On Isomorphic Contact Diffeomorphism Groups

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Abstract

Agustin Banyaga proved, in a series of three papers, that contact structures (in a restricted sense) are determined by their isomorphism groups. In this paper, we arrange the arguments used in these papers in a self-contained manner.

1 Introduction

The goal of this paper is to condense Banyaga's proof that contact structures, in their restricted sense, are determined by their automorphism groups, into a single, self-contained paper. This proof can also be found in [3], drawing from the main results from [1] and [2].

In section 2, we present the proof of the main results drawn from [1] and [2]. Then, in section 3 we present a discussion about Epstein's axioms, which are key to the proof of the main theorem in [3], followed by the original proof of the main theorem. To begin our discussion, we present the essential notation and definitions mentioned in [3].

A *contact structure* ξ on a smooth manifold M is a hyperplane field $\xi \subset TM$ such that there exists an open cover $\{U_i\}$ of M and contact forms α_i on U_i such that $\xi_{U_i} = \ker \alpha_i$ ([8], [9]). The hyperplane field ξ is *co-orientable* if the contact form α is defined globally. A *contact manifold* is a pair (M, ξ) consisting of a smooth manifold M and a contact structure ξ on it. If ξ is a co-orientable contact structure, we call (M, ξ) a *co-orientable contact manifold.*

The automorphism group of the contact structure $\ker \alpha = \xi$ is the group

$$\mathrm{Diff}\,(M, \xi) = \{\varphi \in \mathrm{Diff}^{\infty}\,(M) \mid \varphi^{*}\xi = \xi\}$$

Let $\mathrm{Diff}_K\,(M, \xi)$ be the subgroup of $\mathrm{Diff}\,(M, \xi)$ whose elements have supports in a compact subset K, endowed with the compact-open topology, and let

$$\mathrm{Diff}_c\,(M, \xi) = \lim \mathrm{Diff}_K\,(M, \xi)$$

where K runs over all compact sets, with the direct limit topology. We denote by $G_{\xi}\,(M)$ the identity component in $\mathrm{Diff}_c\,(M, \xi)$. Namely, $G_{\xi}\,(M)$ is the group of compactly supported C^{∞} diffeomorphisms of M isotopic to the identity that preserve the contact structure ξ.

The main result we wish to recreate is the following

Theorem 1.1 ([3])**.** *Let (M_i, ξ_i), $i = 1, 2$, be connected co-orientable contact manifolds of dimension $2n+1$. If $\Phi : G_{\xi_1}\,(M_1) \to G_{\xi_2}\,(M_2)$ is a group isomorphism, then there exists a unique C^{∞} diffeomorphism $w : M_1 \to M_2$ such that $w^{*}\xi_2 = \xi_1$ and*

$$\Phi\,(h) = whw^{-1} \quad \forall h \in G_{\xi_1}\,(M_1)\,.$$

2 Key results

Before we begin the discussion of the main theorem, we shall discuss two results essential for its proof.

The first one guarantees that, given a bijection w between two contact manifolds (M_i, ξ_i) with a co-orientable contact structure, which induces an isomorphism between their groups $G_{\xi_i}\,(M_i)$, this bijection w is furthermore a contact structure exchanging C^{∞} diffeomorphism between the (M_i, ξ_i). This result is contained in [1].

The second one guarantees that given two smooth manifolds M and N and an isomorphism between two of their groups of diffeomorphisms $\phi : G\,(M) \to G\,(N)$, under certain conditions, there exists a unique homeomorphism h between the two manifolds which induces the isomorphism ϕ. This result is contained in [2].

2.1 Isomorphism-inducing bijections are contact structure exchanging C^{∞} diffeomorphisms

Theorem 2.1 ([1])**.** *Let (M_i, ξ_i), $i = 1, 2$, be two co-orientable contact manifolds. Let $w : M_1 \to M_2$ be a bijective map such that for any map $f : M_1 \to M_1$, we have*

$$wfw^{-1} \in G_{\xi_2}\,(M_2) \;\; \text{if and only if } f \in G_{\xi_1}\,(M_1)\,.$$

Then w is a C^{∞} diffeomorphism and $w^{*}\xi_2 = \xi_1$.

Proof. First we show that $w: M_1 \to M_2$ is a homeomorphism, as it is done in [7].

For a function f, let $\mathrm{Fix}(f) = \{x \in M_1 \mid f(x) = x\}$ and let $\mathscr{A} = \{\mathrm{Fix}(f) \mid f \in G_{\xi_1}(M_1)\}$ be the class of fixed subsets of elements of $G_{\xi_1}(M_1)$. Now, let $\mathscr{B} = \{M_1 - A \mid A \in \mathscr{A}\}$ be the set of complements of elements of $\mathscr{A}$. Note that $\mathscr{B}$ consists of open sets in M_1, and that for every $B \in \mathscr{B}$, B is the interior of the support of some diffeomorphism. For each $x \in M_1$, and every open subset $U \subset M_1$ containing x, we can construct and $h \in G_{\xi_1}(M_1)$ such that $x \in \mathrm{Int}(\mathrm{supp}(h))$, and $\mathrm{supp}(h) \subset U$. Namely, there exists $B \in \mathscr{B}$ such that $x \in B \subset U$. Therefore, $\mathscr{B}$ is a basis for the topology on M_1. Note that for $h \in G_{\xi_1}(M_1)$ and $g \in G_{\xi_2}(M_2)$, we have

$$\mathrm{Fix}\left(whw^{-1}\right) = w(\mathrm{Fix}(h)) \quad \text{and} \quad \mathrm{Fix}\left(w^{-1}gw\right) = w^{-1}(\mathrm{Fix}(g)).$$

Therefore w and w^{-1} take open sets to open sets, so both functions are continuous, and therefore w is a homeomorphism.

Let $\mathcal{L}_{\xi_i}(M_i)$ be the Lie algebra of vector fields with compact supports on M_i, generating 1-parameter groups of diffeomorphims h_t belonging to $G_{\xi_i}(M_i)$. Let $X \in \mathcal{L}_{\xi_1}(M_1)$ and let h_t be its 1-parameter diffeomorphisms. For each t, $H_t = wh_tw^{-1} \in G_{\xi_2}(M_2)$ by hypothesis, and the evaluation map

$$\mathcal{H}: \mathbb{R} \times M_2 \to M_2$$

given by $\mathcal{H}(t,x) = H_t(x)$, is continuous. Moreover $H_0 = \mathrm{Id}$ and $H_{t+s} = H_t \circ H_s$. Therefore $\mathcal{H}$ is a continuous action of $\mathbb{R}$ on M_2. By Theorem 3, §5.2 of Montgomery-Zippin [10, p. 212], since $\mathbb{R}$ is a Lie group, this action is C^∞ (so $\mathcal{H}$ is smooth in both variables t and x). Therefore, the 1-parameter group H_t has an infinitesimal generator, namely, a vector field $X_w \in \mathcal{L}_{\xi_2}(M_2)$ such that

$$\frac{d}{dt}H_t(x) = X_w(H_t(x)).$$

Given $f \in C^\infty(M_2)$, we have $X_w \cdot f \in C^\infty(M_2)$. For each $X \in \mathcal{L}_{\xi_1}(M_1)$ and $f \in C^\infty(M_2)$, we have

$$\frac{d}{dt}(f \circ w)(h_t(x))|_{t=0} = (X_w \cdot f)(w(x)).$$

We now want to show that w and w^{-1} are C^∞ maps. It is enough to show that $f \circ w \in C^\infty(M_1)$ for all $f \in C^\infty(M_2)$ and that $g \circ w^{-1} \in C^\infty(M_2)$ for all $g \in C^\infty(M_1)$ [11]. The situation is symmetrical, so it suffices to show that $f \circ w \in C^\infty(M_1)$ for all $f \in C^\infty(M_2)$. We now wish to compute the partial derivatives of $f \circ w$.

Let $\xi_i = \ker \alpha_i$ and let $x \in M_1$ and U be a contractible open neighbourhood of x which is the domain of a local canonical chart $\varphi: U \to \mathbb{R}^{2n+1}$. In this chart, $\alpha_1|_U = \varphi^*\underline{\alpha}$ where

$$\underline{\alpha} = dz - (x_1dy_1 + \cdots + x_ndy_n).$$

The existence of this chart is guaranteed by Darboux's theorem. On U consider the vector fields given by

$$Z = \frac{\partial}{\partial z}, \quad X_k = \frac{\partial}{\partial x_k}, \quad Y_k = \frac{\partial}{\partial y_k} + x_k\frac{\partial}{\partial z}.$$

If η is one of the vector fields above, then $L_\eta\underline{\alpha} = 0$, where $L_\eta\underline{\alpha}$ is the Lie derivative of $\underline{\alpha}$ in the direction of the vector field η. We wish to show that these vector fields defined in U can be extended into elements of $\mathcal{L}_{\xi_1}(M_1)$ [12].

It is well-known that a contact vector field η on a contact manifold (M,ξ) with co-orientable contact structure, with $\ker\alpha = \xi$, is completely determined by the function $\iota_\eta\alpha$ [13], where $\iota_\eta\alpha$ is the interior product of α and η. Therefore, if λ is a C^∞ function which is equal to 1 near x and has compact support in U, the function $\lambda(\iota_\xi\alpha)$ (where again, η is one of the vector fields above) determines contact vector fields $\overline{Z}, \overline{X_k}, \overline{Y_k}$ which have compact supports and coincide with Z, X_k, Y_k near x.

Denote by $\overline{f \circ w}$ the local expression of $f \circ w$ in the chart (φ, U). Namely, $\overline{f \circ w} = f \circ w \circ \varphi^{-1}: \varphi(U) \to \mathbb{R}$. For $a \in U$, denote $\varphi(a) = (a_1, \ldots, a_{2n+1}) = \overline{a}$. Let $h_t^z, h_t^{x_k}, h_t^{y_k}$ be the 1-parameter groups of diffeomorphisms generated by $\overline{Z}, \overline{X_k}, \overline{Y_k}$, respectively. Then, we have that near x, the diffeomorphism $\overline{h}_t^z = \varphi h_t^z \varphi^{-1}: \varphi(U) \to \varphi(U)$ is given by

$$\overline{h}_t^z(x_1, x_2, \ldots, x_{2n+1}) = (x_1 + t, x_2, \ldots, x_{2n+1}).$$

This implies that

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\overline{f \circ w}\right)(\overline{a}) &= \lim_{t\to 0}\frac{\left(f \circ w \circ \varphi^{-1}\right)\left(\varphi h_t^z \varphi^{-1}(\overline{a})\right) - \left(f \circ w \circ \varphi^{-1}\right)(\overline{a})}{t} \\
&= \lim_{t\to 0}\frac{(f \circ w)(h_t^z(a)) - (f \circ w)(a)}{t} \\
&= \frac{d}{dt}(f \circ w)(h_t^z(a))|_{t=0} \\
&= \frac{d}{dt}f(H_t^z(w(a)))|_{t=0} \\
&= \left(\overline{Z}_w \cdot f\right)(w(a)).
\end{aligned}$$

Similarly, we obtain

$$\frac{\partial}{\partial x_k}\left(\overline{f\circ w}\right)(\overline{a}) = \left(\left(\overline{X_k}\right)_w \cdot f\right)(w(a))$$
$$\frac{\partial}{\partial y_k}\left(\overline{f\circ w}\right)(\overline{a}) = \left(\left(\overline{Y_k}\right)_w \cdot f\right)(w(a)) + x_k\left(\overline{Z}_w \cdot f\right)(w(a))$$

and this shows that $f\circ w$ is a C^1 mapping. To compute higher order derivatives, one must simply replace f by $\overline{Z}_w \cdot f$, $\left(\overline{X_k}\right)_w \cdot f$, and $\left(\overline{Y_k}\right)_w \cdot f$. One can see with this that $f\circ w$ is a C^∞ map, hence, w is a C^∞ diffeomorphism.

It remains only to show that w exchanges our contact structures. We have already seen that for each $X \in \mathcal{L}_{\xi_1}(M_1)$ we get $X_w \in \mathcal{L}_{\xi_2}(M_2)$ such that $X_w = w_* X$. Therefore, w induces a Lie algebra isomorphism between the Lie algebras $\mathcal{L}_{\xi_i}(M_i)$.

Now, we use the following theorem due to Omori [12], which is a generalization of a result by Pursell-Shanks [14].

Theorem 2.2 ([12], §X). *Let (M_i, ξ_i), $i = 1, 2$, be connected contact manifolds, with co-orientable contact structures ξ_i. If $\mathcal{L}_{\xi_1}(M_1)$ and $\mathcal{L}_{\xi_2}(M_2)$ are isomorphic, there exists a C^∞ diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*\xi_2 = \xi_1$.*

By theorem 2.2, there exists a C^∞ map $\rho : M_1 \to M_2$, inducing the isomorphism $X \to X_w = w_* X$ with $\rho^*\xi_2 = \xi_1$. Now, we claim that the condition $\rho_*\eta = w_*\eta \quad \forall \eta \in \mathcal{L}_{\xi_1}(M_1)$ implies that $\rho = w$. Let $\varphi = \rho^{-1}w$ and h_t the 1-parameter group of diffeomorphisms generated by η. Then

$$\varphi h_t \varphi^{-1} = h_t.$$

If $\varphi \neq \mathrm{Id}$ then we can take $x \in M_1$ such that $\varphi(x) \neq x$. Take $\eta \in \mathcal{L}_{\xi_1}(M_1)$ such that $\eta(x) \neq 0$, with support not containing $\varphi(x)$. If h is the time-one flow of η, then $h(x) \neq x$ (since $\eta(x) \neq 0$) and $h(\varphi(x)) = \varphi(x)$, but note as well that

$$\left(\varphi h \varphi^{-1}\right)(\varphi(x)) = \varphi(h(x)) \neq \varphi(x) = h(\varphi(x))$$

and therefore $\varphi h \varphi^{-1} \neq h$, which is a contradiction. Therefore $\varphi = \mathrm{Id}$ and $\rho = w$. The proof is complete. □

2.2 Existence of a unique isomorphism-inducing homeomorphism

Before stating our second key result, we need to introduce some notation employed in [2]. Through this subsection, M will be a *smooth manifold*, where we mean a paracompact connected finite dimensional C^∞ manifold without boundary.

Let $\mathrm{Diff}^r(M)$, with $r \in [1, \infty)$, be the group of all C^r-diffeomorphisms of M. Similarly, let $\mathrm{Diff}^r_c(M) \subseteq \mathrm{Diff}^r(M)$ be the subgroup of elements with compact support. A subgroup $G(M) \subseteq \mathrm{Diff}^r(M)$ is called a *group of C^r-diffeomorphisms of M*. We say h is *$G(M)$-isotopic to the identity* if there exists a map $H : [0,1] \to G(M)$ such that $H(0) = \mathrm{Id}$, $H(1) = h$, and the evaluation map $\mathcal{H} : [0,1] \times M \to M$ given by $\mathcal{H}(t, x) = (H(t))(x)$, is C^r.

Now, we list specific subgroups of, and conditions on a given group $G(M)$ we will need further on.

Subgroup	Description
$G_c(M)$	The group of elements $h \in G(M)$ with compact support, $G(M) \cap \mathrm{Diff}^r_c(M)$.
$G(U)$	The group of elements $h \in G(M)$ with compact support in an open subset $U \subseteq M$.
$G(M)_x$	The isotropy group of $x \in M$, that is, the subgroup of $h \in G(M)$ that fix x.
$G^0(M)$	The subgroup of $h \in G(M)$ that are $G(M)$-isotopic to the identity.
$[G(M), G(M)]$	The group generated by commutators $fgf^{-1}g^{-1}$ for all $f, g \in G(M)$; or the commutator subgroup.

Property	Description
A	(Path transitivity) Given $x, y \in M$, $x \neq y$, and $c : [0,1] \to M$ with $c(0) = x$, $c(1) = y$, there is $h \in G(M)$ such that $h(x) = y$ and $\mathrm{supp}(h)$ is contained in an arbitrarily small neighbourhood of $\bigcup_{t\in[0,1]} c(t)$.
B	For any small open ball U in M centered at $x_0 \in U$ there is $h \in G(M)$ such that $\mathrm{Fix}(h) = (M - U) \cup \{x_0\}$. An open $U \subseteq M$ is an *open ball centered at* $x_0 \in U$ if there is an embedding $e : D^n_\rho \to M$ of the open disk of radius ρ centered at the origin in $\mathbb{R}^n$, into M and $U = e(D^n_\sigma)$ for some $\sigma \leq \rho$ and $e(0) = x_0$.
L	(Locality, this property holds on $F \subseteq G(M)$ a subgroup of $G(M)$) For every open cover $\mathscr{U} = (U_i)_{i\in I}$ of M with open balls U_i, we have $\left[G^0_c(M), G^0_c(M)\right] \subseteq F$ if for every U_i, $\left[G^0(U_i), G^0(U_i)\right] \subseteq F$.
C	For any pair (U, x) with $x \in U \subset M$ there is $\mathrm{Id} \neq h \in G(M)$ with $\mathrm{supp}(h) \subset U$ and $x \in \mathrm{Int}(\mathrm{supp}(h))$.
$T(n)$	Given two sets $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ of distinct points, there is $h \in G(M)$ such that $h(x_i) = y_i\ \forall i$.

An immediate consequence of these properties, which we will use in the proof of our result, is the following

Proposition 2.3 ([2]). *If a group of diffeomorphisms satisfies property A (path transitivity), then it satisfies properties C and $T(n)$ for every n, provided* $\dim M > 1$.

Now, we may state our second result.

Theorem 2.4 ([2]). *Let $\phi : G(M) \to G(N)$ be a group isomorphism between two groups of diffeomorphisms of smooth manifolds M and N. If $G(M)$ and $G(N)$ are non-abelian, both satisfy properties A and B, and*

$$\phi^{-1}\left(G(N)_n\right), \phi\left(G(M)_m\right)$$

have the property L for all $m \in M$ and $n \in N$, then there exists a unique homeomorphism $w : M \to N$ with $\phi(f) = wfw^{-1}$.

Proof. First, we will need the following lemma.

Lemma 2.5. *Let $G(M)$ and $G(N)$ be two groups of diffeomorphisms satisfying properties $T(1)$ and C. If $\phi : G(M) \to G(N)$ is an isomorphism such that there exists $x_0 \in M$ and $y_0 \in N$ such that*

$$\phi\left(G(M)_{x_0}\right) = G(N)_{y_0}$$

then there exists a homeomorphism $w : M \to N$ such that $\phi(f) = wfw^{-1}$ for all $f \in G(M)$. Moreover, if $G(M)$ and $G(N)$ satisfy property $T(2)$, then w is unique.

Proof of lemma lemma 2.5. If ϕ takes $G(M)_{x_0}$ to $G(N)_{y_0}$, it induces the following well defined map $w : M \to N$. For $x \in M$, choose $g \in G(M)$ such that $g(x_0) = x$ (we can do so by property $T(1)$), and we define

$$w(x) = \phi(g)(y_0).$$

Note that if another $g' \in G(M)$ satisfies $g(x_0) = g'(x_0) = x$ then $g'^{-1}g \in G(M)_{x_0}$ and $\phi\left(g'^{-1}g\right) = \phi(g')^{-1}\phi(g)$. But recall $\phi\left(g'^{-1}g\right) \in G(N)_{y_0}$ so

$$\phi(g')(y_0) = \phi(g)(y_0).$$

Hence, w is well defined. Now let us see that it is a bijection.

If $w(x_1) = w(x_2)$ then there are $g_1, g_2 \in G(M)$ such that $g_1(x_0) = x_1$ and $g_2(x_0) = x_2$, and $\phi(g_1)(y_0) = \phi(g_2)(y_0)$. Then, we have $\phi(g_1)\phi(g_2)^{-1} = \phi\left(g_1 g_2^{-1}\right) \in G(N)_{y_0}$ so $g_1 g_2^{-1} \in G(N)_{x_0}$. Hence, $g_1(x_0) = g_2(x_0)$, and $x_1 = x_2$. Therefore w is injective. Now, if $y \in N$, we can choose $h \in G(N)$ such that $h(y_0) = y$. Set $x = \phi^{-1}(h)(x_0)$. Then $w(x) = \phi\left(\phi^{-1}(h)\right)(y_0) = h(y_0) = y$. Hence, w is surjective. Since w is both injective and surjective, it is a bijection.

Next, we show that w induces ϕ. Let $y \in N$ and $h \in G(N)$ such that $h(y_0) = y$, and let $x = \phi^{-1}(h)(x_0)$. We have seen that $w(x) = y$. Let $f \in G(M)$ and choose $g \in G(M)$ such that $g(x_0) = f(x)$. Then, $f^{-1}g(x_0) = x = \phi^{-1}(h)(x_0)$, so

$$g^{-1}f\phi^{-1}(h) \in G(M)_{x_0}.$$

This implies that $\phi(g)^{-1}\phi(f)h \in G(N)_{y_0}$, so $\phi(f)h(y_0) = \phi(g)(y_0)$ but $h(y_0) = y = w(x)$ and $\phi(g)(y_0) = w(f(x))$, so

$$\phi(f)w(x) = w(f(x))$$

and since w is a bijection, $\phi(f) = wfw^{-1}$ for all $x \in M$.

To show that w is a homeomorphism, we proceed as in the proof of theorem 2.1. We let $\mathscr{A} = \{\mathrm{Fix}(f) | f \in G(M)\}$, and $\mathscr{B} = \{M - A | A \in \mathscr{A}\}$. Property C implies that $\mathscr{B}$ is a basis for the topology on M. Since $\phi(f) = wfw^{-1}$, it follows that $\mathrm{Fix}(\phi(f)) = w(\mathrm{Fix}(f))$. This implies that w and w^{-1} take open sets to open sets, so both functions are continuous. Hence w is a homeomorphism.

Lastly, we need to show w is unique if property $T(2)$ is satisfied. If there exists another homeomorphism $w' : M \to N$ such that $\phi(f) = w'fw'^{-1} = wfw^{-1}$. Then setting $\rho = w'^{-1}w$ we obtain

$$\rho f \rho^{-1} = f \quad \forall f \in G(M).$$

We show that ρ must be the identity map. Supposing it is not, there is $x \in M$ such that $\rho(x) \neq x$. Consider another point $y \in M$ distinct of x and $\rho(x)$. By property $T(2)$ there is some $f \in G(M)$ such that $f(x) = x$ and $f(\rho(x)) = y$ but

$$\rho f \rho^{-1}(\rho(x)) = \rho f(x) = \rho(x) \quad \text{and} \quad f(\rho(x)) = y \neq \rho(x).$$

Therefore, this contradicts that $\rho f \rho^{-1} = f$, so $\rho = \mathrm{Id}$ and therefore $w = w'$. □

Now we will need a result due by [7], which only uses the property C and $T(3)$.

Lemma 2.6. *Let $G(M)$, $G(N)$ be two groups of diffeomorphisms of smooth manifolds M and N, satisfying properties C and $T(3)$, and let $\phi : G(M) \to G(N)$ be an isomorphism. Let*

$$F = \phi^{-1}\left(G(N)_y\right)$$

for some $y \in N$. If there is a nonempty proper closed subset $A \subset M$ such that $f(A) = A \quad \forall f \in F$. Then $A = \{x\}$ and $F = G(M)_x$.

Note that with lemma 2.5 and lemma 2.6 we only need to construct a nonempty proper subset of M invariant under F.

Proof of the existence of a proper closed subset under F. We follow Filipkiewicz for this.

Let $\phi : G(M) \to G(N)$ be an isomorphism. For $n \in N$ let $\mathscr{M}_n$ be the set of all open balls U of M with

$$\left[G^0(U), G^0(U)\right] \subseteq F_n = \phi^{-1}\left(G(N)_n\right)$$

Similarly, for $m \in M$ let $\mathscr{N}_m$ be the set of all open balls V in N such that

$$\left[G^0(V), G^0(V)\right] \subseteq F'_m = \phi\left(G(M)_m\right)$$

Note that $\mathscr{M}_n$ and $\mathscr{N}_m$ may be empty. Let

$$M_n = M - \bigcup_{v \in \mathscr{M}_n} v \quad \text{and} \quad N_m = N - \bigcup_{v \in \mathscr{N}_m} v.$$

Proposition 2.7. *The subsets M_n and N_m are closed subsets of M and N, respectively. Moreover,*

$$f(M_n) = M_n \; \forall f \in F_n \quad \text{and} \quad g(N_m) = N_m \; \forall g \in F'_m.$$

Assuming that $G(M)$ and $G(N)$ are nonabelian groups satisfying the $T(1)$ property and that F_y, respectively F'_x, have the property L for all $x \in M$, respectively for all $y \in N$, then M_n, respectively N_m, are nonempty.

Proof of proposition 2.7. First, it is clear that M_n and N_m are closed subsets.

Secondly, we show F_n and F'_m fix M_n and N_m, respectively. Let $U \in \mathscr{M}_n$ and $f \in F_n$ and set $V = f(U)$. Clearly, $G(V) = fG(U)f^{-1}$. Therefore,

$$\left[G^0(V), G^0(V)\right] = f\left[G^0(U), G^0(U)\right] \subseteq fF_nf^{-1} \subseteq F_n$$

so $V \in \mathscr{M}_n$. Hence, $f(M_n) = M_n$. The same argument shows $g(N_m) = N_m \; \forall g \in F'_m$.

Suppose that $M_n = \varnothing$. Then $\mathscr{M}_n$ is an open cover of M by balls $\{U_i\}_{i \in I}$ such that

$$\left[G^0(U_i), G^0(U_i)\right] \subset F_n \quad \forall i \in I.$$

Let $y \in N$ be an arbitrary point. Since $G(N)$ has the $T(1)$ property, there is $f \in G(N)$ such that $f(n) = y$. Then

$$G(N)_y = fG(N)_n f^{-1}$$

and

$$F_y = \phi^{-1}\left(G(N)_y\right) = \phi^{-1}\left(fG(N)_n f^{-1}\right) = \phi(f)^{-1}\phi\left(G(N)_n\right)\phi(f) = \rho F_n \rho^{-1}$$

where $F_n = \phi^{-1}\left(G(N)_n\right)$ and $\rho = \phi^{-1}(f)$. Let $\mathscr{V} = \{V_i = \rho(U_i) \,|\, U_i \in \mathscr{M}_n\}$. Then $\mathscr{V}$ is an open cover of M by balls V_i and

$$\left[G^0(V_i), G^0(V_i)\right] = \rho\left[G^0(U_i), G^0(U_i)\right]\rho^{-1} \subseteq \rho F_n \rho^{-1} = F_y.$$

Hence $\left[G^0(V_i), G^0(V_i)\right] \subseteq F_y$ for all $V_i \in \mathscr{V}$. By the property L,

$$\left[G^0_c(M), G^0_c(M)\right] \subseteq F_y.$$

Therefore, $\phi\left(\left[G^0_c(M), G^0_c(M)\right]\right) \subseteq G(N)_y$ for all $y \in N$, and this implies that

$$\phi\left(\left[G^0_c(M), G^0_c(M)\right]\right) \subseteq \bigcap_{y \in N} G(N)_y = \{\mathrm{Id}_M\}.$$

But ϕ is an isomorphism, so this implies that $\left[G^0_c(M), G^0_c(M)\right] = \{\mathrm{Id}_M\}$ which is impossible as $G^0_c(M)$ is nonabelian. Hence, $M_n \neq \varnothing$. The same argument shows that $N_m \neq \varnothing$. □

We only need to know when M_n, respectively N_m, are proper subsets.

Lemma 2.8. *Suppose $G(M)$ and $G(N)$ have the properties B and $T(3)$. Then either M_n is a proper subset, or there exists $m \in M$ such that N_m is a proper subset.*

Proof of lemma 2.8. We use the following result by Filipkiewick [7, Lemma 3.3].

Lemma 2.9. *Let $G(M), G(N)$ be two groups of diffeomorphisms of smooth manifolds M and N and $\phi : G(M) \to G(N)$ is a group isomorphism. Suppose $G(M)$ has property B. Let*

$$F = \phi^{-1}\left(G(N)_y\right)$$

for some $y \in N$. There exists $f \in F, f \neq Id$ such that $Int(Fix(f)) \neq \varnothing$.

This proof will follow Filipkiewicz closely. Property B holds, so applying lemma 2.8 there exists $g_0 \neq \mathrm{Id}$ with $g_0 \in F_n = \phi^{-1}(G(N)_n)$ with

$$X = \mathrm{Int}(\mathrm{Fix}(g_0)) \neq \varnothing.$$

The set $Y = \mathrm{Fix}(\phi(g_0)) \neq \varnothing$ as it contains $n \in Y$. Let

$$H = \phi^{-1}\{h \in G(N) \mid h(Y) = Y\}$$
$$K = \phi^{-1}\{h \in G(N) \mid Y \subset \mathrm{Fix}(h)\}.$$

Then K is a normal subgroup of H. Since K contains g_0, H and K are nontrivial groups. If $h \in \phi(K)$, $h(n) = n$ as $n \in Y \subset \mathrm{Fix}(h)$. This implies $\phi(K) \subset G(N)_n$, namely, $\phi(K) \subseteq F_n$. Now, there are two possibilities:

a) For all $k \in K$, $X \subseteq \mathrm{Fix}(k)$.

b) There is $k \in K$ and $x \in X$ such that $k(x) \neq x$.

Case a). Let $h \in G(N - Y)$. Then $Y \in \mathrm{Fix}(h)$, so $\phi^{-1}(h) \in K$. Then, $\phi^{-1}(h)$ fixes x for all $x \in X$. That is, $\phi^{-1}(h) \in G(M)_x$, or equivalently $h \in \phi(G(X)_x)$. This implies

$$G(N - Y) \subseteq \phi(G(M)_x) \quad \forall x \in X.$$

Let V be any open ball $V \subset N - Y$. Then

$$\left[G^0(V), G^0(V)\right] \subseteq G^0(V) \subseteq G^0(N - Y) \subseteq \phi(G(M)_x) \quad \forall x \in X.$$

By definition of $\mathscr{N}_x$, for all $x \in X$, any open ball $V \subset N - Y$ belongs to $\mathscr{N}_x$. Hence, if *a*) holds, $\mathscr{N}_x \neq \varnothing$ for all $x \in X$, which in turn implies that N_x is proper for all $x \in X$.

Case b). First let us show (exactly like in [7]) that $G(X) \subset H$. Let $g \in G(X)$ with $g \neq \mathrm{Id}$, which exists by property C. Now, let us see that $gg_0g^{-1} = g_0$. If $x \in \mathrm{Fix}(g_0) - X = \partial(\mathrm{Fix}(g_0))$, then $g(x) = x$ and $g_0(x) = x$. Therefore, $gg_0g^{-1}(x) = x = g_0(x)$. If $x \notin \mathrm{Fix}(g_0)$, then $g_0(x) \notin \mathrm{Fix}(g_0)$. Since $\mathrm{supp}(g) \subset X$ then $g(x) = x$, so $gg_0g^{-1}(x) = g(g_0(x)) = g_0(x)$. For $x \in X$, we have $gg_0g^{-1}(x) = x = g_0(x)$. Hence,

$$\phi(g)\phi(g_0)\phi(g)^{-1} = \phi(g_0)$$

and therefore

$$Y = \mathrm{Fix}(\phi(g_0)) = \mathrm{Fix}\left(\phi(g)\phi(g_0)\phi(g)^{-1}\right) = \phi(g)\,\mathrm{Fix}(\phi(g_0)) = \phi(g)(Y)$$

so $g = \phi^{-1}(\phi(g)) \in H$ and $G(X) \subseteq H$.

We have, by assumption of case *b*), that there is $x_0 \in X$ and $k_0 \in K$ such that $k_0(x_0) \neq x_0$. Let $U \subseteq M$ be an open ball contained in X with $x_0 \in U$. We may assume that $k_0(x_0) \in U$, since if it were not originally, then we can choose $f \in G(U) \subseteq G(X) \subseteq H$ with $y_0 = f(x_0)$ with $y_0 \in U$, using property $T(2)$. Then $f^{-1}k_0(x_0) = k_0(x_0)$ since $k_0(x_0) \notin U$ and $\mathrm{supp}(f) = \mathrm{supp}\left(f^{-1}\right) \subset U$. Hence

$$\tilde{k}_0(x_0) = fk_0^{-1}f^{-1}k_0(x_0) = f(x_0) = y_0.$$

Then $\tilde{k}_0(x_0) \neq x_0$, $\tilde{k}_0(x_0) = y_0 \in U$ and $\tilde{k} = \left(fk_0^{-1}f^{-1}\right)k_0 \in K$ since K is a normal subgroup of H and $f \in H$.

Now we show that K acts transitively on U. Let $y \in U$. By property A, there is $\rho \in G(U)$ with $\rho(y) = x_0$ and $\rho(y_0) = y_0$, where $y_0 = \tilde{k}_0(x_0) \in U$. Then

$$\rho\tilde{k}_0(x_0) = \tilde{k}_0(x_0) = y_0, \quad \text{and} \quad \tilde{k}_0^{-1}\rho\tilde{k}_0(x_0) = x_0 = \rho(y).$$

Hence

$$\hat{g}(x_0) = \left(\rho^{-1}\tilde{k}_0^{-1}\rho\right)\tilde{k}_0(x_0) = y$$

and $\hat{g} \in K$ as $\tilde{k}_0 \in K$ and $\rho \in G(U) \subseteq G(X) \subseteq H$. Therefore, as $\hat{g}(x_0) = y$, K acts transitively on U.

Now, given three distinct elements $x_0, x_1, x_2 \in U$ there are $g_1, g_2 \in K$ such that $g_i(x_0) = x_i$ for $i = 1, 2$. Let U_0 be a small open ball containing x_0 and such that the sets

$$\left\{U_0, g_1U_0, g_2U_0, g_1^{-1}U_0, g_2^{-1}U_0\right\}$$

are mutually disjoint and their union is contained in U. An easy argument (by Thurston) shows that if $h_1, h_2 \in G(U_0)$, then $[h_1, h_2] = [[h_1, g_1], [h_2, g_2]]$ where $[h_i, g_i] = \left(h_i g_i h_i^{-1}\right) g_i^{-1} \in K$. This proves that

$$\left[G^0(U_0), G^0(U_0)\right] \subseteq K \subseteq F_n = \phi^{-1}\left(G(N)_n\right)$$

and therefore $U_0 \in \mathscr{M}_n$, so $\mathscr{M}_n \neq \varnothing$ and M_n is proper.

Hence, we have shown that either N_x is proper for all $x \in X$, or M_n is proper. This concludes the proof of lemma 2.8. $\square$

To finish the proof of the existence of our proper closed nonempty subset, we use lemma 2.8. Given $\phi : G(M) \to G(N)$ with the hypothesis of theorem 2.4, starting with any point $y_0 \in N$, we have two possibilities, with which we conclude the proof.

a) $X = M_{y_0}$ is proper nonempty closed and invariant by $F_{y_0} = \phi^{-1}\left(G(N)_{y_0}\right)$.

b) There exists $z_0 \in M$ such that N_{z_0} is proper nonempty closed. In this case, we instead consider the isomorphism $\phi^{-1} : G(N) \to G(M)$, as N_{z_0} is invariant under $F'_{z_0} = \phi\left(G(M)_{z_0}\right)$.

$\square$

In case *a*) above, lemma 2.6 applied to ϕ shows that $M_{y_0} = \{x_0\}$ and $F_{y_0} = G(M)_{x_0}$. In case *b*), lemma 2.6 applied to ϕ^{-1} shows $N_{z_0} = \{u_0\}$ with $u_0 \in N$, and $F'_{z_0} = G(N)_{u_0}$. Hence, in any case, ϕ takes isotropy subgroups of $G(M)_{m_0}$ for some point $m_0 \in M$, to the isotropy subgroups $G(N)_{n_0}$ of some point $n_0 \in N$. Therefore, the hypothesis of lemma 2.5 are satisfied. Hence, lemma 2.5, lemma 2.6 and lemma 2.8 yield a complete proof of theorem 2.4.

$\square$

3 Main theorem

Now, it suffices to prove that for isomorphic $\Phi : G_{\xi_1}(M_1) \to G_{\xi_2}(M_2)$, G_{ξ_1} and G_{ξ_2} are nonabelian, have the A and B property, and $\Phi\left(G_{\xi_1}(M_1)_{m_1}\right)$ and $\Phi^{-1}\left(G_{\xi_2}(M_2)_{m_2}\right)$ have the L property for every $m_1 \in M_1$ and $m_2 \in M_2$.

In fact, we show every subgroup of $G_{\xi_i}(M_i)$ has the L property, hence the result follows. Before proving this, several intermediate steps are required, which include the proof that $G_{\xi_i}(M_i)$ satisfies the Epstein's axioms, and a modification of Filipkiewicz's shrinking lemma.

3.1 Preliminaries

We will make use of the following results.

Theorem 3.1 ([4]). *Let (M, ξ) be a co-orientable contact manifold and let h_t be a contact isotopy. Let $F \subseteq M$ be a closed set, and let $U, W \subseteq M$ be open sets such that*

$$\bigcup_{t\in[0,1]} h_t(F) \subset U \subset \overline{U} \subset W.$$

Then there is a contact isotopy $\tilde{h}_t$ such that $\tilde{h}_t|_F = h_t$ and supp $\left(\tilde{h}_t\right) \subset W$.

Theorem 3.2 (Lychagin's Theorem). *There is a diffeomorphism $\Omega : \mathcal{W} \to \mathcal{V}$, where $\mathcal{W}$ is a neighbourhood of the identity in Diff$_c(M, \xi)$, and $\mathcal{V}$ is a neighbourhood of* 0 *in $\mathcal{L}_\xi(M)$.*

Now we present the Epstein's axioms and theorem. Let G be a group of diffeomorphisms of a paracompact Hausdorff topological space M, and let $\mathscr{U}$ be a basis pf the topology on M. The *Epstein's axioms* for the triple $(G, \mathscr{U}, M)$ are

A1. If $U \in \mathscr{U}$ and $g \in G$, then $gU \in \mathscr{U}$.

A2. G acts transitively on $\mathscr{U}$.

A3. Let $g \in G$, $U \in \mathscr{U}$ and $\mathscr{B}$ be an open cover of M. Then there are $g_1, \ldots, g_n \in G$ and $V_1, \ldots, V_n \in \mathscr{B}$ such that:

- (i) $g = g_n \circ g_{n-1} \circ \cdots \circ g_1$
- (ii) $\operatorname{supp}(g_i) \subseteq V_i$
- (iii) $\operatorname{supp}(g_i) \cup \left(g_{i-1} \circ \cdots \circ g_1 \overline{U}\right) \neq M$.

Theorem 3.3 (Epstein's Theorem). *If the triple $(G, \mathscr{U}, M)$ satisfies Epstein's axioms, then $[G, G]$ is a simple group.*

3.2 Verification of Epstein's axioms for $G_\xi(M)$

Let (M,ξ) be a co-orientable contact manifold and consider $G_\xi(M)$. Fix a point $p \in M$ and let (V,ψ) be a Darboux chart of an open neighbourhood of p such that $\psi(V) = W \subseteq \mathbb{R}^{2n+1}$ and $\psi(p) = 0$. Let $\tau > 0$ such that the open ball $D_{4\tau}$ of radius 4τ centered at $0 \in \mathbb{R}^{2n+1}$, is contained in W. Let

$$\mathbf{B} = \psi^{-1}(D_\tau) \subset M.$$

With this construction, we wish to construct a basis $\mathscr{U}_\xi$ of the topology on M that the triplet $(G_\xi(M), \mathscr{U}_\xi, M)$ satisfies Epstein's axioms.

Lemma 3.4. *The subsets $\mathscr{U}_\xi = \{\varphi\mathbf{B} \mid \varphi \in G_\xi(M)\}$ form a basis for the topology on M.*

Proof. Denote by (x,y,z) a point of $\mathbb{R}^{2n+1}$ where $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. Let

$$\omega_0 = x_1 dy_1 + \cdots + x_n + dy_n + dz$$

be the canonical contact form on $\mathbb{R}^{2n+1}$. For each $t \in \mathbb{R}^+$, let

$$R_t : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1} \quad (x,y,z) \mapsto \left(tx, ty, t^2 z\right)$$

be a contact homothety, which is a contraction for $t < 1$ and a dilation for $t > 1$. Theorem 3.1 applied with $F = \overline{D}_\tau$ yields the following result

Lemma 3.5. *For each small $\sigma, \tau \in \mathbb{R}$ with $3\tau \leq 1$, there is a contact isotopy $\mathcal{R}_t^\sigma$ of $\mathbb{R}^{2n+1}$ such that*

$$\mathcal{R}_t^\sigma|_{\overline{D}_\tau} = R_{\sigma\tau} \quad \textit{and} \quad \textit{supp}\,(\mathcal{R}_t^\sigma) \subset D_{3\tau}.$$

Moreover, $\textit{Fix}\,(\mathcal{R}_1^\sigma) = \left(\mathbb{R}^{2n+1} - D - 3\tau\right) \cup \{0\}$.

Proof of lemma 3.5. Consider a bump function λ_σ positive on $D_{2\tau}$ and given by

$$\lambda_\sigma(x) = \begin{cases} \sigma & x \in D_\tau \\ 0 & x \in \mathbb{R}^{2n+1} \setminus D_{3\tau} \end{cases}$$

Then, the isotopy we look for will be given by the following vector field: Let $\xi = \ker\alpha$. Then there is an isomorphism $\varsigma : \mathcal{L}_\xi(M) \to C^\infty(M)$ given by $\varsigma(X) = \iota_X\alpha$. Hence, one can assign a vector field $\varsigma^{-1}(f)$ to any $f \in C^\infty(M)$. In particular, one can assign a contact vector field to any vector field X defined as

$$\mathcal{C}(X) = \varsigma^{-1}(\iota_X\alpha).$$

Finally, our contact isotopy $\mathcal{R}_t^\sigma$ is defined by the contact vector fields $X_t = \mathcal{C}(\lambda_\sigma \nabla R_t)$. □

Now, we return to our original proof. Consider τ as in the definition of $\mathbf{B}$. Using the chart (V,ψ), we can obtain a contact isotopy ρ_t^σ of M equal to $\psi^{-1}\mathcal{R}_t^\sigma\psi$ on V and the identity outside V. Note that the contact diffeomorphism $\rho_1^\sigma = \rho_\sigma$ fixes p and shrinks $\mathbf{B}$ into an arbitrarily small neighbourhood of $\rho_\sigma(\mathbf{B}) = \mathbf{B}_\sigma \in \mathscr{U}$ of p. Now we are ready to show $\mathscr{U}$ is a basis.

Pick $x \in M$ and an open neighbourhood $\mathcal{O}$ of x. By Boothby's transitivity theorem [5], there is $h \in G_\xi(M)$ such that $h(p) = x$ and $h(\mathbf{B}_\sigma) = U \subset \mathcal{O}$ for small enough σ. Hence, $x \in U$ and $U = h\rho_\sigma(\mathbf{B}) \in \mathscr{U}$. □

Remark 3.6. As a side result, it follows from lemma 3.5 that

$$\operatorname{Fix}(\rho_\sigma) = \left(M - \psi^{-1}(D_{3\tau})\right) \cup \{p\}$$

which implies $G_\xi(M)$ has the property B.

Now that we have a candidate basis $\mathscr{U}_\xi$, we proceed to showing the following

Theorem 3.7. *The triplet $(G_\xi(M), \mathscr{U}_\xi, M)$ satisfies Epstein's axioms.*

Proof. Axioms **A1** and **A2** can be trivially verified.

To verify axiom **A3** we need several intermediate results.

Lemma 3.8. *Let $X \in \mathcal{L}_\xi(M)$ and let $\nu \in C^\infty(M)$ be a compactly supported function. Define the contact vector field*

$$X^\nu := \mathcal{C}(\nu X) = \varsigma^{-1}(\iota_{\nu X}\alpha)$$

where $\ker\alpha = \xi$. Then $\|X^\nu\|_1 \leq m_\nu \|X\|_1$, for some $m_\nu > 0$. Here $\|\cdot\|_1$ denotes the C^1-norm.

Proof of lemma 3.8. Denoting $f_X = \iota_X \alpha$, we have

$$\begin{aligned} X^\nu &= (\nu f_X)\,\eta + \Phi^{-1}\left(\left(\iota_\eta d\left(\nu f_X\right)\right)\alpha - d\left(\nu f_X\right)\right) \\ &= \nu\left(f_X \eta + \Phi^{-1}\left(\left(\iota_\eta d f_X\right)\alpha - d f_X\right)\right) + f_X\left(\Phi^{-1}\left(\left(\iota_\eta d\nu\right)\alpha - d\nu\right)\right) \\ &= \nu X + f_X\left(\Phi^{-1}\left(\left(\iota_\eta d\nu\right)\alpha - d\nu\right)\right) \end{aligned}$$

where Φ is the isomorphism between horizontal vector fields and semi-basic 1-forms. Hence

$$\|X^\nu\|_1 \le \|\nu X\|_1 + \|f_X H_\nu\|_1$$

where H_ν is the horizontal part of the contact vector field $\varsigma^{-1}(\nu)$ corresponding to ν. Let A_ν be such that $\|\nu\|_1 < A_\nu$. Since ς is a linear map (hence bounded, as it is continuous), $\|f_X\|_1 \le c_1 \|X\|_1$, where c_1 depends only on α. Furthermore, Φ, Φ^{-1} are linear operators, we have $\|H_\nu\|_1 \le c_2 \|\nu\|_1 < c_2 A_\nu$ (where c_2 depends only in Φ, which in turn depends on $d\alpha$). Hence

$$\|X^\nu\|_1 \le A_\nu \|X\|_1 + \left(c_1 \|X\|_1\right)\left(c_2 A_\nu\right) = A_\nu\left(1 + c_1 c_2\right)\|X\|_1 .$$

□

Lemma 3.9. *Let $\phi \in G_\xi(M)$ with supp$(\phi) \subset K$ where K is compact. Let $\{U_i\}_{i=1}^n$ be a finite open cover of K. Then*

$$\phi = \phi_1 \circ \cdots \circ \phi_n$$

where supp$(\phi_i) \subset U_i$ and $\phi_i \in G_\xi(M)$ for $i = 1, \ldots, n$.

Proof of lemma 3.9. We may assume ϕ is as close to the identity as we want. For this, consider an isotopy $\phi_t \in \mathrm{Diff}_c(M, \xi)$ from the identity to ϕ, then $\phi = \phi_1$ can be written as $\phi = \phi^n \circ \cdots \circ \phi^n$ where $\phi^i = \phi_{i/n} \circ \phi^{-1}_{i-1/n}$ for a sufficiently large n.

Consider the Lychagin chart $\Omega : \mathcal{W} \to \mathcal{V}$ in theorem 3.2, and choose a neighbourhood of the identity $\mathcal{O} \subset \mathcal{W}$. Let

$$r = \sup\left\{s \mid B_s(0) \subset \Omega(\mathcal{O})\right\}$$

where $B_s(0) = \{X \in \mathcal{L}_\xi(M) \mid \|X\|_1 < s\}$. Now choose a partition of unity $\{\lambda_i\}$ subordinate to the open cover $\{U_i\}$, and define $\mu_i = \sum_{j=1}^i \lambda_j$. By lemma 3.8 we have that $\|\Omega(\phi)^{\mu_i}\|_1 \le m_i \|\Omega(\phi)\|_1$ for each i. Let $k = \max\{m_i\}$. Now, we can choose ϕ close enough to the identity so that $\|\Omega(\phi)\|_1 < \frac{r}{k}$ and hence $\|\Omega(\phi)^{\mu_i}\|_1 \le r$ for all i. This implies that $t\Omega(\phi)^{\mu_i}$ is in the image of the chart Ω.

Note that for $x \notin U_i$, $\mu_i = \mu_{i-1}$, and $\mu_n = 1$. Defining $h_t^i = \Omega^{-1}\left(t\Omega(\phi)^{\mu_i}\right)$, we have $h_i^t = h_{i-1}^t$ outside U_i and $h_n^t = \phi$. Hence, since

$$\phi_i^t = h_i^t \circ \left(h_{i-1}^t\right)^{-1}$$

has support in U_i and by construction, we have $\phi^t = \phi_n^t \circ \cdots \circ \phi_1^t$, our result follows. □

Lemma 3.10. *Let $g \in G_\xi(M)$ where supp$(g) \subset V \in \mathscr{U}_\xi$. Let U be a non-dense open subset of M. Then $g = g_2 \circ g_1$ where $g_i \in G_\xi(M)$, supp$(g_i) \subset V$ and*

$$\text{supp}(g_1) \cup \overline{U} \ne M \quad \text{and} \quad \text{supp}(g_2) \cup g_1\overline{U} \ne M.$$

Proof of lemma 3.10. If supp$(g) \subset V$ then choose $g_1 = \mathrm{Id}$, $g_2 = g$. Else, we can choose $x \in V - \overline{U}$ such that $g(x) \ne x$. Now, choose an isotopy $g_t \in \mathrm{Diff}_c(M, \xi)$ from the identity to $g_1 = g$. Let N_1 be an open neighbourhood of x, such that there are open sets N_2 and N_3 satisfying

$$\bigcup_{t \in [0,1]} g_t\left(\overline{N}_1\right) \subset N \subset \overline{N}_2 \subset N_3 \subset V.$$

By theorem 3.1 there is a contact isotopy h_t such that $h_t|_{\overline{N}_1} = g_t$ and supp$(h_t) \subset N_3$.

Choose a point $y \notin \overline{U} \cup \overline{N}_3$. Then, we define $g_1 = h_1$ and $g_2 = gh_1^{-1}$. It is clear that supp$(g_i) \in V$ for $i = 1, 2$, and since $y \notin \overline{U}$ and $y \notin$ supp(h), we have that supp$(g_1) \cup \overline{U} \ne M$. Finally, $g_1(x) \notin g_1\overline{U}$ since $x \notin \overline{U}$, and for every $z \in N_1$, $g_2 \circ g_1(z) = g(z) = h(z) = g_1(z)$, so $x \notin$ supp(g_2) and therefore, supp$(g_2) \cup g_1\overline{U} \ne M$. □

Now we complete the proof that $(G_\xi(M), \mathscr{U}_\xi, M)$ satisfies axiom **A3**. Take $g \in G_\xi(M)$, $U \in \mathscr{U}_\xi$ and $\mathscr{B}$ an open cover of M. Since g has compact support K, we can cover K by finitely many $V_i \in \mathscr{B}$, say $V_1, \ldots, V_n$. Using lemma 3.9 on g and the cover $\{V_i\}$ we obtain $g = g_n \circ \ldots \circ g_1$ with $g_i \in G_\xi(M)$ and supp$(g_i) \subset V_i$. Finally, we apply lemma 3.10 to each triplet $(g_i, V_i, g_{i-1} \ldots g_1 U)$ to obtain $g_i = g_i^2 g_i^1$ with supp$\left(g_i^j\right) \subset V_i$ for $j = 1, 2$ and supp$\left(g_i^1\right) \cup \overline{U} \ne M$ and supp$\left(g_i^2\right) \cup g_i^1\overline{U} \ne M$. Finally, the decomposition of g we wish is $g = g_n^2 \circ g_n^1 \circ \cdots \circ g_1^2 \circ g_1^1$. □

3.3 The shrinking lemma

We will use a modification of a shrinking lemma of Filipkiewicz [7]. Recall $D_a^{2n+1} \subset \mathbb{R}^{2n+1}$ is the open ball of radius a centered at the origin. Since the dimension we are working on is understood, we simply write $D_a = D_a^{2n+1}$. Also, denote by $B(x,\varepsilon)$ the open ball of radius ε centered at x.

Lemma 3.11. *Let $\mathcal{C}$ be an open cover of $\overline{D}_1$. Then if $a \in (0,1]$, there are $f_i, g_i \in G_{\ker \omega_0}\left(\mathbb{R}^{2n+1}\right)$ for $i = 1, 2, \ldots, m$ such that*

(i) For $i = 1, \ldots, m$, there exists $C_i \in \mathcal{C}$ such that $f_i, g_i \in G_{\ker \omega_0}(C_i)$

(ii) $[f_m, g_m] \circ \cdots \circ [f_1, g_1]\left(\overline{D}_1\right) \subset \overline{D}_a$

where $\omega_0 = \left(\sum_{i=1}^n x_i dy_i\right) + dz$ is the standard contact form on $\mathbb{R}^{2n+1}$.

Proof. Let $A \subset (0,1]$ be the subset of $a \in (0,1]$ for which the lemma is true. Clearly A is non-empty as $1 \in A$, and this can be seen by choosing $f = g = \mathrm{Id}$. Now, let $a_0 = \inf A$. We want to show $a_0 = 0$.

Suppose $a_0 > 0$. Let $\{V_i\}$ be a finite open cover of the boundary $\partial \overline{D}_{a_0}$ of $\overline{D}_{a_0}$ and choose ε less than the Lebesgue number of the cover $\{V_i\}$. For each $x \in \partial \overline{D}_{a_0}$, choose pairs (g_x, U_x) consisting of an open neighbourhood U_x of x and $g \in G_{\ker \omega_0}(B(x,\varepsilon))$ such that

- $U_x \subset B(x,\varepsilon) \cap \left(\overline{D}_{a_0+\frac{\varepsilon}{2}} \setminus \overline{D}_{a_0-\frac{\varepsilon}{2}}\right)$
- $g_x(U_x) \subset \overline{D}_{a_0-\frac{\varepsilon}{2}}$ and $g_x^{-1}(U_x) \subset \mathbb{R}^{2n+1} \setminus \overline{D}_{a_0+\frac{\varepsilon}{2}}$.

This can be done by choosing points

$$p_x \in B(x,\varepsilon) \cap \partial D_{a_0-\frac{3\varepsilon}{4}}$$
$$q_x \in B(x,\varepsilon) \cap \partial D_{a_0+\frac{3\varepsilon}{4}}$$

since by Boothby's transitivity theorem, we can find $u, v \in G_{\ker \omega_0}(B(x,\varepsilon))$ such that $u(x) = p_x$, $u(q_x) = q_x$, $v(q_x) = x$, $v(p_x) = p_x$. Then setting $g_x = v \circ u$, we have $g_x(x) = p_x$ and $g_x^{-1}(x) = q_x$, and it only remains to take a sufficiently small neighbourhood U_x of x.

Clearly the sets U_x cover ∂D_{a_0}, so we can choose a finite cover $\{U_{x_i}\}_{i=1}^k$ with their corresponding $\{g_{x_i}\}$. For simplicity, let $U_{x_i} = U_i$ and $g_{x_i} = g_i$. Now we choose $\varepsilon' < \frac{\varepsilon}{2}$ such that

$$\overline{D_{a_0+\varepsilon'} \setminus D_{a_0-\varepsilon'}} \subset \bigcup_{i=1}^k U_i.$$

Our next step will be constructing an $f \in G_{\ker \omega_0}(D_{a_0+\varepsilon'} \setminus D_{a_0-\varepsilon'})$ such that for some $c > 1$, we have

$$f\left(\overline{D}_{a_0+\frac{\varepsilon'}{c}}\right) \subset \overline{D}_{a_0-\frac{\varepsilon'}{c}}.$$

Consider the homothety $R_\lambda(x,y,z) = \left(\lambda x, \lambda y, \lambda^2 z\right)$ where

$$\lambda = \left(\frac{a_0 - \frac{\varepsilon'}{c}}{a_0 + \frac{\varepsilon'}{c}}\right) < 1$$

with $c > 1$ chosen so that $\left(a_0 - \frac{\varepsilon'}{c}\right)\lambda^2 > a_0 - \varepsilon'$. We can choose such a c since the inequality is equivalent to

$$c^3 a^2 \varepsilon' + \mathcal{O}\left(c^2\right) > 0$$

where the right side is a polynomial in c with positive leading coefficient, hence for c sufficiently large, the inequality holds.

Now, let $p \in \overline{D_{a_0+\frac{\varepsilon'}{c}}}$, $p = (x,y,z)$. Then

$$|R_\lambda(p)| = \left(\lambda^2\left(\sum_{i=1}^n x_i^2 + y_i^2\right) + \lambda^4 z^2\right)^{\frac{1}{2}}$$
$$\leq \left(\lambda^2\left(\left(\sum_{i=1}^n x_i^2 + y_i^2\right) + z^2\right)\right)^{\frac{1}{2}} = \lambda|p| \leq \lambda\left(a_0 + \frac{\varepsilon'}{c}\right) = a_0 - \frac{\varepsilon'}{c}.$$

Hence $R_\lambda\left(\overline{D}_{a_0+\frac{\varepsilon'}{c}}\right) \subset \overline{D}_{a_0-\frac{\varepsilon'}{c}}$. Now, by theorem 3.1 we can find a contact diffeomorphism f which agrees with R_λ on $\overline{D_{a_0+\frac{\varepsilon'}{c}} \setminus D_{a_0-\frac{\varepsilon'}{c}}}$ and $\operatorname{supp}(f) \subset D_{a_0+\frac{\varepsilon'}{c}}$. To verify this, we need to show that if $|p| > a_0 - \frac{\varepsilon'}{c}$ then $|R_\lambda(p)| > a_0 - \varepsilon'$. But

$$\begin{aligned} |R_\lambda(p)| &= \left(\lambda^2\left(\sum_{i=1}^n x_i^2 + y_i^2\right) + \lambda^4 z^2\right)^{\frac{1}{2}} \\ &\geq \left(\lambda^4\left(\left(\sum_{i=1}^n x_i^2 + y_i^2\right) + z^2\right)\right)^{\frac{1}{2}} = \lambda^2 |p| > \lambda^2\left(a_0 - \frac{\varepsilon'}{c}\right) > a_0 - \varepsilon'. \end{aligned}$$

Now, we apply lemma 3.9 to f subject to the open cover $\{U_i\}$ to obtain f_i such that $\operatorname{supp}(f_i) \subset U_i$. Define $h_i = [f_i, g_i]$. Then

$$h_i(x) = \begin{cases} f_i(x) & x \in U_i \\ g_i f_i g_i^{-1} & x \in g_i(U_i) \\ x & \text{otherwise, since } \operatorname{supp}(f_i) \subset U_i \end{cases}$$

The reminder of the proof is as done by Filipkiewicz. If $x \in \overline{D}_{a_0+\varepsilon'} \setminus \overline{D}_{a_0-\varepsilon'}$ then

$$h_k \circ \cdots \circ h_1(x) = f_k \circ \cdots \circ f_1(x)$$

and therefore $h_k \circ \cdots \circ h_1\left(\overline{D}_{a_0}\right) \subset \overline{D}_{a_0-\frac{\varepsilon}{2}}$. Moreover, for each i there is an $x_i \in \partial D_{a_0}$ with $f_i, g_i \in G_{\ker \omega_0}(B(x_i, \varepsilon))$. Since ε is less than the Lebesgue number of the covering $\mathcal{C}$, there is an open $C_i \in \mathcal{C}$ with $f_i, g_i \in G_{\ker \omega_0}(C_i)$ for each i, which shows $a_0 - \frac{\varepsilon'}{2} \in A$ and this is a contradiction to $a_0 = \inf A$. Hence $a_0 = 0$ and we are done. □

3.4 Proof of the main theorem

We shall now prove that the groups $G_{\xi_i}(M_i)$, $i = 1, 2$ have the A and B properties, and every subgroup of them has the L property. Property A is a consequence of Boothby's transitivity theorem and k-transitivity of $G_\xi(M)$ and we have shown property B holds in remark 3.6. Hence, we are only left to show every subgroup of $G_\xi(M)$ has the L property.

Proposition 3.12. *Every subgroup F of $G_\xi(M)$ has the L property.*

Proof. We follow Filipkiewicz method. Let H be a subgroup of $[G_\xi(M), G_\xi(M)]$ generated by the groups $[G_\xi(U), G_\xi(U)]$ as U ranges over $\mathscr{U}_\xi$. H is a normal subgroup of $[G_\xi(M), G_\xi(M)]$, which is simple by theorem 3.3, so $H = [G_\xi(M), G_\xi(M)]$.

Let F be a subgroup of $G_\xi(M)$ satisfying the hypothesis of the L property subject to the open subcover of balls $\mathscr{U}' \subset \mathscr{U}_\xi$. That is, $[G_\xi(U), G_\xi(U)] \subseteq F$ for every $U \in \mathscr{U}'$. We need only show that if $W \in \mathscr{U}_\xi$ then $[G_\xi(W), G_\xi(W)] \subseteq F$. Consider then a Darboux chart ψ with domain U and $\overline{D}_1 \subset U$, and consider $W = \phi\psi(D_1) \in \mathscr{U}_\xi$ where $\phi \in G_\xi(M)$. By hypothesis, we can cover $\overline{W}$ with finitely many $U_1, \ldots, U_s \in \mathscr{U}_\xi$ so that $[G_\xi(U_i), G_\xi(U_i)] \subset F$. Then, define $V_i = \psi^{-1}\phi^{-1}(U_i)$. The sets V_i cover $\overline{D}_1$ and either $V_i \subset U$ or we can choose smaller U_i so that $V_i \subset U$, and in the later case we do so. Assume $0 \in V_1$.

For some $a > 0$, $\overline{D_a} \subset V_1$, and we apply lemma 3.11 to obtain commutators $[f_j, g_j] \in G_{\ker \omega_0}\left(V_{i_j}\right)$ $(1 \leq i \leq k)$ and $[f_r, g_r] \circ \ldots \circ [f_1, g_1]\left(\overline{D}_1\right) \subset \overline{D}_a$. Since each $f_j, g_j \in G_{\ker \omega_0}(U)$ we can define $\hat{f}_j, \hat{g}_j \in G_\xi(M)$ as

$$\hat{f}_j(x) = \begin{cases} (\phi\psi) f_j (\phi\psi)^{-1}(x) & x \in \phi\psi(U) \\ x & \text{otherwise} \end{cases}$$

$$\hat{g}_j(x) = \begin{cases} (\phi\psi) g_j (\phi\psi)^{-1}(x) & x \in \phi\psi(U) \\ x & \text{otherwise} \end{cases}$$

So $\left[\hat{f}_j, \hat{g}_j\right] \in \left[G_\xi\left(U_{i_j}\right), G_\xi\left(U_{i_j}\right)\right] \subset F$. Define $\tau = \left[\hat{f}_r, \hat{g}_j\right] \circ \cdots \circ \left[\hat{f}_1, \hat{g}_1\right]$. Then $\tau \in F$ and $\tau(W) \subset U_1$, and

$$[G_\xi(W), G_\xi(W)] \subset \tau^{-1}[G_\xi(U_1), G_\xi(U_1)]\tau \subset F.$$

□

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