Monodromy of $K3$ surfaces branching over quartic curves

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IMUNAM - Algebraic Topology Seminar

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Goals

- Monodromy of a Family of Varieties

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- Examples

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- Examples
- Explicit computation for a family of $K3$ surfaces

An elliptic fibration over $\mathbb{P}^1$

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$$\begin{array}{ccc} C_\lambda & \hookrightarrow & E \\ & & \downarrow \\ & & \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array} \qquad \begin{array}{c} \{(\lambda, p) \mid p \in C_\lambda\} \\ \downarrow \\ \lambda \end{array}$$

C_λ as a branched cover of $\mathbb{P}^1$

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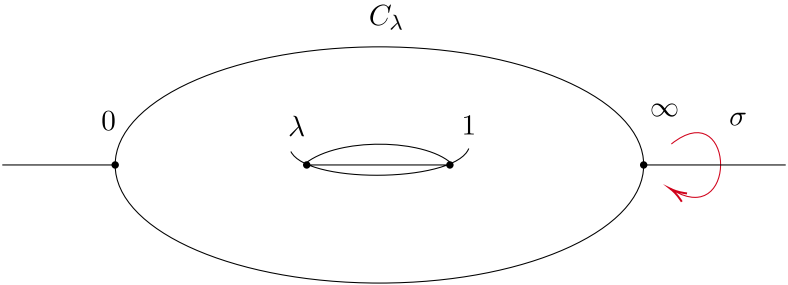
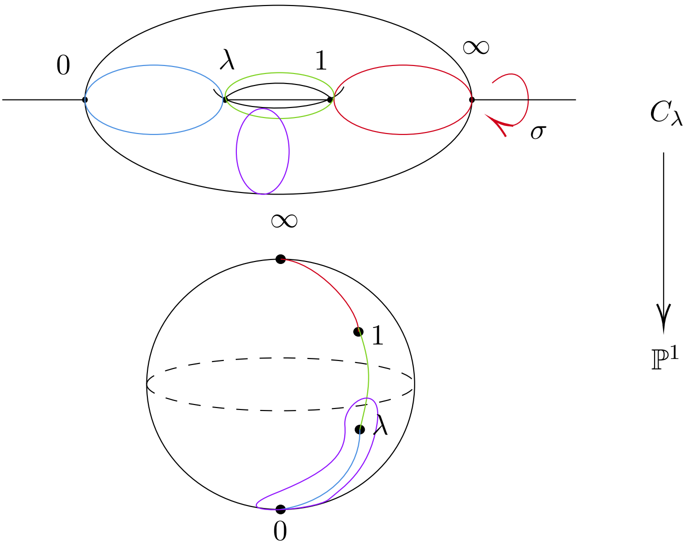
$$\mathrm{Fix}(\sigma) = f^{-1}(\{0,1,\lambda,\infty\})$$

Involution on C_λ



$\mathbb{P}^1$ as a quotient of C_λ



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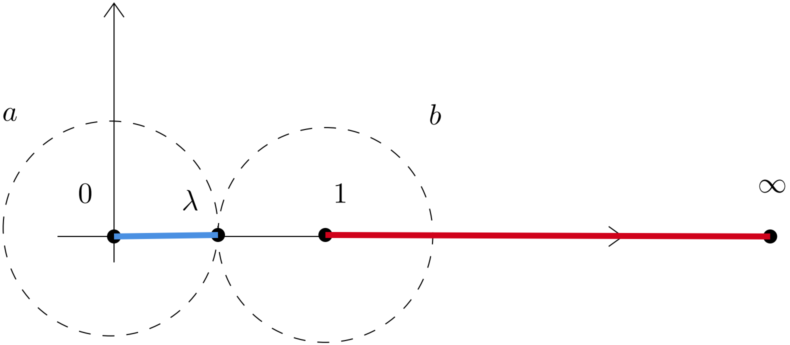
Monodromy

π_1 generators

$$\pi_1\left(\mathbb{P}^1 \setminus \{0,1,\infty\}, \tfrac{1}{2}\right) \cong \langle a, b \rangle$$



Action of generators

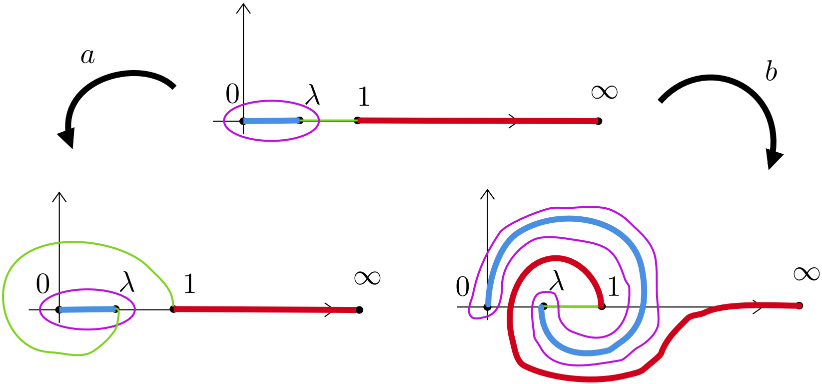


Image of ρ

With respect to our basis of $H_1(C_\lambda; \mathbb{Z})$:

$$\rho(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{y} \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

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where

$$\mathrm{Im}(\rho) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \qquad [\mathrm{SL}_2(\mathbb{Z}) : \mathrm{Im}(\rho)] = 12$$

Spaces of smooth projective hypersurfaces

Hypersurfaces in $\mathbb{P}^n$

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$$\mathbb{P}^{\binom{n+d}{d}-1} = \{V(f) \subset \mathbb{P}^n \mid f \in \mathbb{C}[x_0, \ldots, x_n]_d\}$$

$$[a_i] \mapsto V\left(\sum_i a_i x^i\right)$$

The complements $\mathcal{U}_{n,d}$

The discriminant variety

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What can be said regarding $\pi_1(\mathcal{U}_{n,d})$?

Fundamental group of $\mathcal{U}_{n,d}$

(Lonne) Presents each $\pi_1(\mathcal{U}_{n,d})$ with $(d-1)^n$ generators.

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(Dolgachev-Libgober)

$$\pi_1(\mathcal{U}_{2,3}) \cong \mathcal{H}_3(\mathbb{Z}/3\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$$

where $\mathcal{H}_3(\mathbb{Z}/3\mathbb{Z})$ is the Heissenberg group modulo 3.

The universal family $E_{n,d}$

$$\begin{array}{ccc} V(f) & \hookrightarrow & E_{n,d} \\ & & \downarrow \\ & & \mathcal{U}_{n,d} \end{array} \qquad \begin{array}{c} \{(f,x) \mid x \in V(f)\} \\ \downarrow \\ f \end{array}$$

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$$\rho : \pi_1(\mathcal{U}_{n,d}) \to \mathrm{Aut}\left(H_{n-1}(V(f);\mathbb{Z})\right)$$

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What is the image of this homomorphism?

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- **(Janssen)** If n is even:

$$\operatorname{Im}(\rho) = \begin{cases} \operatorname{Sp}\left(H_{n-1}(V(f);\mathbb{Z})\right) & \text{if } d \text{ is even} \\ \operatorname{SpO}(H_{n-1}(V(f);\mathbb{Z}), q_{V(f)}) & \text{if } d \text{ is odd} \end{cases}$$

Degree 2 del Pezzo surfaces

A bridge between cubic surfaces and quartic curves

Monodromy of $E_{3,3}$

(Klein-Jordan)

$$\mathrm{Im}(\rho : \pi_1(\mathcal{U}_{3,3}) \to \mathrm{Aut}(H_2(V(f);\mathbb{Z}))) \cong W(E_6)$$

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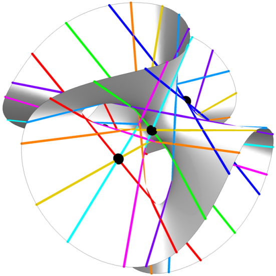
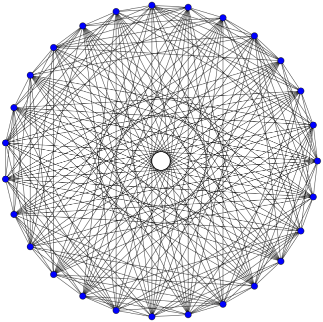


Automorphisms of the 27 lines in a smooth cubic surface.

Monodromy of $E_{3,3}$

Schläfli graph Γ_6



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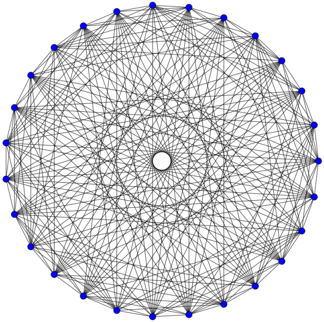

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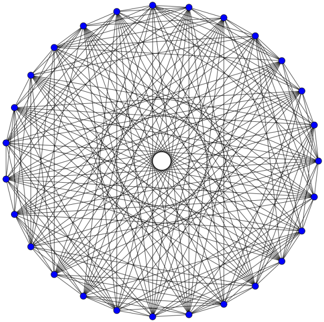

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Intersection pattern of the 27 lines in a smooth cubic surface.

Monodromy of $E_{2,4}$

$$\mathrm{Im}(\rho : \pi_1(\mathcal{U}_{2,4}) \to \mathrm{Aut}(H_1(V(f);\mathbb{Z}))) \cong \mathrm{Sp}_6(\mathbb{Z})$$

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Reduction modulo 2:

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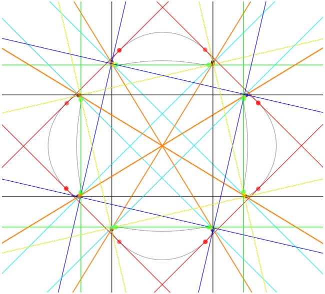

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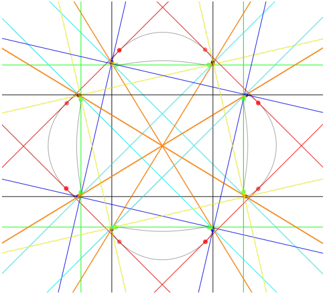
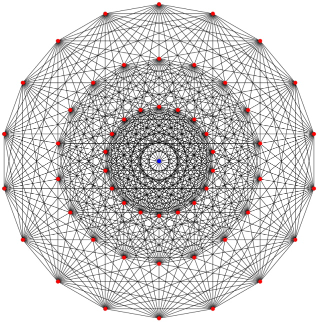


Automorphisms of the 28 bitangents to a smooth quartic curve.

Monodromy of $E_{2,4}$

Gosset graph Γ_7



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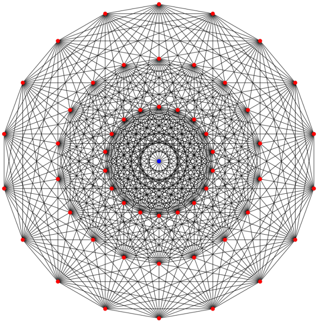

$$\mathrm{Aut}(\Gamma_7) \cong W(E_7) \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{Sp}_6(\mathbb{F}_2) \qquad |W(E_7)| = 56 \cdot |W(E_6)|$$

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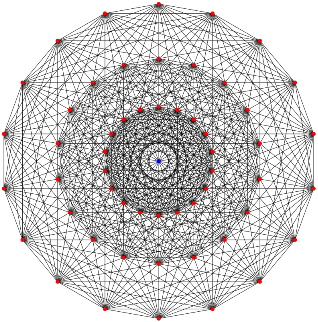

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Which intersection pattern does it give?

Degree 2 del Pezzo surfaces

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$$\{56 \text{ lines in } \mathcal{P}\} \longleftrightarrow \{28 \text{ bitangents to } V(f)\}$$

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- Deck group of $\mathcal{P} \to \mathbb{P}^2$

$$\langle \tau \rangle \cong Z(W(E_7)) \cong \mathbb{Z}/2\mathbb{Z}$$

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(Harris)

$$\mathrm{Stab}(\mathcal{L}) \cong \mathrm{Aut}(\mathcal{B} \setminus \mathcal{L}) \cong \mathrm{Aut}(\mathcal{S}) \cong W(E_6)$$

Families of branched covers

Branching over $V(f)$

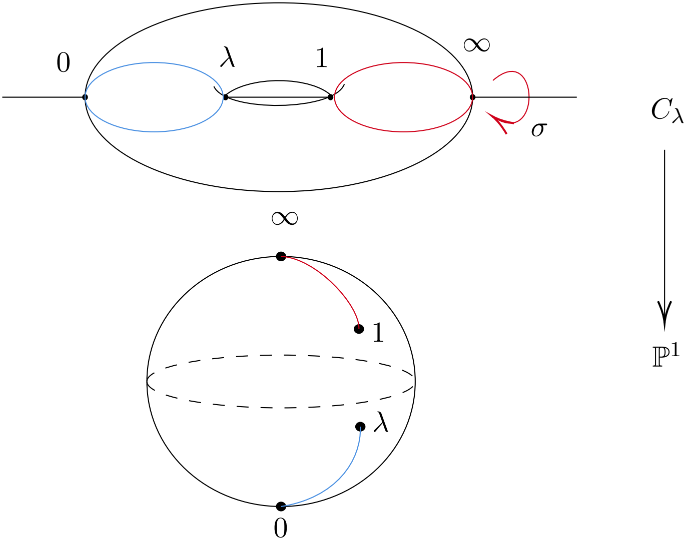
$$\begin{array}{ccc} \sout{V(f)}\ X_f \hookrightarrow \mathcal{E}_{n,d} & & \{(f,p) \mid p \in X_f\} \\ \downarrow & & \downarrow \\ \mathcal{U}_{n,d} & & f \end{array}$$

Branching over $V(f)$

$$\require{cancel}\cancel{V(f)}\ X_f \hookrightarrow \mathcal{E}_{n,d} \qquad \{(f,p) \mid p \in X_f\} \qquad X_f$$

$$\begin{array}{ccc} \mathcal{E}_{n,d} & \{(f,p) \mid p \in X_f\} & X_f \\ \downarrow & \mapsto & \downarrow k \\ \mathcal{U}_{n,d} & f & \mathbb{P}^n \supset V(f) \end{array}$$

Previous example: Double branched cover over 4 points



Branched covers of quartic curves in $\mathbb{P}^2$

Degree $d = 4$ covers

(Hirzebruch) For $n = 2$

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$$V(f) \subset \mathbb{P}^2 \quad \rightsquigarrow \quad \begin{cases} \mathcal{P}_f & k = 2 \quad \textbf{(Degree 2 del Pezzo)} \\ X_f & k = 4 \quad \textbf{(Quartic K3 } V(w^4 - f) \subset \mathbb{P}^3\textbf{)} \end{cases}$$

Degree $d = 4$ covers

(Key) $\mathcal{P}_f$ lets us study X_f

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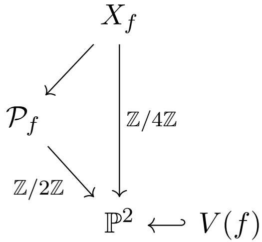

Monodromy of $\mathcal{E}_{2,4}$

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$$\rho_4 : \pi_1(\mathcal{U}_{2,4}) \to \mathrm{Aut}(H^2(V(w^4 - f); \mathbb{Z}))$$

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$$\rho_4 \text{ preserves } \begin{cases} \langle \cdot, \cdot \rangle_{H^2} & \text{intersection form} \\ K_{X_f} & \text{canonical class (this vanishes!)} \end{cases}$$

Monodromy of $\mathcal{E}_{2,4,2}$

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Deck group action of $\mathbb{Z}/4\mathbb{Z}$

$$\mathbb{Z}/4\mathbb{Z} = \langle T \rangle \curvearrowright H^2(X_f;\mathbb{Z}) \qquad \mathbb{Z}/2\mathbb{Z} = \langle T^2 \rangle \curvearrowright H^2(\mathcal{P}_f;\mathbb{Z})$$

$$T : [w \mapsto iw] \qquad T^2 : [w \mapsto -w]$$

$$\mathrm{Im}(\rho_4) \subset C_{H^2(X_f;\mathbb{Z})}(T) \qquad \mathrm{Im}(\rho_2) \subset W(E_7)$$

Theorem (for degree 2 del Pezzo surfaces)

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There is $\gamma \in \pi_1(\mathcal{U}_{2,4})$ realizing the Geiser involution τ via ρ_2:

$$\rho_2(\gamma) = \tau$$

K3 surfaces and Lattices

$K3$ surfaces

(Def) Compact surface X with $\pi_1(X) = 1$, $K_X = 0$

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Mixing decompositions

Intersection of Hodge and Eigenspace decompositions:

	V_1	V_{-1}	V_i	V_{-i}
$H^{2,0}$	0	0	$\mathbb{C}$	0
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$$V_i \cong \mathbb{C}^{1,6} \cong \left(\mathbb{C}^7, |z_0|^2 - |z_1|^2 - \cdots - |z_6|^2\right)$$

L_- as $\mathbb{Z}[i]$-lattice

(Kondo) As a $\mathbb{Z}$-lattice

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(Medrano Martín del Campo)

$$\begin{aligned} h_{L_-} = & -2(|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) \\ & + 2\Re(z_1\overline{z_2} + z_3\overline{z_4} + z_5\overline{z_6}) \\ & + 2\Im(z_1\overline{z_2} + z_3\overline{z_4} + z_5\overline{z_6}) \end{aligned}$$

As a $\mathbb{Z}[i]$-lattice

$$L_- \cong \left(\mathbb{Z}[i]^7, h_{L_-}\right)$$

Computing $\mathbf{Im}(\rho_4)$

Reduction to L_-

(Key) ρ_4 acts on each V_ζ, hence on L_+, L_-

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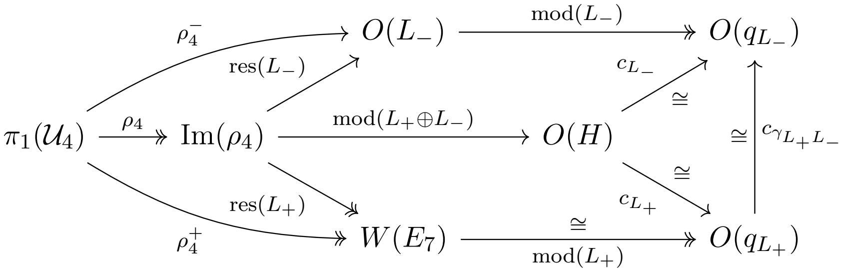
Given by this commutative diagram



Moduli of smooth quartic curves

The moduli $\mathcal{M}_3$ of smooth quartic curves

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$$\mathcal{H} = \bigcup_{\langle \delta, \delta \rangle = -2} H_\delta$$

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(Kondo) $\mathcal{M}_3$ moduli of smooth quartic curves

$$\mathcal{M}_3 \cong (\mathcal{D}_6 - \mathcal{H})/\mathbb{P}(\Gamma)$$

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(Q, λ) defines a $K3$ surface, up to a unit in $\mathbb{Z}[i]$.

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(Torelli) Period map $\wp$ for $K3$ surfaces is an isomorphism

$$\begin{array}{ccc} \widetilde{\mathcal{M}_3} & \xrightarrow[\cong]{\wp} & \mathcal{D}_6 - \mathcal{H} \\ {\scriptstyle \mathbb{P}(\Gamma)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbb{P}(\Gamma)} \\ \mathcal{M}_3 & \xrightarrow[\tilde{\wp}]{\cong} & (\mathcal{D}_6 - \mathcal{H})/\mathbb{P}(\Gamma) \end{array}$$

Reducing loops in $\mathcal{U}_{2,4}$ to $\mathcal{M}_3$

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- Rest of the divisors $\mathcal{O} \subset \mathscr{M}_3$ have higher codimension, thus

$$\pi_1(\mathcal{U}_{2,4}) \to \pi_1(\mathcal{M}_3)$$

is surjective.

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For further details, see:

Monodromy of the family of $K3$ and del Pezzo surfaces branching over smooth quartic curves