(Scranton, PA; Conference Room Meeting) 18/05/2020

4-dimensional lattices with the same ϑ-series

In memory of John Conway

(By Conway, Sloane) (1992)

Recall: Given a lattice Λ, the ϑ-series of the lattice is given by the generating function of its vectors of length n:

$$\boxed{\vartheta_\Lambda(\tau) = \sum_{x \in \Lambda} e^{i\pi\tau \|x\|^2} \qquad (\mathrm{Im}(\tau) > 0) \qquad q = e^{i\pi\tau} \ (\text{nome})}$$

Def: Two lattices are isospectral if they have the same ϑ-series.

Q: Finite vs infinite dimension?

Isospectral operators

$= \{\lambda\}$

Finite dim. ∞ - dim

$$n\left\{\underbrace{\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}}_{n}\right.$$

Study some square matrix!

(We talk about this one today)

Depends on the type of operator:

✓ compact

✓ Δ Laplacian

"Sound of a drum"

D'

D

$$\begin{cases} \Delta u = \lambda u \\ u|_{\partial D} = 0 \end{cases}$$

Q: In the finite dim case, for which n does the following hold:

There are two non-equivalent n-dim lattices Λ, Λ' which are isospectral

There is a bit of history in the discovery of these lattices:

- (Witt, 1941) Found a pair with $n = 16$
- (Kneser, 1967) Found a pair with $n = 12$
- (Kitaoka, 1977) Found a pair with $n = 8$

- (Sloane, 1986) Found a pair with $n = 5, 6$
- (Schiemann, 1990) Found first pair with $n = 4$
- (Earnest, Nipp, 1990) Found another pair

Also; For $n = 2$ we have:

(ϑ-series determine a lattice)

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This tells us the greatest n for which all n-dim lattices are determined by their ϑ-series is: $n = 2$ or 3

On to the construction!

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Theorem

Let e_∞, e_0, e_1, e_2 be vectors with intersection matrix

$$\frac{1}{12}\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \longleftarrow \left\{ \begin{array}{l} e_\infty \cdot e_\infty = \frac{a}{12} \\ \vdots \\ e_2 \cdot e_2 = \frac{d}{12} \end{array} \right.$$

inner product matrix

for $a,b,c,d > 0$. Denote

$$[w,x,y,z] := we_\infty + xe_0 + ye_1 + ze_2$$

and let

$$\left\{ \begin{array}{l} v_\infty^\pm := [\pm 3, -1, -1, -1] \\ v_0^\pm := [1, \pm 3, 1, -1] \\ v_1^\pm := [1, -1, \pm 3, 1] \\ v_2^\pm := [1, 1, -1, \pm 3]. \end{array} \right.$$

8 vectors

Then the lattices:

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Then the lattices:

$$L^+(a,b,c,d) := \langle v_\infty^+, v_0^+, v_1^+, v_2^+ \rangle$$

$$L^-(a,b,c,d) := \langle v_\infty^-, v_0^-, v_1^-, v_2^- \rangle$$

are isospectral (might be equivalent in some cases!) with determinant $abcd$.

(Proof of the theorem)

Note that:

$\vartheta_{L^+} = \vartheta_{L^-} \iff$ There is a 1-1 correspondence of vectors of size n from

$$L^+ \longleftrightarrow L^-$$

we present this correspondence:

Let

$$\mathcal{L}_+ := \begin{array}{cccc} v_\infty^+ & v_0^+ & v_1^+ & v_2^+ \end{array}$$

$$\mathcal{L}_+ := \begin{bmatrix} 3 & 1 & 1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & 1 & 3 & -1 \\ -1 & -1 & 1 & 3 \end{bmatrix} : \langle e_\infty, e_0, e_1, e_2 \rangle \to L^+$$

so

$$L^+ = \operatorname{Im}(\mathcal{L}_+) \qquad \det(\mathcal{L}_+) = 144$$

then, consider the sublattice:

$$L^+ \supset M^+ = \mathcal{L}_+\left(\operatorname{span}\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle\right)$$

$$= \operatorname{span}\left\langle \underbrace{\begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 3 \\ -3 \end{pmatrix}}_{\text{product of signs is } +} \right\rangle$$

Letting

$$= \operatorname{span}\left\langle \underbrace{\begin{pmatrix}3\\3\\3\\3\end{pmatrix}, \begin{pmatrix}-3\\-3\\3\\3\end{pmatrix}, \begin{pmatrix}-3\\3\\-3\\3\end{pmatrix}, \begin{pmatrix}-3\\3\\3\\-3\end{pmatrix}}_{\text{product of signs is } +} \right\rangle$$

Letting

$$M^+ = \begin{bmatrix} 3 & -3 & -3 & -3 \\ 3 & -3 & 3 & 3 \\ 3 & 3 & -3 & 3 \\ 3 & 3 & 3 & -3 \end{bmatrix} \qquad \det(M^+) = 1296$$

Then :

$$\boxed{\begin{aligned} [L^+ : M^+] &= \frac{1296}{144} = 9 \\ L^+/M^+ &= \{0, \pm v_{\infty}^{\overset{-}{+}}, \pm v_0^{\overset{-}{+}}, \pm v_1^{\overset{-}{+}}, \pm v_2^{\overset{-}{+}}\} \end{aligned}}$$

L^-/M^-

The whole construction is analogous for L^-

Then there is a correspondence :

$$\boxed{\begin{gathered} M^+ \longleftrightarrow M^- \\ L^+/M^+ \longleftrightarrow L^-/M^- \\ \Downarrow \\ L^+ \longleftrightarrow L^- \end{gathered}}$$

$\underbrace{\text{(change one sign)} \quad \text{(change sign of the first coordinate divisible by 3)}}_{\text{clearly norm preserving}}$ ☺

$$\boxed{L^+ \longleftrightarrow L^-}$$

clearly norm preserving ☺

Thus, we have our 4-parameter family of isospectral lattices. Some of these lattices were amongst the ones found by Schiemann, and some are equivalent.

In the paper, a plethora of remarks are made:

(A plethora of remarks)

(i) $L^\pm$ classically integral if:

$$\begin{cases} a \equiv b \equiv c \equiv d \pmod 6 \\ a+b+c+d \equiv 0 \pmod 4 \end{cases}$$

$\pmod 8 \Rightarrow$ Even or Type II

(ii) $\pm = \sigma$

$$\left.\begin{aligned} & L^\sigma(a,b,c,d) \\ \cong\ & L^\sigma(a,d,b,c) \\ \cong\ & L^{-\sigma}(b,a,c,d) \end{aligned}\right\}$$

two letters equal $\Rightarrow$ $L^+ \cong L^-$

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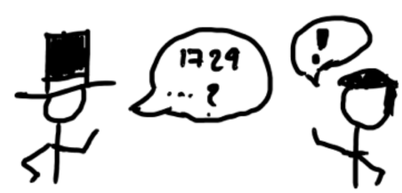
(iii) $(a,b,c,d) = (1,7,13,19)$

First non-trivial Integral case

$$\det = 1 \cdot 7 \cdot 13 \cdot 19 = \boxed{1729}$$

$$12^3 + 1^3 = 10^3 + 9^3$$



Hardy   Ramanujan

(iv) $(a,b,c,d) = \left.\begin{matrix} (1,7,13,31) \\ (1,7,19,25) \\ (2,8,14,20) \end{matrix}\right\}$ New back then (1992)

(v) <u>Conjecture</u> : $a<b<c<d \Longrightarrow L^+ \not\cong L^-$

(vi) $$\boxed{L^\sigma(a,b,c,d)^\vee = L^{-\sigma}(a^{-1},b^{-1},c^{-1},d^{-1})}$$

(vii) Making a,b,c,d close, obtain isospectral pairs close to the cubic lattice I_4

(viii) $\boxed{a=0} \Longrightarrow$ removing norm 0 vectors; get : equivalent 3-dim lattices

span (even sums of $V_{00}^\sigma, V_0^\sigma, V_1^\sigma, V_2^\sigma$) equivalent

(ix) Some examples by Schiemann come from the tetracode.

(x) There are other (2-parameter) families of isospectral 4-dim lattices :

$$\begin{bmatrix} a+6b & -5b & 5b & 2b \\ -5b & a+2b & 6b & 5b \\ 5b & 6b & a-2b & 5b \\ 2b & 5b & 5b & a-6b \end{bmatrix}, \begin{bmatrix} a+6b & -2b & b & 7b \\ -2b & a+2b & 9b & b \\ b & 9b & a-2b & 2b \\ 7b & b & 2b & a-6b \end{bmatrix}$$

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$$\det = (a^2 - 90b^2)^2 \qquad a > \sqrt{90}\,|b|$$

Phew!

Finally, the pairs of isospectral pairs of $\dim = 5, 6$:

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Two binary self dual codes:

generators

110000
001100
000011

generators

110000
101000
111111

weight enumerator:

$$(x^2+y^2)^3$$

$\Longrightarrow$ Get isospectral pairs with ϑ-series:

$$\vartheta_3(q^2)^6 = 1 + 12q^2 + 64q^4 + \cdots$$

which are integral of determinant 64, one is

$$\boxed{\sqrt{2}\, I_6}$$

$$\begin{matrix} 110000 \\ 001100 \\ 000011 \end{matrix} \qquad\qquad \begin{matrix} 110000 \\ 101000 \\ 111111 \end{matrix}$$

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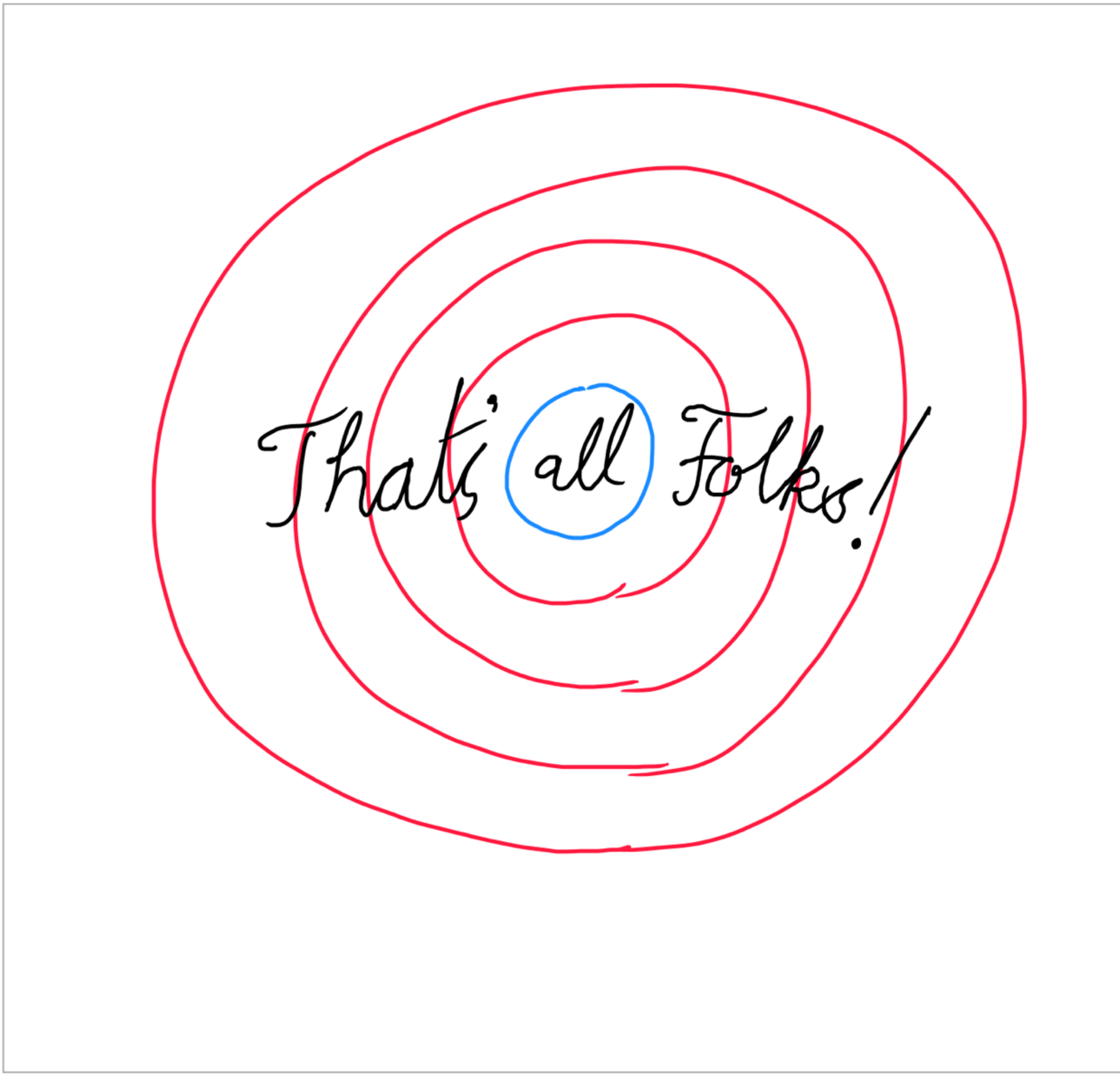
<u>What about the 5-dim?</u> Take

$$\boxed{\langle 1,1,1,1,1,1 \rangle^{\perp}}$$

$$\begin{bmatrix} 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 8 & 4 \\ 2 & 0 & 2 & 4 & 8 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 6 & 4 & 4 \\ 2 & 2 & 4 & 8 & 4 \\ 2 & 2 & 4 & 4 & 8 \end{bmatrix}$$

(Gram Matrices)


That's all Folks!