, 2$, and so $C_1 \cup C_2 = \partial^{-1}(B_T)$. It also follows that each C_i is mapped homeomorphically to B_T for $i = 1, 2$, which proves the claim.

With S^k being a sphere of dimension k, consider a continuous map

$$\Psi : (S^k, *) \to (\mathcal{Z}_n(M;\mathbb{Z}_2), 0), \quad k \geq 2.$$

From the lifting criterion [13, Proposition 1.33] we have that the map Ψ admits a lift

$$\tilde{\Psi} : (S^k, *) \to (\mathbf{I}_{n+1}(M;\mathbb{Z}_2), 0)$$

because S^k is simply connected. From the fact that $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is contractible we obtain that Φ can be homotoped to the zero map. This proves

$$\pi_k(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) = 0$$

for every $k \geq 2$. We now check that

$$\pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) = \mathbb{Z}_2.$$

Given a loop γ in $\mathcal{Z}_{n+1}(M;\mathbb{Z}_2)$ with $\gamma(0) = \gamma(1) = 0$, the unique lifting property [13, Proposition 1.34] says there is a unique lift $\tilde{\gamma}$ to $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $\tilde{\gamma}(0) = 0$. Thus, from Claim 2 one sees that the map

$$P : \pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) \to \{0, M\}$$

which sends the homotopy class of γ to $\tilde{\gamma}(1)$ is well defined. The map is surjective because $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is path-connected and the reader can check that the map is injective.

Finally, we check that $\hat{\Phi}$ induces isomorphisms in every homotopy group. The curve

$$t \mapsto [\cos(\pi t) : \sin(\pi t) : 0 : \cdots], \quad 0 \leq t \leq 1,$$

generates $\pi_1(\mathbb{RP}^\infty, 1)$ and since the loop

$$\gamma(t) = \hat{\Phi}([\cos(\pi t) : \sin(\pi t) : 0 : \cdots]) = \partial\{f \leq -\cot(\pi t)\}, \quad 0 \leq t \leq 1$$

is homotopically non-trivial (because $P(\gamma) = M$), we deduce that the map

$$\hat{\Phi}_* : \pi_1(\mathbb{RP}^\infty, 1) \to \pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0)$$

is an isomorphism. The higher homotopy groups of both spaces are trivial, thus $\hat{\Phi}$ is a weak homotopy equivalence. $\square$

Theorem 2.3 and Hurewicz Theorem imply that

$$H^1(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \bar{\lambda}\}.$$

We call $\overline{\lambda}$ the *fundamental cohomology class*. It has geometric meaning, namely, if $\sigma : S^1 \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a loop then $\overline{\lambda}([\sigma]) = 1$ if and only if σ is homotopically non-trivial ($[\sigma]$ denotes the homology class induced by σ).

Let X denote a finite dimensional cubical subcomplex of some m-dimensional cube I^m. Every such cubical complex is homeomorphic to a finite simplicial complex and vice-versa (see Chapter 4 of [5]).

Definition 2.4. Let $k \in \mathbb{N}$. A continuous map $\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ is called a *k-sweepout* if $\lambda = \Phi^*(\bar{\lambda}) \in H^1(X, \mathbb{Z}_2)$ satisfies

$$\lambda^k = \lambda \smile \cdots \smile \lambda \neq 0 \in H^k(X, \mathbb{Z}_2),$$

where $\smile$ denotes the cup product.

The set of all k-sweepouts Φ is denoted by $\mathcal{P}_k$.

Remark. In the definition above, the parameter space $X = \operatorname{dmn}(\Phi)$ of $\Phi \in \mathcal{P}_k$ is allowed to depend on Φ. Furthermore, every $\mathbf{F}$-continuous map Φ' that is homotopic to Φ in the flat topology is also a k-sweepout.

We now argue that for all $k \in \mathbb{N}$ the set $\mathcal{P}_k$ is nonempty. The map $\Phi_k : \mathbb{RP}^k \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ given by

$$\Phi_k([a_0 : a_1 : \cdots : a_k]) \mapsto \hat{\Phi}([a_0 : a_1 : \cdots : a_k : 0 : \cdots])$$

is such that $\lambda = \Phi_k^*(\bar{\lambda}) \neq 0$ in $H^1(\mathbb{RP}^k; \mathbb{Z}_2)$ and so $\lambda^k \neq 0$ in $H^k(\mathbb{RP}^k; \mathbb{Z}_2)$. Some work would be required to show that Φ_k is continuous in the $\mathbf{F}$-metric and so instead we use Proposition 3.1 of [23] to find Ψ_k continuous in the $\mathbf{F}$-topology and homotopic to Φ_k in the flat topology. Hence $\Psi_k \in \mathcal{P}_k$.

2.3. **Min-max Theorems.** Let

$$\Phi : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$$

be a continuous map. The *homotopy class* of Φ is the class Π of all continuous maps $\Phi' : X \to \mathcal{Z}_n(M; \mathbf{F}; \mathbb{Z}_2)$ such that Φ and Φ' are homotopic to each other in the flat topology.

If Φ is a k-sweepout then the corresponding homotopy class Π is non-trivial. Notice that our definition of homotopy class is slightly unusual, as we allow homotopies that are continuous in a weaker topology.

Definition 2.5. The *width* of Π is defined by:

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup\{\mathbf{M}(\Phi(x)) : x \in X\}.$$

It is implicitly assumed that every homotopy class Π being considered has $\mathbf{L}(\Pi) < \infty$.

Lemma 2.6. *If Π is a non-trivial homotopy class then $\mathbf{L}(\Pi) > 0$.*

Proof. Consider $\varepsilon > 0$ given by Lemma 2.2. If $\mathbf{L}(\Pi) = 0$, we can find a map $\Phi \in \Pi$ so that Φ is a k-sweepout and $\mathbf{M}(\Phi(x)) < b_M^{-1}\varepsilon^{\frac{n}{n+1}}$ for all $x \in X = \operatorname{dmn}(\Phi)$, where b_M is the constant given by Federer-Fleming Isoperimetric Inequality. As a result we deduce from (2.1) that $\mathcal{F}(\Phi(x)) < \varepsilon$ for all $x \in X$. From Lemma 2.2 we have the existence of a unique $\Omega(x) \in \mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ so that $\Phi(x) = \partial\Omega(x)$ and $\mathbf{M}(\Omega(x)) < \varepsilon$ for all $x \in X$, which means that Φ admits a lift $\tilde{\Phi}$ to $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ that is continuous in flat topology. But $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ is contractible and so the map Φ is homotopic to a constant map, which contradicts Π being non-trivial.

$\square$

The following version of the Min-max Theorem, that follows from combining the existence theory of Almgren-Pitts [3, 27] with the regularity theory of Schoen-Simon [28], can be found in Section 3 of [22].

Min-max Theorem 2.7. *Suppose* $\mathbf{L}(\Pi) > 0$*. There exists an integer* n*-varifold* V *with* $||V||(M) = \mathbf{L}(\Pi)$ *and support a closed minimal hypersurface that is smooth and embedded outside a set of Hausdorff dimension less than or equal to* $n-7$*.*

This theorem has the following celebrated consequence (after combining with Lemma 2.6) which corresponds to Theorem 1.1.

Corollary 2.8. *Every closed Riemannian manifold* (M^{n+1}, g) *has a closed minimal hypersurface that is smooth and embedded outside a set of Hausdorff dimension less than or equal to* $n-7$*.*

Consider a smooth embedded minimal cycle V so that $V = m_1|\Sigma_1| + \cdots + m_N|\Sigma_N|$ for a set $\{\Sigma_1, \ldots, \Sigma_N\}$ of closed, smooth, embedded, minimal hypersurfaces in M and a set $\{m_1, \ldots, m_N\} \subset \mathbb{N}$. The *Morse index* of V is the number

$$\mathrm{index}(V) = \sum_{i=1}^{N} \mathrm{index}(\Sigma_i).$$

If $m_1 = \cdots = m_N = 1$, we say V has *multiplicity one*.

From the definition of width, one sees that we maximize over a cubical complex X of dimension k and then minimize over an infinite dimensional space. Thus it is natural to expect that the Morse index of the smooth embedded minimal cycle should be bounded from above by k. This question was initially left unanswered in the original work of Almgren and Pitts. In [22] the authors showed that

Theorem 2.9. *Assume that* $3 \leq (n+1) \leq 7$*. There exists a smooth embedded minimal cycle* V *so that*

$$||V||(M) = \mathbf{L}(\Pi) \quad \textit{and} \quad \mathrm{index}(V) \leq k.$$

We expect that a similar result should hold for dimensions higher than seven.

Lower bounds on the Morse index of smooth embedded minimal hypersurfaces is a subtler issue for the following reason: We can simply add some artificial parameters to the parameter space X so that we increase its dimension but the homotopy class of Π does not change. Thus, Morse index lower bounds have to be given in terms of the some topological property of Π rather than the dimension of the cubical complex X.

It turns out that obtaining optimal lower bounds for the Morse index is related with Multiplicity Once Conjecture made by authors in [23] (see also [22]) which states that

Multiplicity One Conjecture 2.10. *For generic metrics on* M^{n+1}*,* $3 \leq (n+1) \leq 7$*, any component of a closed, minimal hypersurface obtained by min-max methods is two-sided and has multiplicity one.*

After this lectures were completed, Zhou [38, Theorem A] made a serious contribution to the min-max theory and, using a novel regularization of the area functional (developed by him and Zhu in [39]) proved the Multiplicity One Conjecture. Previously, in another tour-de-force, Chodosh-Mantoulidis [6] had proved this conjecture in the 3-dimensional case using the Allen-Cahn functional.

Multiplicity One Theorem 2.11 (Zhou)**.** *Let (M^{n+1}, g) be a closed Riemannian manifold, $3 \leq (n+1) \leq 7$, with a bumpy metric.*

If Π is a non-trivial homotopy class there is an embedded, two-sided, multiplicity one, minimal hypersurface Σ with

$$\mathbf{L}(\Pi) = \mathrm{vol}(\Sigma).$$

The result still holds if g is assumed simply to have positive Ricci curvature.

Remark. Theorem A in [38] is stated assuming that (i) the homotopy class Π realizes the volume spectrum $\omega_k(M)$ (to be defined in Section 3.1) and that (ii) the maps in Π are defined on a cubical complex of dimension k. An inspection of the proof shows that (i) is not necessary and that (ii) can be dropped if one is not concerned about having sharp upper bounds on the Morse index of Σ that are also proven in [38, Theorem A].

The extension of the result to metrics of positive Ricci curvature is stated in [38, Remark 0.1] and the idea is to consider a sequence of bumpy metrics $\{g_i\}_{i \in \mathbb{N}}$ converging to g with $\mathrm{Ric}(g) > 0$, apply Theorem A in [38] to obtain a sequence of embedded, two-sided, multiplicity one, minimal hypersurfaces Σ_i (with respect to metric g_i) and then use Sharp Compactness Theorem [29] to deduce the result for the metric g.

In [23], the authors showed optimal Morse index lower bounds assuming the Multiplicity One Conjecture. After Zhou's work we were able to remove that requirement (see [23, Addendum]) and showed

Theorem 2.12. *Let (M^{n+1}, g) be a closed Riemannian manifold, $3 \leq (n+1) \leq 7$ with a bumpy metric.*

Let Π be the homotopy class of a k-sweepout Φ defined on a k-dimensional cubical complex. There is an embedded, two-sided, multiplicity one, minimal hypersurface Σ with

$$\mathbf{L}(\Pi) = \mathrm{vol}(\Sigma) \quad \textit{and} \quad \mathrm{index}(\Sigma) = k.$$

Remark. In the Addendum of [23] the result is stated assuming that Π realizes the volume spectrum but that condition is not necessary.

3. Volume Spectrum and Weyl Law

Gromov [10] introduced the notion of volume spectrum, which will become extremely useful when paired with the min-max theory for minimal hypersurfaces.

3.1. **Volume spectrum.** Recall the definition of k-sweepouts $\mathcal{P}_k$ given in Definition 2.4.

Definition 3.1. The *k-width* of (M,g) is defined to be

$$\omega_k(M,g) := \inf_{\Phi\in\mathcal{P}_k} \sup_{x\in \mathrm{dmn}(\Phi)} \mathbf{M}(\Phi(x)).$$

The non-increasing sequence $\{\omega_k(M,g)\}_{k\in\mathbb{N}}$ is called the *volume spectrum* of (M,g).

When there is no risk of ambiguity, we denote the k-width simply by $\omega_k(M)$.

Remark. Because of Proposition 3.1 in [23], the above definition of k-width coincides with the definition of k-width of [20] (Section 4.3) (or in [14, 17, 24]) where continuity in the $\mathbf{F}$-metric in the definition of a k-sweepout is replaced by continuity in the flat topology together with a no concentration of mass property.

The following analogy with the Laplacian spectrum is instructive. The Rayleigh quotient is defined as

$$E : W^{1,2}(M)\setminus\{0\} \to [0,\infty), \quad E(f) = \frac{\int_M |\nabla f|^2 dV_g}{\int_M f^2 dV_g}$$

and the k^{th}-eigenvalue $\lambda_k(M)$ of (M,g) is defined via the following min-max characterization:

$$\lambda_k(M) = \inf_{(k+1)-\text{plane } P} \max_{f\in P-\{0\}} E(f).$$

The Rayleigh quotient is scale invariant, meaning that $E(cf) = E(f)$ for all $c \neq 0$ and thus, considering the projectivization $\mathbb{P}W^{1,2}(M)$, where an element $[f] \in \mathbb{P}W^{1,2}(M)$, $f \neq 0$, represents the line $\{cf : c \in \mathbb{R}\} \subset W^{1,2}(M)$, we see that the Rayleigh quotient descends to a map

$$E : \mathbb{P}W^{1,2}(M) \to [0,\infty), \quad E([f]) = \frac{\int_M |\nabla f|^2 dV_g}{\int_M f^2 dV_g}.$$

Note that $\mathbb{P}W^{1,2}(M)$ is homeomorphic to $\mathbb{RP}^\infty$. In the same vein, a $(k+1)$-plane in $W^{1,2}(M)$ projects to a k-dimensional projective subspace of $\mathbb{P}W^{1,2}(M)$ that we denote simply by $\mathbb{P}^k$ and thus $\lambda_k(M)$ is given by

$$\lambda_k(M) = \inf_{\mathbb{P}^k \subset \mathbb{P}W^{1,2}(M)} \max_{[f]\in\mathbb{P}^k} E([f]).$$

This identity has a striking similarity with Definition 3.1 and so, in that sense, $\{\omega_k(M)\}_{k\in\mathbb{N}}$ can be regarded as a non-linear spectrum.

It is worthwhile to point out that, unlike the spectrum for the Laplacian, the volume spectrum has not been computed on any specific example. Considering the unit 3-sphere S^3 with the standard metric, it is fairly straightforward to show that

$$\omega_1(S^3) = \omega_2(S^3) = \omega_3(S^3) = \omega_4(S^3) = \max_{\theta\in\mathbb{RP}^4} \mathbf{M}(\Phi_4(\theta)) = 4\pi$$

but to show that $\omega_5(S^3) = \omega_6(S^3) = \omega_7(S^3) = 2\pi^2$, Nurser [26] had to use some of the work done by the authors in the solution to the Willmore conjecture [18], which should illustrate the subtleties of the problem. All other widths for S^3 are unknown.

The next result was essentially proven by Gromov [10, Section 4.2.B] and Guth [12] . A proof can also be found in Theorem 5.1 and Theorem 8.1 in [20].

Theorem 3.2. *There is a constant $C = C(M, g) > 0$ so that for all $k \in \mathbb{N}$*

$$C^{-1}k^{\frac{1}{n+1}} \leq \omega_k(M) \leq Ck^{\frac{1}{n+1}}.$$

We postpone the proof of the lower bound to Corollary 3.7. Regarding the upper bound, we choose to present the different proof given in Theorem 3 of [4] which relies on a connection with the nodal sets of eigenfunctions that was made by authors in [20, Section 9].

Proof of upper bound. Let $\bar{g}$ be an analytic metric on M and so we have $g \leq c_1\bar{g}$ for some constant c_1. With $\phi_0, \ldots, \phi_p$ denoting the first $(p+1)$-eigenfunctions for the Laplace operator of $(M, \bar{g})$, where ϕ_0 is the constant function, we can consider the map

$$\Phi_k : \mathbb{RP}^p \to \mathcal{Z}_n(M; \mathbb{Z}_2),$$
$$\Phi_k([a_0, \ldots, a_k]) = \partial\{x \in M : a_0\phi_0(x) + \ldots + a_k\phi_k(x) < 0\}.$$

The map is well defined because we are considering mod 2 cycles and it was shown in [4] that the map is continuous in the flat topology and has no concentration of mass. This last part is relevant because we can then invoke Proposition 3.1 of [23] to find a map $\Psi_k \in \mathcal{P}_k$ so that

$$\sup_{y\in\mathbb{RP}^k} \mathbf{M}(\Psi_k(y)) \leq 2 \sup_{y\in\mathbb{RP}^k} \mathbf{M}(\Phi_k(y)).$$

Building on the volume estimates of nodal sets for analytic metrics of Donnely and Fefferman [7], it was shown in [15] that for some constant c_2,

$$\mathrm{vol}_{\bar{g}}(\{x \in M : a_0\phi_0(x) + \ldots + a_k\phi_k(x) = 0\} \leq c_2k^{\frac{1}{n+1}}$$

and thus

$$\mathrm{vol}_g(\{x \in M : a_0\phi_0(x) + \ldots + a_k\phi_k(x) = 0\} \leq c_1^{\frac{n}{2}}c_2k^{\frac{1}{n+1}}$$

for all $k \in \mathbb{N}$ and $[a_0, \ldots, a_k] \in \mathbb{RP}^k$ so that $a_0\phi_0 + \ldots a_k\phi_k \neq 0$. Therefore

$$\sup_{y\in\mathbb{RP}^k} \mathbf{M}(\Psi_k(y)) \leq 2c_1^{\frac{n}{2}}c_2k^{\frac{1}{n+1}} \quad \text{for all } k \in \mathbb{N}.$$

□

Consider the C^0-topology on the space of all metrics. The next proposition says that the map $g \mapsto k^{-\frac{1}{(n+1)}}\omega_k(M, g)$ is Lipschitz on sets of uniformly equivalent metrics, with a Lipschitz constant that does not depend on k .

Proposition 3.3. *Let $\tilde{g}$ be a Riemannian metric on M, and let c be a positive constant. Then there exists $K = K(\tilde{g}, c) > 0$ such that*

$$|k^{-\frac{1}{(n+1)}}\omega_k(M,g) - k^{-\frac{1}{(n+1)}}\omega_k(M,g')| \leq K \cdot |g-g'|_{\tilde{g}}$$

for any Riemannian metrics $c^{-1}\tilde{g} \leq g, g' \leq c\tilde{g}$ and any $k \in \mathbb{N}$.

Proof. Given g, g' as above, we have

$$\sup_{v\neq 0} \frac{g'(v,v)}{g(v,v)} \leq 1 + \sup_{v \neq 0} \frac{|g(v,v) - g'(v,v)|}{g(v,v)} \leq 1 + c|g-g'|_{\tilde{g}}.$$

Assume $g \neq g'$ and choose a k-sweepout $\Phi : X \to \mathcal{Z}_n(M;;\mathbf{F};\mathbb{Z}_2)$ with

$$\sup\{\mathbf{M}_g(\Phi(x)) : x \in X\} \leq \omega_k(M,g) + |g-g'|_{\tilde{g}},$$

where $\mathbf{M}_g$ is the mass with respect to g. Then, considering the constant $C = C(M, \tilde{g})$ given by Theorem 3.2, we have

$$\begin{aligned}
\omega_k(M,g') - \omega_k(M,g) &\leq \sup\{\mathbf{M}_{g'}(\Phi(x)) : x \in X\} - \omega_k(M,g) \\
&\leq \left(\sup_{v\neq 0}\frac{g'(v,v)}{g(v,v)}\right)^{\frac{n}{2}} \sup\{\mathbf{M}_g(\Phi(x)) : x \in X\} - \omega_k(M,g) \\
&\leq \left(\sup_{v\neq 0}\frac{g'(v,v)}{g(v,v)}\right)^{\frac{n}{2}} (\omega_k(M,g) + |g-g'|_{\tilde{g}}) - \omega_k(M,g) \\
&\leq ((1+c|g-g'|_{\tilde{g}})^{\frac{n}{2}} - 1)\omega_k(M,g) + c^{\frac{n}{2}}|g-g'|_{\tilde{g}} \\
&\leq ((1+c|g-g'|_{\tilde{g}})^{\frac{n}{2}} - 1)c^{\frac{n}{2}}\omega_k(M,\tilde{g}) + c^{\frac{n}{2}}|g-g'|_{\tilde{g}} \\
&\leq ((1+c|g-g'|_{\tilde{g}})^{\frac{n}{2}} - 1)c^{\frac{n}{2}}Ck^{\frac{1}{(n+1)}} + c^{\frac{n}{2}}|g-g'|_{\tilde{g}},
\end{aligned}$$

from which the result follows. □

3.2. **Weyl Law for Volume Spectrum.** A celebrated result concerning the spectrum of a manifold is the so called Weyl Law, which states that

$$\lim_{k\to\infty} \lambda_k(M)k^{-\frac{2}{n+1}} = a(n)\mathrm{vol}(M)^{-\frac{2}{n+1}},$$

where $a(n) = 4\pi^2\mathrm{vol}(B)^{-\frac{2}{n+1}}$ and B is the unit ball in $\mathbb{R}^{n+1}$. This was proven by Weyl [32] in 1911 for domains that are regions of space. The proof for closed manifolds came later in 1949, by Minakshisundaram and Pleijel, and uses the asymptotic expansion for the trace of the heat kernel.

Gromov conjectured ([11, 8.4]) that the volume spectrum $\{\omega_p(M)\}_{p\in\mathbb{N}}$ satisfies a Weyl's asymptotic law. Jointly with Liokumovich, the authors confirmed this conjecture and showed in [17] the following result.

Weyl Law for the Volume Spectrum 3.4. *There exists a constant $a(n) > 0$ such that, for every compact Riemannian manifold (M^{n+1}, g) with (possibly empty) boundary, we have*

$$\lim_{k\to\infty} \omega_k(M)k^{-\frac{1}{n+1}} = a(n)\mathrm{vol}(M)^{\frac{n}{n+1}}.$$

Before sketching its proof it is worthwhile to make some comments. Unlike the spectrum of the Laplacian that is known in several cases (like round spheres or cubes) the volume spectrum, due in part to being a non-linear spectrum, has not been computed on any specific example. Nonetheless, we were able to prove a universal asymptotic law without knowing the value of the universal constant $a(n)$, which is in stark contrast with both Weyl and Minakshisundaram-Pleijel proofs. Moreover, Minakshisundaram-Pleijel proof for closed manifolds uses techniques that do not seem to have an analogue for the volume spectrum and so a new approach had to be developed.

In order to prove the Weyl Law we need to introduce relative cycles and mention their basic features.

Let (Ω, g) be Riemannian compact $(n+1)$-manifold with Lipschitz boundary $\partial\Omega$ and $H_{n+1}(\Omega, \partial\Omega, \mathbb{Z}_2) = \mathbb{Z}_2$. We denoted them by *connected Lipschitz domains*.

Consider the space

$$\mathbf{I}_n(\Omega, \partial\Omega; \mathbb{Z}_2) = \{T \in \mathbf{I}_n(\Omega; \mathbb{Z}_2) : \mathrm{support}(\partial T) \subset \partial\Omega\}.$$

We say that $T, S \in \mathbf{I}_n(\Omega, \partial\Omega; \mathbb{Z}_2)$ are equivalent if $T - S \in \mathbf{I}_n(\partial\Omega; \mathbb{Z}_2)$ and the connected component containing zero of the space of such equivalence classes, called *mod 2 relative n-cycles*, is denoted by $\mathcal{Z}_n(\Omega, \partial\Omega; \mathbb{Z}_2)$. We abuse notation and use $T \in \mathbf{I}_n(\Omega, \partial\Omega; \mathbb{Z}_2)$ to denote its equivalence class as a mod 2 relative n-cycle. In [17, Section 2.2] a further subscript appears in the notation of mod 2 relative n-cycles.

The mass, flat metric, and $\mathbf{F}$-metric on $\mathcal{Z}_n(\Omega, \partial\Omega; \mathbb{Z}_2)$ are defined as

$$\mathbf{M}(T) = \inf\{\mathbf{M}(T + R) : R \in \mathbf{I}_n(\partial\Omega; \mathbb{Z}_2)\},$$

$$\mathcal{F}(S, T) = \inf\{\mathcal{F}(S + R, T) : \ R \in \mathbf{I}_n(\partial\Omega; \mathbb{Z}_2)\}$$

and

$$\mathbf{F}(S, T) = \inf\{\mathbf{F}(S + R, T), \mathbf{F}(T + R, S) : R \in \mathbf{I}_n(\partial\Omega; \mathbb{Z}_2)\}$$

for all S, T in $\mathcal{Z}_n(\Omega, \partial\Omega; \mathbb{Z}_2)$. These metrics induced the flat and $\mathbf{F}$-topology, respectively.

The theory for $\mathcal{Z}_n(\Omega, \partial\Omega; \mathbb{Z}_2)$ mimics the theory for $\mathcal{Z}_n(M; \mathbb{Z}_2)$ (see [17, Section 2] for details). Namley,

$$H^1(\mathcal{Z}_n(\Omega, \partial\Omega; \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2 = \{0, \bar{\lambda}\},$$

and the set of k-sweepouts $\mathcal{P}_k$ is defined as in Definition 2.4. One has $\mathcal{P}_k \neq \emptyset$ for all $k \in \mathbb{N}$ and the k-width $\omega_k(\Omega)$ is defined exactly like in Definition 3.1. In [17], k-sweepouts and k-width are defined in terms of maps that are continuous in the flat topology and have no concentration of mass but using the approximation results of [17, Section 2.9] one can show that the value for $\omega_k(\Omega)$ remains the same if one requires the maps to be continuous in the $\mathbf{F}$-topology instead. Finally, similarly to Theorem 3.2, there is a constant $C = C(\Omega, g)$ so that

$$\omega_k(\Omega) \leq C k^{\frac{1}{n+1}} \quad \text{for all } k \in \mathbb{N}. \tag{3.1}$$

We first prove Theorem 3.4 for connected Lipschitz domains using a superadditivity property for the k-widths. The key ingredients are the min-max definition of k-width and the vanishing property of the cup product. Weyl's proof is also based on similar properties for the Laplacian eigenvalues.

Lusternick-Schnirelman Superadditivity 3.5. *Consider connected Lipschitz domains Ω_0, $\{\Omega_i^*\}_{i=1}^N$ such that*

- $\Omega_i^* \subset \Omega_0$ *for all $i = 1, \ldots, N$ and the interiors of $\{\Omega_i^*\}_{i=1}^N$ are pairwise disjoint.*

Then, given positive integers so that $k_i + \ldots + k_N \leq k$, we have

$$\omega_k(\Omega_0) \geq \sum_{i=1}^N \omega_{k_i}(\Omega_i^*).$$

Sketch of proof. Set $\bar{k} = \sum_{i=1}^N k_i$. Given Φ a k-sweepout of Ω_0 (with $X = \mathrm{dmn}(\Phi)$) and $\lambda = \Phi^* \bar{\lambda} \in H^1(X, \mathbb{Z}_2)$, we assume for simplicity that the set

$$U_i = \{x \in X : \mathbf{M}(\Phi(x) \llcorner \Omega_i^*) < \omega_{k_i}(\Omega_i^*)\}$$

is open and the map

$$\Phi_i : X \to \mathcal{Z}_n(\Omega_i^*, \partial \Omega_i^*; \mathbf{F}; \mathbb{Z}_2) \quad x \mapsto \Phi(x) \llcorner \Omega_i^*$$

is well defined for all $1 \leq i \leq N$. The general argument can be found in Theorem 3.1 of [17].

Fix $1 \leq i \leq N$. With $\iota : U_i \to X$ the inclusion map, we have from the definition of U_i that $\Phi_i \circ \iota$ is not a k_i-sweepout of Ω_i, which means that $\iota^* \lambda^{k_i} = (\Phi_i \circ \iota)^* \bar{\lambda}^{k_i} = 0$ in $H^{k_i}(U_i; \mathbb{Z}_2)$. Therefore λ^{k_i} vanishes on U_i for all $1 \leq i \leq N$. The vanishing property for the cup product [13, page 209] implies that

$$\lambda^{\bar{k}} = \lambda^{k_1} \smile \ldots \smile \lambda^{k_N}$$

vanishes on $\cup_{i=1}^N U_i$. But $\lambda^{\bar{k}} \neq 0$ on X because $\bar{k} \leq k$ and so $X \neq \cup_{i=1}^N U_i$.

Choose $x \in X \setminus \cup_{i=1}^N U_i$. Combining the definition of U_i with the fact that the interiors of $\{\Omega_i^*\}_{i=1}^N$ are pairwise disjoint we have that

$$\Phi(x) \geq \sum_{i=1}^N \mathbf{M}(\Phi(x) \llcorner \Omega_i^*) \geq \sum_{i=1}^N \omega_{k_i}(\Omega_i^*).$$

$\square$

We use $\lfloor x \rfloor$ to denote the integer part.

Corollary 3.6. *Under the same conditions of the Lusternick-Schnirelman Superadditivity, assume also that*

- Ω_0 *has unit volume and* $\Omega_0, \{\Omega_i^*\}_{i=1}^N \subset \mathbb{R}^{n+1}$.

For all $i = 1, \ldots, N$, denote by Ω_i a scaling of Ω_i^* with unit volume.

Then, with $V = \min\{\mathrm{vol}(\Omega_i^*)\}_{i=1}^N$ *and* $k_i = \lfloor k\mathrm{vol}(\Omega_i^*)\rfloor$, $i = 1, \dots, N$, *we have for all* $k \in \mathbb{N}$

$$k^{-\frac{1}{n+1}}\omega_k(\Omega_0) \geq \sum_{i=1}^N \mathrm{vol}(\Omega_i^*){k_i}^{-\frac{1}{n+1}}\omega_{k_i}(\Omega_i) + O\left(\frac{1}{kV}\right).$$

Proof. The p-width scales like n-dimensional area and so we have $\omega_p(\Omega_i^*) = \mathrm{vol}(\Omega_i^*)^{\frac{n}{n+1}}\omega_p(\Omega_i)$ for all $i = 1, \dots, N$ and $p \in \mathbb{N}$.

We have $\sum_{i=1}^N k_i \leq k\mathrm{vol}(\Omega_0) = k$ and so, using Lusternick-Schnirelman Superadditivity and (3.1), we deduce

$$\begin{aligned}
k^{-\frac{1}{n+1}}\omega_k(\Omega_0) &\geq k^{-\frac{1}{n+1}}\sum_{i=1}^N \omega_{k_i}(\Omega_i^*) \\
&= k^{-\frac{1}{n+1}}\sum_{i=1}^N \mathrm{vol}(\Omega_i^*)^{\frac{n}{n+1}}\omega_{k_i}(\Omega_i) \\
&= \sum_{i=1}^N \mathrm{vol}(\Omega_i^*)\left(\frac{k_i}{k\mathrm{vol}(\Omega_i^*)}\right)^{\frac{1}{n+1}} {k_i}^{-\frac{1}{n+1}}\omega_{k_i}(\Omega_i) \\
&\geq \sum_{i=1}^N \mathrm{vol}(\Omega_i^*)\left(1 - \frac{1}{k\mathrm{vol}(\Omega_i^*)}\right)^{\frac{1}{n+1}} {k_i}^{-\frac{1}{n+1}}\omega_{k_i}(\Omega_i) \\
&= \sum_{i=1}^N \mathrm{vol}(\Omega_i^*){k_i}^{-\frac{1}{n+1}}\omega_{k_i}(\Omega_i) + O\left(\frac{1}{kV}\right).
\end{aligned}$$

□

From the Lusternick-Schnirelman Superadditivity 3.5 we can also deduce the lower bounds for the k-width stated in Theorem 3.2.

Corollary 3.7. *There is a constant* $C = C(M, g) > 0$ *so that for all* $k \in \mathbb{N}$

$$\omega_k(M) \geq C^{-1}k^{\frac{1}{n+1}}.$$

Proof. Given $p \in M$, let $B_r(p)$ denote the geodesic ball in M of radius r and centered at p, and consider $\omega_1(B)$, where B is the unit ball in $\mathbb{R}^{n+1}$. Lemma 2.6 extends to the context of relative cycles to conclude that $\omega_1(B) > 0$.

There is $\bar{r}$ small so that for all $r \leq \bar{r}$ and $p \in M$ we have

$$\omega_1(B_r(p), g) \geq r^n\omega_1(B)/2.$$

Moreover, there exists some constant $\nu = \nu(M) > 0$ such that, for every $k \in \mathbb{N}$, one can find a collection of k disjoint geodesic balls $\{B_j\}_{j=1}^k$ of radius $r = \nu k^{-\frac{1}{n+1}}$. Hence, we deduce from Lusternick-Schnirelman Superadditivity 3.5

$$\omega_k(M, g) \geq \sum_{j=1}^k \omega_1(B_j, g) \geq kr^n\frac{\omega_1(B)}{2} = kk^{-\frac{n}{n+1}}\nu^n\frac{\omega_1(B)}{2} = k^{\frac{1}{n+1}}\nu^n\frac{\omega_1(B)}{2}.$$

□

We are now ready to show

Weyl Law for Domains 3.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a connected Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$. There is a universal constant $a(n)$ so that for every $\Omega \subset \mathbb{R}^{n+1}$ we have*

$$\lim_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(\Omega) = a(n)\mathrm{vol}(\Omega)^{\frac{1}{n+1}}.$$

In [17, Theorem 3.2] this result was also proven for higher codimension relative cycles.

Proof. Without loss of generality we assume that $\mathrm{vol}(\Omega) = 1$. Let C denote the unit cube in $\mathbb{R}^{n+1}$ and set

$$a_-(n) = \liminf_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(C) \quad \text{and} \quad a^+(n) = \limsup_{p\to\infty} k^{-\frac{1}{n+1}}\omega_k(C).$$

Claim 1: $a_-(n) = a^+(n)$ and so define $a(n)$ to be that common value.

Choose $\{k_l\}_{l\in\mathbb{N}}$, $\{p_j\}_{j\in\mathbb{N}}$ so that

$$\lim_{l\to\infty} k_l^{-\frac{1}{n+1}}\omega_{k_l}(C) = a^+(n) \quad \text{and} \quad \lim_{j\to\infty} p_j^{-\frac{1}{n+1}}\omega_{p_j}(C) = a_-(n).$$

With l fixed and for all j large enough so that $\delta_j := k_l/p_j < 1$, consider N_j to be the maximum number of cubes $\{C_i^*\}_{i=1}^{N_j}$ with pairwise disjoint interiors contained in C and all with the same volume δ_j. We must have $\delta_j N_j \to 1$ as $j \to \infty$. From Corollary 3.6 we obtain

$$\begin{aligned} p_j^{-\frac{1}{n+1}}\ \omega_{p_j}(C) &\geq \sum_{i=1}^{N_j} \mathrm{vol}(C_i^*) k_l^{-\frac{1}{n+1}}\omega_{k_l}(C) + O(k_l^{-1}) \\ &= \delta_j N_j k_l^{-\frac{1}{n+1}}\omega_{k_l}(C) + O(k_l^{-1}). \end{aligned}$$

Making $j \to \infty$ and then $l \to \infty$ we deduce the claim.

Claim 2: $\liminf_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(\Omega) \geq a(n)$.

Given any $\varepsilon > 0$, one can find a family of cubes $\{C_i^*\}_{i=1}^N$ with pairwise disjoint interiors contained in Ω, all with the same volume δ, and such that

$$\sum_{i=1}^N \mathrm{vol}(C_i^*) \geq 1 - \varepsilon.$$

From Corollray 3.6 we obtain, with $k_\delta = \lfloor k\delta \rfloor$,

$$k^{-\frac{1}{n+1}}\omega_k(\Omega) \geq \sum_{i=1}^N \mathrm{vol}(C_i^*) k_\delta^{-\frac{1}{n+1}}\omega_{k_\delta}(C) + O\left(\frac{1}{k\delta}\right)$$

and thus making $k \to \infty$ we have

$$\liminf_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(\Omega) \geq (1-\varepsilon) \liminf_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(C) = (1-\varepsilon)a(n).$$

The claim follows from the arbitrariness of ε.

Claim 3: $\limsup_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(\Omega) \leq a(n)$.

Define $\tilde{\omega}_k(U) := k^{-\frac{1}{n+1}}\omega_k(U)$ for every connected Lipschitz domain U. Choose $\{k_j\}_{j\in\mathbb{N}}$ so that

$$\beta := \lim_{j\to\infty} \tilde{\omega}_{k_j}(\Omega) = \limsup_{k\to\infty} \tilde{\omega}_k(\Omega).$$

Choose $\varepsilon > 0$. From Lemma 3.5 in [17] we can choose domains $\{\Omega_i^*\}_{i=1}^N$ contained in C so that (i) every Ω_i^* is a scaling of Ω, (ii) their interiors are pairwise disjoint, and (iii) $\sum_{i=1}^N \mathrm{vol}(\Omega_i^*) \geq 1-\varepsilon$.

Fix $j \in \mathbb{N}$ and pick $p_j \in \mathbb{N}$ so that $\lfloor p_j \mathrm{vol}(\Omega_1^*)\rfloor = k_j$ (such choice is possible because $\mathrm{vol}(\Omega_1^*) \leq 1$). With $V_N = \min\{\mathrm{vol}(\Omega_i^*)\}_{i=1}^N$ and $k_{i,j} = \lfloor p_j vol(\Omega_i^*)\rfloor$, we have from Corollary 3.6 that

$$\begin{aligned}
\tilde{\omega}_{p_j}(C) &\geq \sum_{i=1}^N \mathrm{vol}(\Omega_i^*)\tilde{\omega}_{k_{i,j}}(\Omega) + O\left(\frac{1}{p_j V_N}\right)\\
&= \mathrm{vol}(\Omega_1^*)\tilde{\omega}_{k_j}(\Omega) + \sum_{i=2}^N \mathrm{vol}(\Omega_i^*)\tilde{\omega}_{k_{i,j}}(\Omega) + O\left(\frac{1}{p_j V_N}\right).
\end{aligned}$$

Making $j \to \infty$ and using Claim 2 we have

$$a(n) \geq \mathrm{vol}(\Omega_1^*)\beta + a(n)\sum_{i=2}^N \mathrm{vol}(\Omega_i^*) \geq \mathrm{vol}(\Omega_1^*)\beta + a(n)(1-\varepsilon-\mathrm{vol}(\Omega_1^*)).$$

Making $\varepsilon \to 0$ we obtain

$$a(n) \geq \mathrm{vol}(\Omega_1^*)\beta + a(n)(1-\mathrm{vol}(\Omega_1^*)),$$

and so $a(n) \geq \beta$, which finishes the claim. $\square$

We now explain the key ideas to show the theorem below

Weyl Law for Compact Manifolds 3.9. *For every closed Riemannian manifold (M^{n+1}, g) with (possibly empty) boundary, we have*

$$\lim_{k\to\infty} \omega_k(M)k^{-\frac{1}{n+1}} = a(n)\mathrm{vol}(M)^{\frac{n}{n+1}}.$$

Sketch of proof. Without loss of generality we assume that $\mathrm{vol}(M) = 1$. We also assume that $\partial M = \emptyset$ for simplicity.

The idea to show

$$\lim_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(M) \geq a(n)$$

is the following: We find a sufficiently large number of small pairwise disjoint geodesic balls $\{B_i\}_{i=1}^N \subset M$ so that $\sum_{i=1}^N \mathrm{vol}(B_i) \simeq 1$ and the metric g on B_i is close to being the Euclidean metric on a ball. Due to the Weyl Law for domains 3.8, we have $\omega_k(B_i)k^{-\frac{1}{n+1}} \simeq a(n)\mathrm{vol}(B_i)^{\frac{n}{n+1}}$ for all k very large.

Thus from the Lusternick-Schnirelman Superadditivity 3.5 we deduce, for all k sufficiently large,

$$k^{-\frac{1}{n+1}}\omega_k(M) \geq k^{-\frac{1}{n+1}}\sum_{i=1}^{N}\omega_{\lfloor k\mathrm{vol}(B_i)\rfloor}(B_i) = \sum_{i=1}^{N}k^{-\frac{1}{n+1}}\omega_{\lfloor k\mathrm{vol}(B_i)\rfloor}(B_i)$$
$$\simeq \sum_{i=1}^{N}a(n)\mathrm{vol}(B_i) \simeq a(n).$$

The reader can see the details in Theorem 4.1 of [17].

To prove the other inequality, the first step consists in decomposing M into regions $\{\mathcal{C}_i\}_{i=1}^N$ so that:

- Each $\mathcal{C}_i$ is $(1+\varepsilon)$-bilipschitz diffeomorphic to a Lipschitz domain C_i in $\mathbb{R}^{n+1}$;
- The regions $\{\mathcal{C}_i\}_{i=1}^N$ cover M;
- $\{\mathcal{C}_i\}_{i=1}^N$ and $\{C_i\}_{i=1}^N$ have mutually disjoint interiors, respectively.

We then connect the disjoint regions $\{C_i\}_{i=1}^N \subset \mathbb{R}^{n+1}$ with tubes of very small volume so that we obtain a connected Lipschitz domain Ω. By making ε smaller, we can make the volume of Ω arbitrarily close to $\mathrm{vol}(M)$.

In what follows we will be content with producing sweepouts that are only continuous with respect to the flat topology (instead of continuous with respect to the $\mathbf{F}$-topology). The reader can see the general argument in [17, Theorem 4.2].

Consider Φ a k-sweepout of Ω with $X = \mathrm{dmn}(\Phi)$, which then induces k-sweepouts on each C_i given by

$$\Phi_i : X \to \mathcal{Z}_n(C_i, \partial C_i; \mathbb{Z}_2), \quad \Phi_i(x) = \Phi(x)\llcorner C_i, \quad i = 1, \ldots, N.$$

In [17, Section 4] we show that, after a possibly small perturbation, the map Φ_i is well defined and a k-sweepout with

$$\lambda := \Phi_i^*\bar{\lambda} = \Phi^*\bar{\lambda} \quad \text{for all } i = 1, \ldots, N.$$

The general idea is to use the maps $\{\Phi_i\}_{i=1}^N$ to construct a k-sweepout of M as follows: For every $x \in X$ the elements $\Phi_i(x)$ have boundary in ∂C_i and we show in [17, Lemma 4.3] the existence of $Z_i(x) \in \mathbf{I}_{n+1}(C_i; \mathbb{Z}_2)$ so that the cycle $\partial Z_i(x)$ coincides with $\Phi_i(x)$ on the interior of C_i. Because the choice of $Z_i(x)$ is not unique ($C_i + Z_i(x)$ would have also been a valid choice) it is not always possible to construct a continuous map $x \mapsto \partial Z_i(x)$. Nonetheless, we will argue that a choice of $Z_1(x)$ for a given x will induce choices of $Z_2(x), \ldots, Z_N(x)$ so that if $\tilde{Z}_i(x)$ denotes the image of $Z_i(x)$ in $\mathcal{C}_i$ under the respective bilipschitz diffeomorphism, then $\partial\tilde{Z}_1(x) + \ldots + \partial\tilde{Z}_N(x)$ is a cycle in M that does not depend on the choice of $Z_1(x)$ and we use that to conclude that the map $x \mapsto (\partial\tilde{Z}_1 + \ldots + \partial\tilde{Z}_N)(x)$ is continuous. We now provide some of the details.

For each $i = 1, \ldots, N$ set

$$SX_i = \{(x, Z) : x \in X, \Phi_i(x) - \partial Z \in \mathbf{I}_n(\partial C_i; \mathbb{Z}_2)\} \subset X \times \mathbf{I}_{n+1}(C_i; \mathbb{Z}_2).$$

From the Constancy Theorem we have that $\mathbf{I}_n(\partial C_i;\mathbb{Z}_2) = \{0, \partial C_i\}$ for all $i = 1,\dots,N$. Thus if $(x,Z),(x,Z') \in SX_i$ we have that either $Z = Z'$ or $Z = C_i - Z'$. There is a natural projection $\tau_i : SX_i \to X$ and in Lemma 4.3 of [17] we show that τ_i is a 2-cover of X for all $i = 1,\dots,N$.

Claim 1: *SX_1 is isomorphic to SX_i for all $i = 1,\dots,N$.*

The isomorphism classes of double covers of X are in a bijective correspondence with $\mathrm{Hom}(\pi_1(X),\mathbb{Z}_2)$, which is homeomorphic to $H^1(X;\mathbb{Z}_2)$. It suffices to see that, for all $i = 1,\dots,N$, the element $\sigma_i \in H^1(X;\mathbb{Z}_2)$ that classifies SX_i is identical to λ. Indeed given $\gamma : S^1 \to X$ nontrivial in $\pi_1(X)$, consider a lift to SX_i given by $\theta \mapsto (\gamma(\exp(i\theta)), Z_\theta)$, $0 \le \theta \le 2\pi$. Then $\sigma_i(\gamma)$ is 1 if $Z_0 = C_i - Z_{2\pi}$ and 0 if $Z_0 = Z_{2\pi}$. Thus $\sigma_i(\gamma)$ is non-zero if and only if $\Phi_i \circ \gamma$ is a sweepout.

As a result we obtain that SX_1 is isomorphic to SX_i for all $i = 1,\dots,N$ and let $F_i : SX_1 \to SX_i$ be the corresponding isomorphism. Given an element $v = (x,T)$ in SX_i, we denote by $\Xi_i(v) \in \mathbf{I}_{n+1}(\mathcal{C}_i;\mathbb{Z}_2)$ the image of T under the bilipschitz diffeomorphism from C_i to $\mathcal{C}_i$, $i = 1,\dots,N$. Using this notation we consider the continuous map in the flat topology

$$\hat\Psi : SX_1 \to \mathcal{Z}_n(M;\mathbb{Z}_2), \quad \hat\Psi(y) = \sum_{i=1}^N \partial\, \Xi_i(F_i(y)).$$

If $(x,Z) \in SX_1$, then $\Xi_i(F_i(x, C_1 + Z)) = \mathcal{C}_i + \Xi_i(F_i(x,Z))$ for all $i = 1,\dots,N$, and so

$$\begin{aligned}\hat\Psi(x, C_1 + Z) = \sum_{i=1}^N \partial(\mathcal{C}_i + \Xi_i(F_i(x,Z))) &= \sum_{i=1}^N \partial\mathcal{C}_i + \hat\Psi(x,Z)\\ &= \partial M + \hat\Psi(x,Z) = \hat\Psi(x,Z).\end{aligned}$$

Thus $\hat\Psi(x, C_1 + Z) = \hat\Psi(x,Z)$ in $\mathcal{Z}_{n,}(M;\mathbb{Z}_2)$, which means that $\hat\Psi$ descends to a continuous map in the flat topology $\Psi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$.

Claim 2: *Ψ is a p-sweepout.*

Choose $\gamma : S^1 \to X$ nontrivial in $\pi_1(X)$ and denote by γ_1 its lift to SX_1. Then $\gamma_i = F_i \circ \gamma_1$ gives a lift to SX_i for all $i = 1,\dots,N$ and we consider the continuous map in the flat topology

$$B : [0,2\pi] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2), \quad B(\theta) = \sum_{i=1}^N \Xi_i(\gamma_i(\theta)).$$

We have $(\Psi \circ \gamma)(\theta) = \partial B(\theta)$ for all $0 \le \theta \le 2\pi$.

Hence $\Psi^*\bar\lambda = \lambda$ because, recalling that $\sigma_i = \lambda$ for all $i = 1,\dots,N$,

$$\begin{aligned}\lambda(\gamma) = 0 &\implies \sigma_i(\gamma) = 0 \text{ for all } i = 1,\dots,N\\ &\implies \Xi_i(\gamma_i(2\pi)) = \Xi_i(\gamma_i(0)) \text{ for all } i = 1,\dots,N\\ &\implies B(2\pi) = B(0)\end{aligned}$$

and

$$\begin{aligned}\lambda(\gamma)=1 &\implies \sigma_i(\gamma)=1 \text{ for all } i=1,\dots,N\\ &\implies \Xi_i(\gamma_i(2\pi))=\mathcal{C}_i+\Xi_i(\gamma_i(0)) \text{ for all } i=1,\dots,N\\ &\implies B(2\pi)=M+B(0),\end{aligned}$$

where in the last line we used the fact that $\{\mathcal{C}_i\}_{i=1}^N$ are pairwise disjoint and cover M. This implies that Ψ is a p-sweepout because $\lambda^p\neq 0$.

Throughout the rest of the proof we ignore the ε-dependence in some of the constants for simplicity.

Claim 3: *For all $x\in X$ we have*

$$\mathbf{M}(\Psi(x))\lesssim \mathbf{M}(\Phi(x))+\sum_{i=1}^N \mathrm{vol}(\partial C_i).$$

Given $(x,Z)\in SX_i$, $i=1,\dots,N$, we have that

$$\mathbf{M}(\Xi_i(\partial Z))\simeq \mathbf{M}(\partial Z)\leq \mathbf{M}(\Phi_i(x))+\mathbf{M}(\partial C_i)$$

Hence, recalling the definition of the maps Φ_i, we have

$$\begin{aligned}\mathbf{M}(\Psi(x)) &\leq \sum_{i=1}^N \mathbf{M}(\partial\,\Xi_i(F_i(y)))\lesssim \sum_{i=1}^N(\mathbf{M}(\Phi_i(x))+\mathbf{M}(\partial C_i))\\ &\leq \sum_{i=1}^N \mathbf{M}(\Phi(x)\llcorner C_i)+\sum_{i=1}^N \mathrm{vol}(\partial C_i)\leq \mathbf{M}(\Phi(x))+\sum_{i=1}^N \mathrm{vol}(\partial C_i).\end{aligned}$$

Combining Claim 2 with Claim 3 we deduce that for all $k\in\mathbb{N}$

$$\omega_k(M)\lesssim \omega_k(\Omega)+\sum_{i=1}^N \mathrm{vol}(\partial C_i).$$

From Theorem 3.8 we have that

$$\limsup_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(M)\lesssim \lim_{k\to\infty} k^{-\frac{1}{n+1}}\omega_k(\Omega)\leq a(n)\mathrm{vol}(\Omega)^{\frac{n}{n+1}}\simeq a(n)\mathrm{vol}(M)^{\frac{n}{n+1}}.$$

$\square$

3.3. **Min-max Theory and the volume spectrum.** Combining the Weyl Law for the Volume Spectrum 3.4 with Theorem 2.9 and Theorem 2.12 we obtain

Theorem 3.10. *Assume (M^{n+1},g) is a closed Riemannian manifold, $3\leq (n+1)\leq 7$.*

For each $k\in\mathbb{N}$ there exist a smooth embedded cycle V so that

$$\mathrm{vol}(V)=\omega_k(M)\simeq a(n)\mathrm{vol}(M)^{\frac{n}{n+1}}k^{\frac{1}{n+1}}\quad \textit{and}\quad \mathrm{index}(V)\leq k,$$

where $a(n)$ is a universal constant.

If the metric g is bumpy there is an embedded, two-sided, multiplicity one, minimal hypersurface Σ_k with

$$\mathrm{vol}(\Sigma_k) = \omega_k(M) \simeq a(n)\mathrm{vol}(M)^{\frac{n}{n+1}} k^{\frac{1}{n+1}} \quad \textit{and} \quad \mathrm{index}(\Sigma_k) = k.$$

Proof. Choose a sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}_k$ such that

$$\lim_{i\to\infty} \sup\{\mathbf{M}(\Phi_i(x)) : x \in X_i = \mathrm{dmn}(\Phi_i)\} = \omega_k(M).$$

Denote by $X_i^{(k)}$ the k-dimensional skeleton of X_i. Then $H^k(X_i, X_i^{(k)}; \mathbb{Z}_2) = 0$ and hence the long exact cohomology sequence gives that the natural pullback map from $H^k(X_i; \mathbb{Z}_2)$ into $H^k(X_i^{(k)}; \mathbb{Z}_2)$ is injective. This implies $(\Phi_i)_{|X_i^{(k)}} \in \mathcal{P}_k$. The definition of $\omega_k(M)$ then implies

$$\lim_{i\to\infty} \sup\{\mathbf{M}(\Phi_i(x)) : x \in X_i^{(k)}\} = \omega_k(M).$$

We denote by Π_i the homotopy class of $(\Phi_i)_{|X_i^{(k)}}$. Its width $\mathbf{L}(\Pi_i)$ satisfies

$$\omega_k(M) \le \mathbf{L}(\Pi_i) \le \sup\{M(\Phi_i(x)) : x \in X_i^{(k)}\}, \quad i \in \mathbb{N}$$

and in particular $\lim_{i\to\infty} \mathbf{L}(\Pi_i) = \omega_k(M)$.

Theorem 2.9 implies, for all $i \in \mathbb{N}$, the existence of smooth embedded cycles V_i so that

$$\mathbf{L}(\Pi_i) = \mathrm{vol}(V_i) \quad \text{and} \quad \mathrm{index}(V_i) \le k.$$

The Compactness Theorem of Sharp (Theorem 2.3 of [29]) gives the existence of a smooth embedded cycles V with $\mathrm{index}(V) \le k$ such that, after passing to a subsequence, $\mathrm{vol}(V_i) \to \mathrm{vol}(V)$ as $i \to \infty$, which finishes the proof in the general case.

When g is bumpy we have from Sharp (Theorem 2.3 and Remark 2.4, [29]) that the set of connected, closed, smooth, embedded minimal hypersurfaces in (M, g) with both area and index uniformly bounded is finite and so there must exist some $j \in \mathbb{N}$ so that $\omega_k(M) = \mathbf{L}(\Pi_j)$. The result follows from Theorem 2.12. $\square$

This theorem, when combined with the Multiplicity One Theorem 2.11 has the following corollary and corresponds to Theorem B in [38].

Corollary 3.11. *Assume (M^{n+1}, g) is a closed Riemannian manifold, $3 \le (n+1) \le 7$ with either a bumpy metric or a metric with positive Ricci curvature. Then there exists infinitely many smooth, connected, closed, embedded, minimal hypersurfaces.*

The argument used to prove Theorem 1.4 in [20] used a different reasoning from the one presented above because neither the Multiplicity One Theorem nor the index estimates were available at the time. A detailed sketch of the argument can be found in [21].

4. Denseness and Equidistribution of Minimal Hypersurfaces

4.1. **Denseness of minimal hypersurfaces.** Let $\mathcal{M}$ be the set of all smooth metrics with the C^∞-topology. We now present the following result, due to Irie and the authors [14].

Denseness Theorem 1.5. *Let M^{n+1} be a closed manifold of dimension $(n+1)$, with $3 \leq (n+1) \leq 7$.*

For a C^∞-generic Riemannian metric g on M, the union of all closed, smooth, embedded minimal hypersurfaces is dense.

Proof. Given a metric $g \in \mathcal{M}$, let $\mathcal{S}(g)$ denote the set of all connected, smooth, embedded minimal hypersurfaces with respect to the metric g. An element $\Sigma \in \mathcal{S}(g)$ is nondegenerate if every Jacobi vector field vanishes.

Chose an open set $U \subset M$ and set

$$\mathcal{M}_U = \{g \in \mathcal{M} : \exists\, \Sigma \in \mathcal{S}(g) \text{ with } \Sigma \cap U \neq \emptyset \text{ and } \Sigma \text{ is nondegenerate}\}.$$

The set $\mathcal{M}_U$ is open because if $\Sigma \in \mathcal{S}(g)$ is nondegenerate, an application of the Inverse Function Theorem implies that for every Riemannian metric g' sufficiently close to g, there exists a unique nondegenerate closed, smooth, embedded minimal hypersurface Σ' close to Σ. In particular, $\Sigma' \cap U \neq \emptyset$ if g' is sufficiently close to g.

If we show that $\mathcal{M}_U$ is dense in $\mathcal{M}$ then the result follows: Indeed, choose $\{U_i\}_{i\in\mathbb{N}}$ a countable basis of M and consider the set $\cap_i \mathcal{M}_{U_i}$, which is C^∞ Baire-generic in $\mathcal{M}$ because each $\mathcal{M}_{U_i}$ is open and dense in $\mathcal{M}$. Thus if g is a metric in $\cap_i \mathcal{M}_{U_i}$ then for every open set $V \subset M$ there is $\Sigma \in \mathcal{S}(g)$ that intersects V and this proves the theorem.

Consider the set

$$\mathcal{M}_U^* = \{g \in \mathcal{M} : \exists\, \Sigma \in \mathcal{S}(g) \text{ with } \Sigma \cap U \neq \emptyset\}.$$

In Proposition 2.3 of [14] it is shown that $\mathcal{M}_U$ is dense in $\mathcal{M}_U^*$ and so it suffices to see that $\mathcal{M}_U^*$ is dense in $\mathcal{M}$.

Let g be an arbitrary smooth Riemannian metric on M and $\mathcal{B}$ be an arbitrary neighborhood of g in the C^∞-topology. By the Bumpy Metrics Theorem of White (Theorem 2.1, [33]), there exists $g' \in \mathcal{B}$ such that every closed, smooth immersed minimal hypersurface with respect to g' is nondegenerate.

Since g' is bumpy, it follows from Sharp (Theorem 2.3 and Remark 2.4, [29]) that the set of connected, closed, smooth, embedded minimal hypersurfaces in (M, g') with both area and index uniformly bounded from above is finite, which means that the set $\mathcal{S}(g')$ is countable and thus

$$\mathcal{C} = \{\mathrm{vol}_{g'}(V) : V \text{ a smooth embedded cycle}\}$$

is also countable.

Consider a small perturbation $(g'(t))_{0\leq t\leq t_0}$ of g' that is supported in U and so that $\mathrm{vol}(M, g'(t_0)) > \mathrm{vol}(M, g')$. For instance, choose $h : M \to \mathbb{R}$ a smooth nonnegative function such that $\mathrm{supp}\,(h) \subset U$ and $h(x) > 0$ for some

$x \in U$, define $g'(t) = (1 + th)g'$ for $t \geq 0$, and let $t_0 > 0$ be sufficiently small so that $g'(t) \in \mathcal{B}$ for every $t \in [0, t_0]$. Because $\mathrm{vol}(M, g'(t_0)) > \mathrm{vol}(M, g')$ it follows from the Weyl Law for the Volume Spectrum 3.4 that there exists $k \in \mathbb{N}$ such that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$.

Assume by contradiction $\mathcal{B} \cap \mathcal{M}^*_U = \emptyset$. In this case, for every $t \in [0, t_0]$, every closed, smooth, embedded minimal hypersurface in $(M, g'(t))$ is contained in $M \setminus U$. Since $g'(t) = g'$ outside a compact set contained in U we have $\mathcal{S}(g') = \mathcal{S}(g'(t))$ and so we conclude from Theorem 3.10 that $\omega_k(M, g'(t)) \in \mathcal{C}$ for all $t \in [0, t_0]$. But $\mathcal{C}$ is countable and we know from Proposition 3.3 that the function $t \mapsto \omega_k(M, g'(t))$ is continuous. Hence $t \mapsto \omega_k(M, g'(t))$ is constant in the interval $[0, t_0]$. This contradicts the fact that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$.

Therefore $\mathcal{B} \cap \mathcal{M}^*_U \neq \emptyset$ and hence $\mathcal{M}^*_U$ is dense in $\mathcal{M}$.

□

4.2. **Equidistribution of minimal hypersurfaces.** In this section we explain the key ideas behind the following result.

Equidistribution Theorem 1.6. *Let M^{n+1} be a closed manifold of dimension $(n + 1)$, with $3 \leq (n + 1) \leq 7$.*

For a C^∞-generic Riemannian metric g on M, there exists a sequence $\{\Sigma_j\}_{j\in\mathbb{N}}$ of closed, smooth, embedded, connected minimal hypersurfaces that is equidistributed in M: for any $f \in C^0(M)$ we have

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^{q} \mathrm{vol}_g(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} f \, d\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M f dV.$$

Before we sketch its proof we discuss an heuristic argument. Fix $g \in \mathcal{M}$, choose $f \in C^\infty(M)$ and a small closed interval $I \subset \mathbb{R}$ containing the origin in its interior. For each $t \in I$ define $g(t) = \exp(tf)g$ and, for each $k \in \mathbb{N}$, consider the function

$$t \mapsto \mathcal{W}_k(t) = \ln \omega_k(M, g(t)) - \frac{n}{n+1} \ln \mathrm{vol}_{g(t)}(M) - \ln k^{\frac{n}{n+1}}. \tag{4.1}$$

From Proposition 3.3 we have that $\mathcal{W}_k$ is uniformly Lipschitz on I (independently of $k \in \mathbb{N}$). Thus the Weyl Law for the Volume Spectrum 3.4 implies that

$$\lim_{k\to\infty} \max\{\mathcal{W}_k(t) - \ln a(n) : t \in I\} = 0. \tag{4.2}$$

Recall the definition of minimal embedded cycles in Definition 2.1. We now assume the following strong assumption: For all $k \in \mathbb{N}$ and $t \in I$ there is a *unique* smooth embedded minimal cycle $\Sigma_k(t)$ (with respect to $g(t)$) so that

- $\mathrm{vol}_{g(t)}(\Sigma_k(t)) = \omega_k(M, g(t))$;
- $\Sigma_k(t)$ is two-sided and multiplicity one.

The uniqueness of $\Sigma_k(t)$ and the Sharp Compactness Theorem [29] implies that, for all $k \in \mathbb{N}$, the deformation $t \in I \mapsto \Sigma_k(t)$ is smooth and so, using the fact that $\partial_t g(t) = fg(t)$, we have

$$\frac{d}{dt}\omega_k(M, g(t)) = \frac{d}{dt}\mathrm{vol}_{g(t)}(\Sigma_k(t)) = \frac{n}{2}\int_{\Sigma_k(t)} f d\Sigma_k(t)$$

$$\frac{d}{dt}\mathrm{vol}_{g(t)}(M) = \frac{n+1}{2}\int_M f dV_{g(t)}.$$

Hence, we have that for all $t \in I$

$$\mathcal{W}_k'(t) = \frac{n}{2}\left(\frac{1}{\mathrm{vol}_{g(t)}(\Sigma_k(t))}\int_{\Sigma_k(t)} f d\Sigma_k(t) - \frac{1}{\mathrm{vol}(M)}\int_M f dV_{g(t)}\right).$$

Thus we deduce from (4.2) that, after passing to a subsequence, $\mathcal{W}_{k_j}'(t) \to 0$ for almost all $t \in I$. Hence setting $\Sigma^j = \Sigma_{k_j}(t)$ we deduce

$$\lim_{j\to\infty} \frac{1}{\mathrm{vol}_g(\Sigma^j)}\int_{\Sigma^j} f d\Sigma^j = \frac{1}{\mathrm{vol}(M)}\int_M f dV.$$

Without assuming the strong assumption above, the function $\mathcal{W}_k$ is only uniformly Lipschitz and one can surely find a sequence of uniformly Lipschitz functions converging to a constant whose derivative (whenever it is well defined) is uniformly away from zero. Overcoming the lack of differentiability everywhere of $\mathcal{W}_k$ is at the core of the proof of the Equidistribution Theorem 1.6.

Given a Lipschitz function ϕ on a cube $I^m \subset \mathbb{R}^m$ we know from Rademacher Theorem that ϕ is differentiable almost everywhere. Let us consider the *generalized derivative* of ϕ that is defined as

$$\partial^*\phi(t) = \mathrm{Conv}\{\lim_{i\to\infty} \nabla\phi(t_i) : \nabla\phi(t_i) \text{ exists and } \lim t_i = t\},$$

where $\mathrm{Conv}(K)$ denotes the convex hull of $K \subset \mathbb{R}^m$. If the function is C^1, the generalized derivative coincides with the classical derivative. The result we need is the following.

Lemma 4.1. *There is a constant C (depending on I^m) so that for every Lipschitz function ϕ on I^m with*

$$|\phi(x) - \phi(y)| \le \varepsilon \quad \textit{for all } x, y \in I^m,$$

there is $\bar{t} \in I^m$ with dist$(0, \partial^*\phi(\bar{t})) \le C\varepsilon$.

Proof. Let us first assume that ϕ achieves its maximum at an interior point $\bar{t} \in I^m$. We now argue that $0 \in \partial^*\phi(\bar{t})$.

Consider the compact set

$$K = \{\lim_{i\to\infty} \nabla\phi(t_i) : \nabla\phi(t_i) \text{ exists and } \lim t_i = \bar{t}\}.$$

For almost all unit vector w in $\mathbb{R}^m$ we have that the function $\phi_w(s) = \phi(\bar{t} + sw)$ is defined in an open neighborhood U of zero and $\nabla\phi(\bar{t} + sw)$ is well defined for almost all $s \in U$. The function ϕ_w has an absolute maximum at $s = 0$, its derivative is given by $\phi_w'(s) = \nabla\phi(\bar{t} + sw).w$ for s almost everywhere, and thus there is $v \in K$ with $v.w \le 0$. Hence we deduce that for every unit vector $w \in \mathbb{R}^m$ there is $v \in K$ with $v.w \geq 0$. If $0 \notin \mathrm{Conv}(K)$ there is $0 \neq p \in \mathrm{Conv}(K)$ that minimizes the distance to the origin and so $v.p \geq |p|^2$ for all $v \in \mathrm{Conv}(K)$, which is a contradiction.

We now handle the general case. Choose a smooth function $\eta : I^m \to \mathbb{R}$ with $\eta(0) = 0$ and $\eta = 2$ on ∂I^m. Set $\psi = \phi - \varepsilon\eta$. Then $\psi(x) \leq \psi(0) - \varepsilon$ for all $x \in \partial I^m$ and so ψ must have an interior maximum $\bar{t}$. Thus $0 \in \partial^*\psi(\bar{t})$ and so $\mathrm{dist}(0, \partial^*\phi(\bar{t})) \leq C\varepsilon$, where C bounds the gradient of η. $\square$

Proof of Equidistribution Theorem 1.6. Given a metric $g \in \mathcal{M}$, $\mathcal{S}(g)$ denotes the set of all connected, smooth, embedded minimal hypersurfaces with respect to g and $\mathcal{V}(g)$ denotes the set of all smooth embedded minimal cycles. Given $S \in \mathcal{V}(g)$ we define by μ_S and μ_M the unit Radon measures on M given by, respectively,

$$\mu_S(f) = \frac{||S||(f)}{||S||(M)} \text{ and } \mu_M(f) = \frac{1}{\mathrm{vol}_g(M)} \int_M f dV, \quad f \in C^0(M).$$

Finally, we define by $\mathrm{Conv}(\mathcal{V}(g))$ the unit Radon measures μ that are given by convex linear combinations of Radon measures μ_S, i.e.,

$$\left\{ \sum_{i=1}^{J} a_i \mu_{S_i} : 0 \leq a_i \leq 1, S_i \in \mathcal{V}(g), i = 1, \ldots, J \text{ and } a_1 + \ldots + a_J = 1 \right\}.$$

We say that $\mu = \sum_{i=1}^{J} a_i \mu_{S_i} \in \mathrm{Conv}(\mathcal{V}(g))$ is non-degenerate if the support of each S_i is a non-degenerate minimal hypersurface.

Choose a subset $\{\psi_i\}_{i\in\mathbb{N}} \subset C^\infty(M)$ that is dense in $C^0(M)$ and set

$$\begin{aligned}\mathcal{M}(m) = \{g \in \mathcal{M} : \exists\, \mu \in \mathrm{Conv}(\mathcal{V}(g)) &\text{ non-degenerate such that} \\ &|\mu(\psi_i) - \mu_M(\psi_i)| < m^{-1} \quad \text{for all } i = 1, \ldots, m\}.\end{aligned}$$

A standard perturbation argument based on the Inverse Function Theorem shows that $\mathcal{M}(m)$ is open in the C^∞-topology for all $m \in \mathbb{N}$.

We now explain that if $\mathcal{M}(m)$ is dense in $\mathcal{M}$ for all $m \in \mathbb{N}$ then the desired result follows. If so $\mathcal{M}_\infty = \cap_{m\in\mathbb{N}} \mathcal{M}(m)$ is a residual set (in the Baire sense) and we choose $g \in \mathcal{M}_\infty$. In this case we can find a sequence $\{\mu_m\}_{m\in\mathbb{N}}$ of elements of $\mathrm{Conv}(\mathcal{V}(g))$ so that

$$|\mu_m(\psi_i) - \mu_M(\psi_i)| < m^{-1} \quad \text{for all } i = 1, \ldots, m.$$

Hence every accumulation point ν (in the weak topology) is a Radon measure with $\nu(\psi_i) = \mu_M(\psi_i)$ for all $i \in \mathbb{N}$. Thus, from the way $\{\psi_i\}_{i\in\mathbb{N}}$ was chosen, the sequence $\{\mu_m\}_{m\in\mathbb{N}}$ converges weakly to μ_M.

We have for some $J_m \in \mathbb{N}$

$$\mu_m = \sum_{j=1}^{J_m} a_{j,m} \mu_{S_{j,m}}, 0 \leq a_{j,m} \leq 1, S_{j,m} \in \mathcal{V}(g) \text{ for all } j = 0, \ldots, J_m$$

and $\sum_{j=1}^{J_m} a_{j,m} = 1$. Choose integers $d_m, b_{j,m}$, $j = 1, \dots, J_m$ so that

$$\left| \frac{a_{j,m}}{||S_{j,m}||(M)} - \frac{b_{j,m}}{d_m} \right| < \frac{1}{mJ_m||S_{j,m}||(M)} \tag{4.3}$$

and set $V_{j,m} = b_{j,m} S_{j,m} \in \mathcal{V}(g)$. We claim that for all $f \in C^0(M)$ we have

$$\lim_{m\to\infty} \frac{\sum_{j=1}^{J_m} ||V_{j,m}||(f)}{\sum_{j=1}^{J_m} ||V_{j,m}||(M)} = \mu_M(f). \tag{4.4}$$

With $f \in C^0(M)$ fixed and $K = \sup_M |f|$ we have, using (4.3),

$$\begin{aligned}\mu_m(f) &= \sum_{j=1}^{J_m} a_{j,m}\mu_{S_{j,m}}(f) = \sum_{j=1}^{J_m} \frac{b_{j,m}}{d_m}||S_{j,m}||(f) + \sum_{j=1}^{J_m} O\left(\frac{K}{mJ_m}\right)\\ &= \sum_{j=1}^{J_m} \frac{b_{j,m}}{d_m}||S_{j,m}||(f) + O\left(\frac{K}{m}\right) = \frac{\sum_{j=1}^{J_m} ||V_{j,m}||(f)}{d_m} + O\left(\frac{K}{m}\right).\end{aligned}$$

Furthermore, combining $\sum_{j=1}^{J_m} a_{j,m} = 1$ with (4.3) we have

$$\frac{\sum_{j=1}^{J_m} ||V_{j,m}||(M)}{d_m} = \frac{\sum_{j=1}^{J_m} b_{j,m}||S_{j,m}||(M)}{d_m} = 1 + O\left(\frac{1}{m}\right),$$

which when combined with the previous identities gives

$$\mu_m(f) = \left(1 + O\left(\frac{1}{m}\right)\right) \frac{\sum_{j=1}^{J_m} ||V_{j,m}||(f)}{\sum_{j=1}^{J_m} ||V_{j,m}||(M)} + O\left(\frac{K}{m}\right).$$

Making $m \to \infty$ we deduce (4.4). One immediate consequence is that we obtain the existence of a finite sequence $\{\Sigma_{i,m}\}_{i=1}^{P_m}$ of elements in $\mathcal{S}(g)$ so that for all $f \in C^0(M)$ we have

$$\lim_{m\to\infty} \frac{1}{\sum_{i=1}^{P_m} \mathrm{vol}_g(\Sigma_{i,m})} \sum_{i=1}^{P_m} \int_{\Sigma_{i,m}} f d\Sigma_{i,m} = \frac{1}{\mathrm{vol}_g(M)} \int_M f dV.$$

Using this identity, a further combinatorial argument (see [24, pages 15, 16]) shows that we can extract a sequence $\{\Sigma_i\}_{i\in\mathbb{N}}$ of elements of $\cup_{m\in\mathbb{N}}\{\Sigma_{j,m}\}_{j=1}^{P_m}$ so that for all $f \in C^0(M)$

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^{q} \mathrm{vol}_g(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} f\, d\Sigma_j = \frac{1}{\mathrm{vol}_g M} \int_M f dV.$$

We now show that $\mathcal{M}(m)$ is dense in $\mathcal{M}$ with respect to the C^∞-topology. Consider the slightly larger set

$$\begin{aligned}\mathcal{M}^*(m) = \{g \in \mathcal{M} : \exists\, \mu \in \mathrm{Conv}(\mathcal{V}(g)) \text{ such that}&\\ |\mu(\psi_i) - \mu_M(\psi_i)| < m^{-1} \quad \text{for all } i = 1, \dots, m\}.&\end{aligned}$$

The first remark is that using Lemma 4 of [24] one can see that $\mathcal{M}^*(m)$ and $\mathcal{M}(m)$ have the same closure in $\mathcal{M}$ (the reader can see the details at the end of Section 3 of [24]). Thus it suffices to see that $\mathcal{M}^*(m)$ is dense in $\mathcal{M}$.

Choose $I^m \subset \mathbb{R}^m$ a small cube centered at the origin. For every vector $t \in I^m$ define $g(t) = \exp(\sum_{i=1}^m t_i\psi_i)g$ and, for each $k \in \mathbb{N}$, consider the functions $\mathcal{W}_k(t)$ defined in (4.1). Like it was explained during the heuristic argument, we have from Proposition 3.3 that $\mathcal{W}_k$ is uniformly Lipschitz on I^m (independently of $k \in \mathbb{N}$) and the Weyl Law for the Volume Spectrum 3.4 implies that

$$\lim_{k\to\infty} \max\{\mathcal{W}_k(t) - \ln a(n) : t \in I^m\} = 0. \tag{4.5}$$

From Lemma 2 of [24] (which is based on Smale's Transversality Theorem) there are arbitrarily small smooth perturbations of the map $t \mapsto g(t)$ that are bumpy for almost all $t \in I^m$. We leave the details for the reader to see in [24] and instead assume, for simplicity, that $g(t)$ is a bumpy metric for almost all $t \in I^m$. Using the fact that $\partial_{t_i} g(t) = \psi_i g(t)$, it is shown in [24, Lemma 2] that for a full measure set $t \in A \subset I^m$ we have

$$\frac{\partial}{\partial_{t_i}}\omega_k(M, g(t)) = \frac{n}{2}||S_k(t)||(\psi_i), \quad i = 1, \ldots, m, \tag{4.6}$$

where $S_k(t) \in \mathcal{V}(g(t))$ is a smooth embedded minimal cycle with

$$\omega_k(M, g(t)) = ||S_k(t)||(M) \quad \text{and} \quad \text{index}(S_k(t)) \le k.$$

Combining with the fact that

$$\frac{\partial}{\partial_{t_i}}\text{vol}_{g(t)}(M) = \frac{n+1}{2}\int_M \psi_i dV_{g(t)}$$

we have that for almost all $t \in I^m$

$$\frac{\partial}{\partial_{t_i}}\mathcal{W}_k(t) = \frac{n}{2}\left(\mu_{S_k(t)}(\psi_i) - \mu_M(\psi_i)\right) \quad i = 1, \ldots, m. \tag{4.7}$$

The sequence of functions $\{\mathcal{W}_k\}_{k\in\mathbb{N}}$ converges uniformly to a constant (4.5) as $k \to \infty$ and so we deduce from Lemma 4.1 the existence of $k \in \mathbb{N}$ and $\bar{t} \in I^m$ so that $\text{dist}(0, \partial^*\mathcal{W}_k(\bar{t})) < n/(2m)$.

Choose $v \in \partial^*\mathcal{W}_k(\bar{t})$ with $|v| < n/(2m)$ and set

$$K = \{\lim_{i\to\infty} \nabla\mathcal{W}_k(t_i) : \nabla\mathcal{W}_k(t_i) \text{ exists and } \lim t_i = \bar{t}\} \subset \mathbb{R}^m.$$

From Caratheodory Theorem, there are $v_0, \ldots, v_m \in K$ and $0 \le a_0, \ldots, a_m \le 1$ so that

$$v = \sum_{l=0}^m a_l v_l \quad \text{and} \quad a_0 + \ldots + a_l = 1.$$

Claim: *For each $l = 0, \ldots, m$ there is a smooth embedded cycle V_l so that*

$$v_l.e_i = \frac{n}{2}\left(\mu_{V_l}(\psi_i) - \mu_M(\psi_i)\right) \quad i = 1, \ldots, m,$$

where e_i is the i^{th} coordinate vector.

There is a sequence $\{t_j\}_{j\in\mathbb{N}} \in A \subset I^m$ so that $t_j \to \bar{t}$ and $\nabla\mathcal{W}_k(t_j) \to v_l$ as $t \to \infty$. Considering the smooth embedded minimal cycles $S_k(t_j)$ that appear in (4.7), we have from Sharp Compactness Theorem [29, Theorem 2.3] the existence of a smooth embedded minimal cycle V_l so that, after passing to a subsquence, $S_k(t_j)$ converges to V_l in the varifold sense. Thus $||S_k(t_j)||(M) \to ||V_l||(M)$ and $||S_k(t_j)||(\psi_i) \to ||V_l||(\psi_i)$, $i = 1, \dots, m$, as $j \to \infty$. Therefore we have from (4.7) that

$$\begin{aligned} v_l.e_i &= \lim_{j\to\infty} \nabla\mathcal{W}_k(t_j).e_i = \lim_{j\to\infty} \frac{\partial}{\partial_{t_i}} W_k(t_j) \\ &= \lim_{j\to\infty} \frac{n}{2}\left(\mu_{S_k(t_j)}(\psi_i) - \mu_M(\psi_i)\right) = \frac{n}{2}\left(\mu_{V_l}(\psi_i) - \mu_M(\psi_i)\right) \end{aligned}$$

for all $i = 1, \dots, m$.

The claim implies that if we consider $\mu = \sum_{l=0}^m a_l \mu_{V_l} \in \mathrm{Conv}(\mathcal{V}(g(\bar{t})))$ then

$$v.e_i = \sum_{l=0}^m a_l v_l.e_i = \sum_{l=0}^m a_l \frac{n}{2}\left(\mu_{V_l}(\psi_i) - \mu_M(\psi_i)\right) = \frac{n}{2}\left(\mu(\psi_i) - \mu_M(\psi_i)\right)$$

for all $i = 1, \dots, m$. Recalling that $|v| < n/(2m)$, the identity above implies that $g(\bar{t}) \in \mathcal{M}^*(m)$. Because the cube $I^m \subset \mathbb{R}^m$ can be chosen arbitrarily small, we deduce that $\mathcal{M}^*(m)$ is dense in $\mathcal{M}$. $\square$

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