The University of Chicago
Differential Topology Final Project

Banchoff-Chmutov Surfaces

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Abstract

In this paper, we study a particular family of surfaces which arises from the Chebyshev polynomials, the Banchoff-Chmutov surfaces. To do so, we employ techniques from Morse theory and the trigonometric functional equations which define the Chebyshev polynomials.

1 Introduction

This paper is an exposition of the characterization of the even Banchoff-Chmutov Surfaces Z_n as topological 2-manifolds. To do so, we employ techniques from Morse theory which will enable us to understand the topology of Z_n by producing a Morse function on it. Our goal will be to prove the following theorem:

Theorem 1.1. For even n, Z_n is a smooth closed orientable surface of genus $\frac{n^2(n-3)}{4}+1$.

Put simply, Morse theory is the study of the topology of manifolds via calculus of a function on them. The reason we want to use Morse theory is that from a Morse function, one may recover a great amount of information of the topology of the manifold in question. In general, once we have a Morse function on a manifold, we analyze its critical points and its gradient vector field.

These two mathematical devices permit us reconstruct our manifold in the following way: We build our manifold from top to bottom, following the gradient vector field's integral curves, and every time we encounter a critical value of our function, the corresponding critical points of that critical value tell us that a certain type of handle (depending on the Morse index of the critical point in question) is to be attached to our manifold. Thus critical points tell us what is to be attached, and the gradient vector field tells us how is it going to be attached.

It is often the case that there is more than one critical point corresponding to each critical value of a Morse function. The family of surfaces we study in this paper provide an example where, with the Morse function we will endow Z_n, our critical points are clustered within few critical values. For instance, every maxima and every minima will be critical points corresponding the the critical value (which can be thought of as the maximal and minimal height of Z_n, respectively). Since we will have many critical points for a single critical value, understanding the gradient flow of our Morse function is going to be fundamental to know how the handles are attaching at each step of the construction of Z_n.

Since we are working with surfaces, we will naturally prove that the genus of Z_n is as stated in 1.1, and this equivalent to knowing the Euler characteristic of our surface. One more benefit from Morse theory is that the understanding of the critical points permits us understand the ranks of the homology groups of Z_n via the Morse inequalities. In this paper, we simply need the beautiful fact that the Euler characteristic is given by the alternating sum of the number of points of each Morse index.

2 Morse Theory

First, we illustrate the principles of Morse theory necessary for this paper. For this section, let M be a smooth compact Riemannian m-manifold with Riemannian metric $\langle\cdot,\cdot\rangle$, and $f: M \to \mathbb{R}$ a smooth function.

Definition 2.1. [2] A *critical point* p of f is a point $p \in M$ such that $D_p f = 0$.

Definition 2.2. [2] The *gradient flow vector field* of f is the unique vector field $\nabla f : M \to TM$ defined by

$$\langle \nabla f_p, V_p \rangle = D_p f\,(V_p)$$

for every smooth vector field $V \in \mathcal{X}^\infty(M)$.

Definition 2.3. [2] The *Hessian* of f at a critical point p is the symmetric bilinear form on the tangent space $H_p f : T_pM \times T_pM \to \mathbb{R}$ defined by $H_p f\,(v,w) = V\,(W\,(f))\,(p)$ where V and W are extensions of v and w, respectively, to vector fields in a neighbourhood of p.

Definition 2.4. [2] A critical point p of f is called *non-degenerate* if $H_p f$ is non-degenerate.

Definition 2.5. [2] Let p be a non-degenerate critical point of f. The *Morse index* of p is the maximal dimension of a subspace of T_pM on which $H_p f$ is negative-definite.

Definition 2.6. [2] f is a *Morse function* if and only if all its critical points are non-degenerate.

Definition 2.7. An *l-handle* attached to M is a tubular neighbourhood of an l-disk D^l such that $\partial D^l \subset M$.

2.1 Three theorems in Morse Theory

Theorem 2.1. [2] Let f be a Morse function on M and let $\phi(t)$ be the flow lines of ∇f.

(i) f is strictly increasing along the non-degenerate flow lines $\phi(t)$, and constant on degenerate flow lines corresponding to the critical points.

(ii) If M is compact and $x \in M$, then $\lim_{t\to-\infty} \phi_x(t)$ and $\lim_{t\to\infty} \phi_x(t)$ are critical points of f.

Theorem 2.2. [2] Let f be a Morse function on M and let $a < b$ be such that $f^{-1}[a,b]$ is compact.

(i) If $f^{-1}[a,b]$ contains no critical point of f, then $M^a = \{f(p) \le a\}$ is homotopy equivalent to M^b.

(ii) If $f^{-1}[a,b]$ contains exactly k critical points $p_1, p_2, \ldots, p_k$ of f of indices $l_1, l_2, \ldots, l_k$ respectively, corresponding to the same critical value, then M^b is homotopy equivalent to M^a with an attached l_i-handle for each $i = 1, 2, \ldots, k$.

Theorem 2.3. [2] Let f be a Morse function on a compact m-manifold M with n_k critical points of Morse index k for $k = 0, 1, \ldots, m$. Then

$$\chi(M) = \sum_{k=0}^{m} (-1)^k n_k.$$

3 Chebyshev Polynomials

In this section we introduce the family of Chebyshev polynomials. These polynomials are given by trigonometric identities and are characterized, along with their derivatives, as the solutions of a second order differential equation, as we shall see.

Definition 3.1. The *n-th Chebyshev polynomial of the first kind* is the uniquely determined degree n polynomial in one variable T_n such that

$$T_n(\cos(x)) = \cos(nx) \quad \forall x \in \mathbb{R}.$$

Definition 3.2. The *n-th Chebyshev polynomial of the second kind* is the uniquely determined degree n polynomial U_n such that

$$U_n(\cos(x)) = \frac{\sin((n+1)x)}{\sin(x)} \quad \forall x \in \mathbb{R}.$$

Proposition 3.1. The Chebyshev polynomials satisfy the following relations for all $n \ge 0$:

(i) $\frac{d}{dx} T_n(x) = nU_{n-1}(x)$.

(ii) $T_n(x)^2 - (x^2 - 1) U_{n-1}(x)^2 = 1$.

Proof. Proving these results locally will suffice to prove the results globally, since T_n, U_n are polynomials.

(i) Using the chain rule,

$$-n\sin(nx) = \frac{d}{dx}(T_n(\cos(x))) = -\frac{d}{dx}(T_n(\cos(x)))\sin(x)$$

and thus

$$\frac{d}{dx}(T_n(\cos(x))) = \frac{n\sin(nx)}{\sin(x)} = nU_{n-1}(\cos(x)).$$

Hence the equality holds on $[-1, 1]$, so it holds in $\mathbb{R}$.

(*ii*) We have that

$$T_n\left(\cos\left(x\right)\right)^2-\left(\cos\left(x\right)^2-1\right)U_{n-1}\left(\cos\left(x\right)\right)^2=1.$$

Hence the equality holds on $[-1,1]$, so it holds in $\mathbb{R}$.

□

Note that T_n and U_{n-1} each have n simple zeros, which are, respectively,

$$\{x\in\mathbb{R}\mid T_n\left(x\right)=0\}=\left\{\cos\left(\frac{\left(2m+1\right)\pi}{2n}\right)\mid m=0,1,\ldots,n-1\right\}\subset[-1,1]$$
$$\{x\in\mathbb{R}\mid U_n\left(x\right)=0\}=\left\{\cos\left(\frac{m\pi}{n+1}\right)\mid m=1,\ldots,n\right\}\subset(-1,1)$$

Moreover, since the zeros of $\frac{d}{dx}T_n=nU_{n-1}$ are simple, so the critical points of T_n are non-degenerate.

3.1 Second order ODE of T_n

Theorem 3.1. The Chebyshev polynomial T_n satisfies the second order differential equation

$$\left(1-x^2\right)\frac{d^2}{dx^2}T_n\left(x\right)-x\frac{d}{dx}T_n\left(x\right)+n^2T_n\left(x\right)=0.$$

Proof. Note that

$$\left(\frac{d}{dx}\right)^2T_n\left(\cos\left(x\right)\right)=\left(\frac{d}{dx}\right)^2\cos\left(nx\right)=-n^2\cos\left(nx\right)^2=-n^2T_n\left(\cos\left(x\right)\right)$$

and hence, if we let $F_n\left(t\right)=T_n\left(\cos\left(t\right)\right)$, we have F_n satisfies the differential equation

$$\frac{d^2}{dt^2}F_n\left(t\right)+n^2F_n\left(t\right)=0.$$

Letting $t=\arccos\left(x\right)$, it follows that

$$T_n\left(x\right)=F_n\left(t\right)=-\frac{1}{n^2}\left(\frac{d^2F_n}{dt^2}\right)\left(t\right)$$
$$\frac{d}{dx}T_n\left(x\right)=\frac{d}{dx}F_n\left(t\right)=\left(\frac{dF_n}{dt}\right)\left(t\right)\frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d^2}{dx^2}T_n\left(x\right)=\frac{d^2}{dx^2}F_n\left(t\right)=\left(\frac{d^2F_n}{dt^2}\right)\left(t\right)\frac{1}{1-x^2}-\left(\frac{dF_n}{dt}\right)\left(t\right)\frac{1}{1-x^2}\cdot\frac{x}{\sqrt{1-x^2}}$$

so, putting these three equations together we can obtain the differential equation

$$\left(1-x^2\right)\frac{d^2}{dx^2}T_n\left(x\right)-x\frac{d}{dx}T_n\left(x\right)+n^2T_n\left(x\right)=0.$$

□

4 Banchoff-Chmutov Surfaces Z_n

Now, we are ready to introduce the main object of this paper, the Banchoff-Chmutov surfaces.

Definition 4.1. The *Banchoff-Chmutov surface of degree* n is defined as

$$Z_n = \left\{(x,y,z) \in \mathbb{R}^3 \mid T_n(x) + T_n(y) + T_n(z) = 0\right\}$$

Some examples for small (and even) n look as follows:

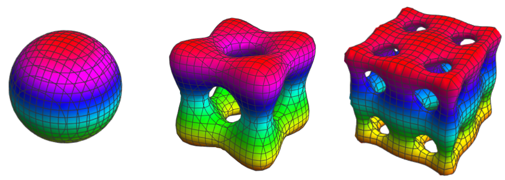

Figure 1: Illustration of The Banchoff-Chmutov surfaces Z_2, Z_4 and Z_6.

Theorem 4.1. Z_n is a smooth submanifold of $\mathbb{R}^3$ for every n.

Proof. Let $f_n(x,y,z) = T_n(x) + T_n(y) + T_n(z)$. Then we have that

$$D_{(x,y,z)} f_n = \begin{pmatrix} \frac{dT_n}{dt}(x) & \frac{dT_n}{dt}(y) & \frac{dT_n}{dt}(z) \end{pmatrix}.$$

Suppose $(x,y,z) \in Z_n$ and rank $D_{(x,y,z)} f_n = 0$, then

$$\frac{dT_n}{dt}(x) = \frac{dT_n}{dt}(y) = \frac{dT_n}{dt}(z) = 0$$

and therefore, since $\frac{d}{dt} T_n(t) = nU_{n-1}(t)$, we have

$$U_{n-1}(x) = U_{n-1}(y) = U_{n-1}(z) = 0.$$

Finally, since $T_n^2(t) - \left(t^2 - 1\right) U_{n-1}(t) = 1$, this implies that

$$T_n(x), T_n(y), T_n(z) = \pm 1$$

which is a contradiction, since this tells us $T_n(x) + T_n(y) + T_n(z)$ is odd, but $T_n(x) + T_n(y) + T_n(z) = 0$. Hence, for all $(x,y,z) \in Z_n$, we have that rank $D_{(x,y,z)} f_n = 1$ and by the submersion theorem, it follows that $Z_n = f_n^{-1}(0)$ is a smooth 2-dimensional submanifold of $\mathbb{R}^3$. ☐

4.1 Compactness

Theorem 4.2. For even n, Z_n is compact.

Proof. It suffices to show that Z_n is bounded (since Z_n is closed by continuity of f_n), so Z_n is compact by the Heine-Borel theorem. Since n is even, we have that

$$\lim_{|x| \to \infty} T_n(x) = \infty$$

hence, for sufficiently large N, T_n is positive on $\mathbb{R} \setminus [-N, N]$, and therefore, f_n is positive on $\mathbb{R}^3 - [-N, N]^3$. Hence, $Z_n \subset [-N, N]^3$, so Z_n is bounded. ☐

For odd n, Z_n is unbounded since T_n is an odd degree polynomial, so Z_n is not compact. For instance

$$T_1(t) = t \implies Z_1 = \{x + y + z = 0\}$$

so Z_1 is a plane in $\mathbb{R}^3$. Nevertheless, we are interested in the compact case, so we will assume that n is even for the rest of the paper.

4.2 Orientability

Theorem 4.3. Z_n is orientable.

Proof. Since Z_n is a closed surface embedded in $\mathbb{R}^3$, it follows that Z_n is orientable. $\square$

4.3 A Morse function on Z_n

Consider the projection $p : Z_n \to \mathbb{R}$ onto the z coordinate. We will show that this is a Morse function, for which we must simply show that its critical points are isolated and non-degenerate. Hence, we proceed to characterize the critical points of p.

Proposition 4.1. The function p has $\frac{n^2(n-1)}{2}$ critical points, of which $\frac{1}{4}n^2$ are maxima, $\frac{1}{4}n^2$ are minima, and the rest are saddle points.

Proof. The critical points of h are those points $(x, y, z) \in Z_n$ such that

$$D_{(x,y,z)}p : T_{(x,y,z)}Z_n \to T_z\mathbb{R}$$

vanishes. But

$$\begin{aligned} T_{(x,y,z)}Z_n &= \ker D_{(x,y,z)}f_n \\ &= \left\{(a, b, c) \in \mathbb{R}^3 \mid a\frac{dT_n}{dt}(x) + b\frac{dT_n}{dt}(y) + c\frac{dT_n}{dt}(z) = 0\right\}. \end{aligned}$$

Then, $D_{(x,y,z)}p\left((a, b, c)\right) = c$. Hence, if z is a critical value of p, then $c = 0$ for all $(a, b, c) \in T_{(x,y,z)}Z_n$, implying that

$$\frac{dT_n}{dt}(x) = \frac{dT_n}{dt}(y) = 0$$

and therefore

$$U_{n-1}(x) = U_{n-1}(y) = 0.$$

This implies that

$$x, y \in \left\{\cos\left(\frac{m\pi}{n}\right) \mid m = 1, \ldots, n-1\right\} \subset (-1, 1)$$

and note that

$$\begin{aligned} T_n(x) &= T_n\left(\cos\left(\frac{m_1\pi}{n}\right)\right) \\ &= \cos(m_1\pi) \\ &= (-1)^{m_1}. \end{aligned}$$

Similarly, $T_n(y) = (-1)^{m_2}$. Hence, we have that $T_n(z) = -(-1)^{m_1} - (-1)^{m_2}$, so

$$T_n(z) \in \{-2, 0, 2\}.$$

If $T_n(z) = 0$ then we have already proved that

$$z \in \left\{\cos\left(\frac{(2m+1)\pi}{2n}\right) \mid m = 0, 1, \ldots, n-1\right\} \subset [-1, 1].$$

Now suppose that $T_n(z) = \pm 2$. Since $T_n(\cos(x)) = \cos(nx) \in [-1, 1]$, then ± 2 cannot be achieved on $[-1, 1]$. Moreover, T_n is decreasing on $(-\infty, -1]$ and increasing on $[1, \infty)$. This implies that the value -2

cannot be achieved, while the value 2 is achieved exactly twice (once on $(-\infty, 1]$ and once on $[1, \infty)$. Hence, we have obtained every critical point, and these can be characterized as follows:

$$
\begin{aligned}
C_0 &= \left\{ \left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), \cos\left(\frac{(2m_3+1)\pi}{2n}\right) \right) \mid m_1+m_2 \text{ odd}, 0<m_1,m_2<n \right\} \\
C_2^+ &= \left\{ \left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), z \right) \mid m_1, m_2 \text{ odd}, 0<m_1,m_2<n, T_n(z)=2, z>1 \right\} \\
C_2^- &= \left\{ \left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), z \right) \mid m_1, m_2 \text{ odd}, 0<m_1,m_2<n, T_n(z)=2, z<-1 \right\}
\end{aligned}
$$

Since n is even, so the number of critical points is

$$
\begin{aligned}
|C_0| + \left|C_2^+\right| + \left|C_2^-\right| &= \frac{n^2(n-2)}{2} + \frac{n^2}{4} + \frac{n^2}{4} \\
&= \frac{n^2(n-1)}{2}.
\end{aligned}
$$

Now, we proceed to classify these critical points.

$\boxed{C_2^+}$ Let $(x,y,z) \in C_2^+$. We have shown that $T_n(x) = T_n(y) = -1$, so x, y are local minima for T_n. Hence, on a neighbourhood $U \subset Z_n$ of (x,y,z), we have that for any $(x_0, y_0, z_0) \in U$ then $T_n(x_0) \geq T_n(x)$ and $T_n(y_0) \geq T_n(y)$. Hence, it follows that

$$T_n(z_0) \leq T_n(z).$$

Since $z \in [1, \infty)$ and T_n is increasing in that interval, this imples $z_0 \leq z$. Therefore, (x,y,z) is a local maxima of p.

$\boxed{C_2^-}$ Similar to the previous case, for $(x,y,z) \in C_2^-$, we have $T_n(x) = T_n(y) = -1$, so x, y are local minima for T_n. Hence, on a neighbourhood $U \subset Z_n$ of (x,y,z), we have that for any $(x_0, y_0, z_0) \in U$ then $T_n(x_0) \geq T_n(x)$ and $T_n(y_0) \geq T_n(y)$. Hence, it follows that

$$T_n(z_0) \leq T_n(z).$$

Since $z \in (-\infty, -1]$ and T_n is decreasing in that interval, this imples $z_0 \geq z$. Therefore, (x,y,z) is a local minima of p.

$\boxed{C_0}$ For $(x,y,z) \in C_0$, we have that $T_n(x), T_n(y) \in \{-1, 1\}$ and $T_n(x) + T_n(y) = 0 = T_n(z)$. Note that x, y are local minima and maxima of T_n. Assume for now that x is a local minima and y is a local maxima of T_n. Then, choose a neighbourhood $U \subset Z_n$ of (x,y,z).

Fixing x, we see that for $(x, y_0, z_0) \in U$ we have $T_n(y_0) \leq T_n(y)$ and thus $T_n(z_0) \geq 0 = T_n(z)$. Since T_n has simple zeros, this implies that either $z_0 \geq z$ or $z_0 \leq z$ for all such $(x, y_0, z_0) \in U$. A similar calculation fixing y shows that for every $(x_0, y, z_0) \in U$ we a have that either $z_0 \leq z$ or $z_0 \geq z$ for every $(x_0, y, z_0) \in U$. Therefore, $p : U \to \mathbb{R}$ satisfies that

$$p\,|_{x=x_0} \quad \text{and} \quad p\,|_{y=y_0}$$

have local minima and maxima (in some order) at (x,y,z). Since $T_n(z) = 0$, we have that $\frac{dT_n}{dt}(z) \neq 0$, and by the implicit function theorem z is a function of x, y, which is precisely p. Hence, (x,y,z) is a saddle point of p.

□

In particular, every critical point of p is non-degenerate, so the following corollary is immediate.

Corollary 4.1. p is a Morse function on Z_n.

4.4 Gradient flow of p

Now, we will obtain the gradient vector field of p.

Proposition 4.2. The gradient flow vector field of p is

$$\nabla p(x,y,z) = \frac{\left(-U_{n-1}(x)\,U_{n-1}(z) \quad -U_{n-1}(y)\,U_{n-1}(z) \quad U_{n-1}^2(x)+U_{n-1}^2(y)\right)}{U_{n-1}^2(x)+U_{n-1}^2(y)+U_{n-1}^2(z)}.$$

Proof. Let

$$V(x,y,z) = \frac{\left(-U_{n-1}(x)\,U_{n-1}(z) \quad -U_{n-1}(y)\,U_{n-1}(z) \quad U_{n-1}^2(x)+U_{n-1}^2(y)\right)}{U_{n-1}^2(x)+U_{n-1}^2(y)+U_{n-1}^2(z)}.$$

Since $\frac{dT_n}{dt} = nU_{n-1}$, it is clear that

$$\left\langle V(x,y,z), \left(\frac{dT_n(x)}{dt}, \frac{dT_n(y)}{dt}, \frac{dT_n(z)}{dt}\right)\right\rangle = 0$$

and therefore, if $(x,y,z)\in Z_n$, then $V(x,y,z)\in T_{(x,y,z)}Z_n$, so $V\mid_{Z_n}$ defines a vector field on Z_n. Moreover, for any $(a,b,c)\in T_{(x,y,z)}Z_n$ we have that

$$\begin{aligned}
\langle V(x,y,z),(a,b,c)\rangle &= c - \frac{U_{n-1}(z)}{U_{n-1}^2(x)+U_{n-1}^2(y)+U_{n-1}^2(z)}\left(aU_{n-1}(x)+bU_{n-1}(y)+cU_{n-1}(z)\right)\\
&= c - \frac{\frac{1}{n}U_{n-1}(z)}{U_{n-1}^2(x)+U_{n-1}^2(y)+U_{n-1}^2(z)}\left(anU_{n-1}(x)+bnU_{n-1}(y)+cnU_{n-1}(z)\right)\\
&= c - \frac{\frac{1}{n}U_{n-1}(z)}{U_{n-1}^2(x)+U_{n-1}^2(y)+U_{n-1}^2(z)}\left(a\frac{dT_n}{dt}(x)+b\frac{dT_n}{dt}(y)+c\frac{dT_n}{dt}(z)\right)\\
&= c\\
&= D_{(x,y,z)}p(a,b,c).
\end{aligned}$$

Hence, for all $(x,y,z)\in Z_n$ we have that

$$V(x,y,z) = \nabla p(x,y,z)$$

so $V=\nabla p$ is the gradient flow vector field of p on Z_n. ☐

4.5 Connectedness

Now we prove that Z_n is connected using the handle decomposition of Z_n and the gradient flow of ∇p.

Theorem 4.4. Z_n is connected.

Proof. Consider the subsets of Z_n of the form

$$Z_n^\alpha = \{(x,y,z)\in Z_n \mid z\le\alpha\}.$$

Since p is a Morse function, Z_n^α is a smooth manifold whenever α is not a critical value of p. If α is a critical value, then by 2.2 for small $\varepsilon>0$, we will have that $Z_n^{\alpha+\varepsilon}$ is homotopic to a manifold obtained from $Z_n^{\alpha-\varepsilon}$ by attaching a d-handle to $Z_n^{\alpha-\varepsilon}$ for every critical point of Morse index d with critical value α.

Hence, consider $\alpha_0 < -1$ such that $T_n(\alpha_0)=2$. Then α_0 is a critical value of p, and our critical points of index 0 correspond exactly to this critical value. Hence, for small $\varepsilon>0$, $Z_n^{\alpha_0+\varepsilon}$ is a disjoint union of $\frac{1}{4}n^2$ 0-handles, or equivalently $\frac{1}{4}n^2$ disjoint manifolds diffeomorphic to the disk D_2. The 1-handle corresponding to the critical point $q \in C_0$ will attach to the handles corresponding to the (not necessarily distinct) critical points q_1, q_2 such that there is an integral curve $\gamma : \mathbb{R} \to Z_n$ whose closure $\overline{\gamma}$ is diffeomorphic a segment and its endpoints $\partial\overline{\gamma}$ are q and q_i, for $i = 1, 2$. It will suffice to analyze the attachment of the 1-handles corresponding to the critical points

$$\left\{ \left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), \cos\left(\frac{(2n-1)\pi}{2n}\right) \right) \mid 0 < m_1, m_2 < n, m_1 + m_2 \text{ odd} \right\}.$$

Suppose $m \in \mathbb{Z}$. Then $U_{n-1}\left(\cos\left(\frac{m\pi}{n}\right)\right) = 0$ and therefore

$$\nabla p\left(\cos\left(\frac{m\pi}{n}\right), y, z\right) = \frac{U_{n-1}(y)}{U_{n-1}^2(y) + U_{n-1}^2(z)}\left(0, -U_{n-1}(z), U_{n-1}(y)\right)$$
$$\nabla p\left(x, \cos\left(\frac{m\pi}{n}\right), z\right) = \frac{U_{n-1}(x)}{U_{n-1}^2(y) + U_{n-1}^2(z)}\left(-U_{n-1}(z), 0, U_{n-1}(y)\right).$$

Moreover, for $x \in [-1, 1]$ and $x \neq \cos\left(\frac{M\pi}{n}\right)$ for all $M \in \mathbb{Z}$, there is a unique $m \in \{0, 1, \ldots, n-1\}$ such that

$$\cos\left(\frac{m\pi}{n}\right) > x > \cos\left(\frac{(m+1)\pi}{n}\right).$$

Since $\cos(t)$ is bijective and monotonically decreasing on $[-1, 1]$, then it follows that there is a unique t such that $\cos(t) = x$ and $t \in \left(\frac{m\pi}{n}, \frac{(m+1)\pi}{n}\right)$. Hence,

$$U_{n-1}(x) = U_{n-1}(\cos(t)) = \frac{\sin(nt)}{\sin(t)} \begin{cases} > 0 & m \text{ even} \\ < 0 & m \text{ odd} \end{cases}$$

We also have that T_n is monotonically decreasing on $\left(-\infty, \cos\left(\frac{(2n-1)\pi}{2n}\right)\right)$, so $U_{n-1}(z) < 0$.

This together with 2.1 we have integral curves of ∇p near critical points in $p^{-1}\left(\cos\left(\frac{(2n-1)\pi}{2n}\right)\right)$:

$$\gamma^+_{m_1,2l_2}, \gamma^-_{m_1,2l_2} : \mathbb{R} \to Z_n \cap \left\{x = \cos\left(\frac{m_1\pi}{n}\right)\right\}, \quad m_1, 2l_2 \in (0, n) \text{ and } m_1 \text{ odd}$$
$$\gamma^+_{2l_1,m_2}, \gamma^-_{2l_1,m_2} : \mathbb{R} \to Z_n \cap \left\{y = \cos\left(\frac{m_2\pi}{n}\right)\right\}, \quad 2l_1, m_2 \in (0, n) \text{ and } m_2 \text{ odd}$$

such that

Integral Curve	$\lim_{t\to-\infty} \gamma(t)$	$\lim_{t\to\infty} \gamma(t)$
$\gamma^+_{2l_1,m_2}$	$\left(\cos\left(\frac{(2l_1-1)\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), \alpha_0\right)$	$\left(\cos\left(\frac{(2l_1)\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), \cos\left(\frac{(2n-1)\pi}{2n}\right)\right)$
$\gamma^-_{2l_1,m_2}$	$\left(\cos\left(\frac{(2l_1+1)\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), \alpha_0\right)$	$\left(\cos\left(\frac{(2l_1)\pi}{n}\right), \cos\left(\frac{m_2\pi}{n}\right), \cos\left(\frac{(2n-1)\pi}{2n}\right)\right)$
$\gamma^+_{m_1,2l_2}$	$\left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{(2l_2-1)\pi}{n}\right), \alpha_0\right)$	$\left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{(2l_2)\pi}{n}\right), \cos\left(\frac{(2n-1)\pi}{2n}\right)\right)$
$\gamma^-_{m_1,2l_2}$	$\left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{(2l_2+1)\pi}{n}\right), \alpha_0\right)$	$\left(\cos\left(\frac{m_1\pi}{n}\right), \cos\left(\frac{(2l_2)\pi}{n}\right), \cos\left(\frac{(2n-1)\pi}{2n}\right)\right)$

and thus we obtain a grid-like attachment of handles, which produces a connected manifold Z_n^α. Since Z_n is obtained by attaching more d-handles to Z_n^α with $d > 0$ (because we already included all 0-handles), it follows that Z_n is connected.

Remark 4.1. For $n = 2$, there are no 1-handles, and $Z_2 \cong S^2$, which is connected.

□

Figure 2: $\frac{n(n-2)}{2}$ 1-handles are attached to the $\frac{1}{4}n^2$ 0-handles, shown for $n = 4, 6$.

4.6 Genus

Now that we know that Z_n is a connected closed orientable surface, we are only left to determine its genus. This is equivalent to determining its Euler characteristic, since

$$\chi(Z_n) = 2 - 2g(Z_n).$$

For this, we will employ our Morse function p, and the characterization of its critical points.

Theorem 4.5. The genus of Z_n is $\frac{n^2(n-3)}{4} + 1$.

Proof. Since p is a Morse function on Z_n, by 2.3 and 4.1 we have that

$$\begin{aligned}
\chi(Z_n) = 2 - 2g(Z_n) &= \sum_{i=0}^{2} (-1)^i \left|\text{Morse index } i \text{ points of } p\right| \\
&= \left|C_2^+\right| - \left|C_0\right| + \left|C_2^-\right| \\
&= \frac{n^2}{4} - \frac{n^2(n-2)}{2} + \frac{n^2}{4} \\
&= \frac{n^2(3-n)}{2}
\end{aligned}$$

Hence, we conclude that

$$g(Z_n) = \frac{n^2(n-3)}{4} + 1.$$

□

Hence, we have completely characterized the topology of Z_n, summarized as our main theorem:

Theorem 4.6. For even n, Z_n is a smooth closed orientable surface of genus $\frac{n^2(n-3)}{4} + 1$.

5 Bibliography

References

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[2] K. Burns and M. Gidea, *Differential Geometry and Topology With a View to Dynamical Systems*, Chapman & Hall/CRC (Studies in Advanced Mathematics), 2005, ISBN 1-58488-253-0, 978-1-58488-253-4, Hardcover, IX+389 pp.

A Handle decomposition of Z_4

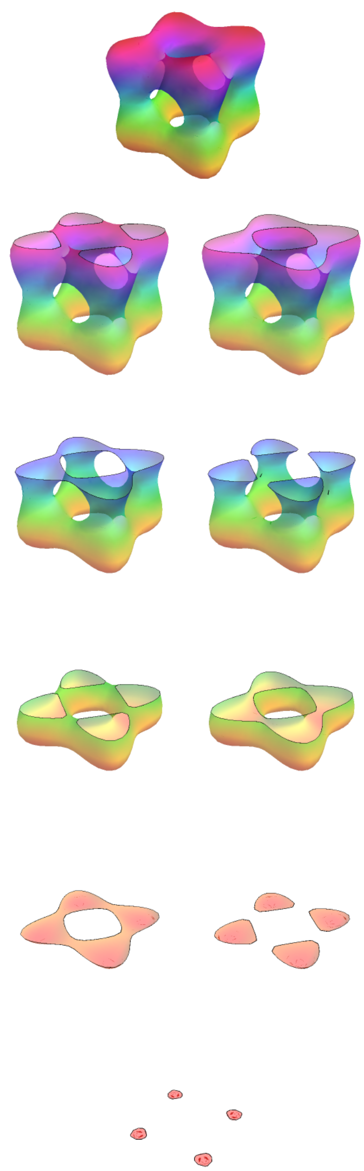

Figure 3: Handle decomposition of Z_4 via some manifolds Z_4^α with α near a critical value of p.

B Handle decomposition of Z_6

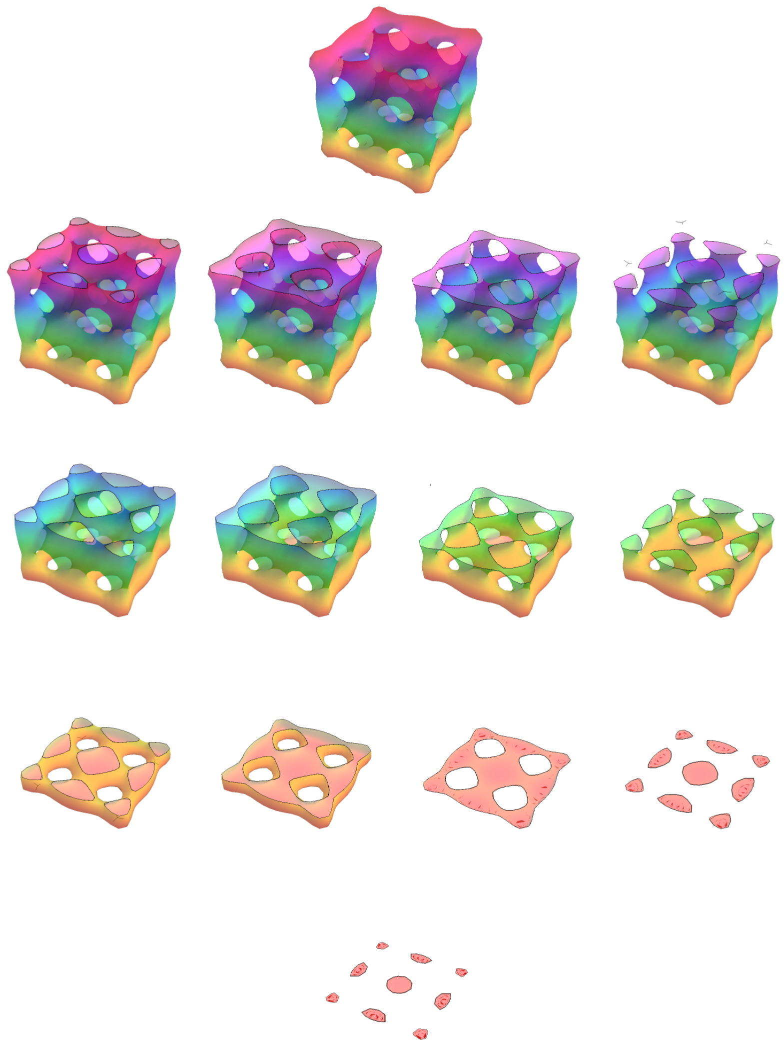

Figure 4: Handle decomposition of Z_6 via some manifolds Z_6^α with α near a critical value of p.