Topic Proposal

Cohomology of Configuration Spaces and Representation Stability

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1 Introduction

Configuration spaces provide a common ground for interactions between topology, combinatorics and representation theory. An important example is the space of n distinct unordered points in the plane. This is an S_n-quotient of the space of n distinct *ordered* points in the plane. These are classifying spaces of the Braid group and the Pure Braid group, respectively.

The study of the cohomology of configuration spaces leads to an important phenomenon: representation stability. In this topic, we study such cohomology, how representation stability realizes in the case of the braid groups, and how it can be studied as seen through the glass of FI-modules.

1.1 Configuration spaces and braid groups

Definition 1.1. Let X be a topological space. Define

$$F_n(X) = \{(x_1, x_2, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}$$

as the **ordered configuration space** of n-tuples of distinct points in X. The symmetric group S_n acts on $F_n(X)$ by permuting the coordinates. Define the quotient

$$C_n(X) = F_n(X)/S_n = \{\{x_1, x_2, \dots, x_n\} \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}$$

as the **unordered configuration space** of n-tuples of dictinct points in X.

Definition 1.2. The **braid group** B_n is defined as the fundamental group of $C_n(\mathbb{C})$, the unordered configuration space of n-tuples of complex numbers. Similarly, the **pure braid group** P_n is defined as the fundamental group of $F_n(\mathbb{C})$.

Since the action of S_n on $F_n(\mathbb{C})$ is free and proper discontinuous, $F_n(\mathbb{C}) \to C_n(\mathbb{C})$ is a covering map. This gives an associated exact sequence

$$1 \to P_n \to B_n \to S_n \to 1$$

where the map $B_n \to S_n$ takes a braid to the permutation associated to its endpoints.

Proposition 1.3. The spaces $F_n(\mathbb{C})$ and $C_n(\mathbb{C})$ are classifying spaces of P_n and B_n, respectively.

Proof. The fibration $F_{n+1}(\mathbb{C}) \to F_n(\mathbb{C})$ given by forgetting a coordinate has fiber homotopic to $\bigvee^n S^1$. Thus, induct on n, using the associated homotopy long exact sequence of this fibration. ☐

Corollary 1.4. Let $n \geq 1$ and R be a commutative ring. Then

$$H^*(P_n; R) \cong H^*(F_n(\mathbb{C}); R) \quad \text{and} \quad H^*(B_n; R) \cong H^*(C_n(\mathbb{C}); R)$$

2 Cohomology of the Braid Groups

2.1 The integral cohomology of P_n

We provide a sketch proof of an explicit presentation of $H^*(F_n(\mathbb{C});\mathbb{Z})$, due to Arnol'd.

Theorem 2.1 ([Arn69])**.** The cohomology ring $H^*(F_n(\mathbb{C});\mathbb{Z})$ is isomorphic to the quotient exterior algebra

$$\Lambda^*[\omega_{ij}]/\langle R_{i,j,k}\rangle \quad \text{for distinct } i,j,k \text{ with } 1 \leq i,j,k \leq n$$

where $R_{i,j,k} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij}$ and $\omega_{ij} \in H^1$ are generators of degree 1.

Sketch proof. There is a section of the fibration $F_n(\mathbb{C}) \to F_{n-1}(\mathbb{C})$ given by the map

$$(z_1,\ldots,z_{n-1}) \mapsto (z_1,\ldots,z_{n-1},1+\max\{|z_1|,\ldots,|z_{n-1}|\}).$$

Using this and induction on the Serre spectral sequence associated to this fibration, Arnol'd proves that

$$H^*(F_n(\mathbb{C});\mathbb{Z}) \cong \bigotimes_{i=1}^{n-1} H^*\left(\bigvee^{i} S^1;\mathbb{Z}\right). \tag{1}$$

To show eq. (1), we use that because we have a section, the spectral sequence degenerates on the E_2 page, and since we have trivial monodromy action on the fiber, we have

$$\begin{aligned} H^k(F_n(\mathbb{C});\mathbb{Z}) &\cong \bigoplus_{p+q=k} H^p\left(F_n(\mathbb{C});H^q\left(\bigvee^{n-1} S^1;\mathbb{Z}\right)\right) \\ &\cong \bigoplus_{p+q=k} H^p(F_{n-1}(\mathbb{C});\mathbb{Z}) \otimes H^q\left(\bigvee^{n-1} S^1;\mathbb{Z}\right) \end{aligned}$$

Moreover, we can also identify $\omega_{1,n},\omega_{2,n},\ldots,\omega_{n-1,n}$ with the generators of $H^*\left(\bigvee^{n-1} S^1;\mathbb{Z}\right)$. Hence

$$H^p(F_n(\mathbb{C});\mathbb{Z}) = \left\{\omega_{i_1j_1}\cdots\omega_{i_pj_p} \mid j_1 < j_2 < \cdots < j_p \text{ and } i_s < j_s \text{ for } s=1,\ldots,p\right\}.$$

As for the ring structure, it suffices to identify ω_{ij} with the de Rham cocycle

$$\frac{1}{2\pi i}\left(\frac{dz_i - dz_j}{z_i - z_j}\right) \in H^1(F_n(\mathbb{C});\mathbb{Z})$$

and this gives a ring homomorphism $\varphi : \Lambda^*[\omega_{ij}]/\langle R_{i,j,k}\rangle \to H^*(F_n(\mathbb{C});\mathbb{Z})$, since the relations $R_{i,j,k}$ are satisfied by these cocycles. Both rings are additively generated by the products of $\omega_{i_lj_l}$, so by a rank count, φ must be both injective and surjective, thus an isomorphism. ☐

2.2 The rational cohomology of B_n

Theorem 2.1 implies that $H^i(F_n(\mathbb{C});\mathbb{Z})$ is free for all $i \geq 0$. The vector spaces $H^i(F_n(\mathbb{C});\mathbb{Q})$ become S_n-representations via $\sigma \cdot \omega_{ij} = \omega_{\sigma(i)\sigma(j)}$. For $n \geq 2$ the only S_n-invariant subspace of $H^1(F_n(\mathbb{C});\mathbb{Q})$ is that spanned by

$$\Omega = \sum_{1 \leq i < j \leq n} \omega_{ij}.$$

Applying the transfer homomorphism gives

$$H^1(C_n(\mathbb{C});\mathbb{Q}) \cong H^1(F_n(\mathbb{C});\mathbb{Q})^{S_n} \cong \mathbb{Q}$$

It has been proved by Arnol'd in [Arn68] that for $i > 1$, the integral cohomology groups of P_n are finite, and therefore $H^i(C_n(\mathbb{C});\mathbb{Q}) \cong 0$. Hence, the next theorem follows.

Theorem 2.2. For $n \geq 2$, we have

$$H^k\left(C_n\left(\mathbb{C}\right);\mathbb{Q}\right)=\begin{cases}\mathbb{Q} & k=0,1\\ 0 & k\neq 0,1\end{cases}$$

2.3 The cohomology of B_n with $\mathbb{F}_2$ coefficients

We now turn to the computation of $H^*\left(B_n;\mathbb{F}_2\right)$. The natural group homomorphisms

$$P_n \to B_n \to S_n \hookrightarrow O\left(n\right)$$

induces maps of classifying spaces $C_n\left(\mathbb{C}\right) \to \mathrm{Gr}\left(n,\mathbb{R}^\infty\right)$, where $\mathrm{Gr}\left(n,\mathbb{R}^\infty\right)$ is the Grassmanian of n-planes in $\mathbb{R}^\infty$. The pullback of the canonical bundle in $\mathrm{Gr}\left(n,\mathbb{R}^\infty\right)$ is a rank n real vector bundle ξ_n on $C_n\left(\mathbb{C}\right)$. This can be further pulled back to a trivial bundle over $F_n\left(\mathbb{C}\right)$ since the sequence $P_n \to B_n \to S_n$ is exact. Thus, one can view the bundle ξ_n as

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook] & F_n\left(\mathbb{C}\right)\times_{S_n}\mathbb{R}^n \arrow[d, "\xi_n"] \\
 & F_n\left(\mathbb{C}\right)/S_n
\end{tikzcd}$$

with diagonal action of S_n on the product. We associate Stifel-Whitney classes $w_i \in H^i\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$ to the bundle ξ_n. In [Fuk70], Fuks computes $H^*\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$, and shows it is generated by the $w_i\left(\xi_n\right)$.

Theorem 2.3 ([Fuk70])**.** Let $n \geq 2$. For all $k \geq 0$ the vector space $H^k\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$ has a basis indexed by sequences of non-negative integers $\{r_1, r_2, \ldots\} = \{r_i\}$ such that $\sum r_i 2^i \leq n$ and $\sum r_i\left(2^j-1\right)=k$. The algebra $H^*\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$ is generated by $w_1,\ldots,w_n$ with multiplication given by

$$\{r_i\} \smile \{s_i\} = \prod_{i\geq 1}\binom{r_i+s_i}{r_i}\{r_i+s_i\}.$$

Sketch proof. Consider the points $\{z_1,\ldots,z_n\} \in C_n\left(\mathbb{C}\right)$ such that the real parts $\Re\left(z_j\right)$ come in multiplicities $m_1,\ldots,m_r$. This means that for some real $x_1 < \cdots < x_r$, exactly m_j elements the set $\{z_1,\ldots,z_n\}$ have real part x_j, for $j = 1, 2, \ldots, r$. The set $e\left(m_1,\ldots,m_r\right)$ of all such points is homeomorphic to $\mathbb{R}^{n+r}$. Thus, every partition $m_1,\ldots,m_r$ of n defines a set $e\left(m_1,\ldots,m_r\right)$. Considering the one-point compactification $C_n\left(\mathbb{C}\right)\cup\{\infty\}$, the sets $e\{m_1,\ldots,m_r\}\cup\{\infty\} \cong S^{n+r}$ give a cell structure. As for the boundary maps,

$$e\left(m_1,\ldots,m_s+m_{s+1},\ldots,m_r\right) \text{ is in the boundary of } e\left(m_1,\ldots,m_s,m_{s+1},\ldots,m_r\right).$$

The degree of this map is $\binom{m_s+m_{s+1}}{m_s}$, which corresponds to the number of ways in which m_s+m_{s+1} points with the equal real part can split into m_s points with equal real part and m_{s+1} points with (distinct to the previous m_s points) equal real part. Using this cell structure we can compute $H^*\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$. For the additive structure, we reduce binomial coefficients $\pmod 2$, and we see that every cocycle is cohomologous to a sum of cells corresponding to partitions consisting of powers of 2.

In order to calculate the ring structure, we use that the sections of $F_{n+1}\left(\mathbb{C}\right) \to F_n\left(\mathbb{C}\right)$ descend to compatible homotopy classes of maps via

$$\begin{tikzcd}
C_n\left(\mathbb{C}\right)\times C_m\left(\mathbb{C}\right) \arrow[r] \arrow[d] & C_{n+m}\left(\mathbb{C}\right) \arrow[dr] & \\
 & & C_{n+m+1}\left(\mathbb{C}\right) \arrow[dl] \\
C_{n+1}\left(\mathbb{C}\right)\times C_{m+1}\left(\mathbb{C}\right) \arrow[r] & C_{n+m+2}\left(\mathbb{C}\right) &
\end{tikzcd}$$

Defining $B_\infty = \varinjlim B_n$, we obtain that $H^*\left(B_\infty;\mathbb{F}_2\right) = \varprojlim H^*\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$ has a Hopf algebra structure surjecting onto each $H^*\left(C_n\left(\mathbb{C}\right);\mathbb{F}_2\right)$, and the cup product is obtained for each n using properties of Hopf algebras. $\square$

2.4 Integral homological stability of B_n

Arnol'd proved as well that the groups $H_k(B_n;\mathbb{Z})$ satisfy homological stability.

Theorem 2.4 ([Arn70]). Fix a homology degree k. For $n \geq 2k$, we have isomorphisms

$$H_k(C_n(\mathbb{C});\mathbb{Z}) \xrightarrow{\sim} H_k(C_{n+1}(\mathbb{C});\mathbb{Z})$$

On the other hand, by Theorem 2.1, we can see that if $n > 2k$ this is not true, even for $k=1$. Since H^1 is generated by the elements ω_{ij}, it follows that

$$H^1(F_n(\mathbb{C});\mathbb{Z}) \cong H_1(F_n(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{2}}.$$

Nevertheless, not all hope is lost. Looking at $H^*(F_n(\mathbb{C});\mathbb{Q})$, we previously showed that $H^1(F_n(\mathbb{C});\mathbb{Q})$ has one copy of the trivial S_n-representation, spanned by $\Omega = \sum_{i<j}\omega_{ij}$. Moreover, the elements

$$\omega_i = \sum_{i \neq j} \omega_{ij}$$

span an $(n-1)$-dimensional irreducible S_n-representation along with $\langle\Omega\rangle$, since $\sigma \cdot \omega_i = \omega_{\sigma(i)}$. Finally, there remains only one irreducible component in this representation, which is a $\frac{1}{2}n(n-3)$-dimensional representation. These irreducible representations correspond to the partitions of n of the form $\{n\}, \{1, n-1\}, \{2, n-2\}$. This is no coincidence, as we will see in Section 4, where we introduce the notion of representation stability.

3 Configuration Spaces of Oriented Manifolds

Now we turn to oriented manifolds X of real dimension m. In his paper, Totaro [Tot96] proved that the cohomology of $F_n(X)$ can be calculated from the cohomology of X via a Leray spectral sequence induced by the inclusion $F_n(X) \hookrightarrow X^n$.

Definition 3.1. Let $a \neq b \in \{1, 2, \ldots, n\}$. Let

$$p_a^* : H^*(X;\mathbb{Z}) \to H^*(X^n;\mathbb{Z}) \quad \text{and} \quad p_{ab}^* : H^*\left(X^2;\mathbb{Z}\right) \to H^*(X^n;\mathbb{Z})$$

be the pullback maps defined by the canonical projections p_a and p_{ab}. Let $\Delta \in H^m\left(X^2;\mathbb{Z}\right)$ be the diagonal class of X. For a partition J of $\{1, 2, \ldots, n\}$ into $n-r$ sets define

$$X_J^{n-r} = \{(x_1, x_2, \ldots, x_n) \in X^n \mid x_i = x_j \text{ if } i, j \text{ belong to the same partition in } J\}$$

Theorem 3.2 ([Tot96]). Let X be an oriented manifold of real dimension m. The inclusion $F_n(X) \hookrightarrow X^n$ induces a Leray spectral sequence converging to $H^*(F_n(X);\mathbb{Z})$ as an algebra. The E_2 page is a quotient of the graded commutative algebra $H^*(X^n;\mathbb{Z})[G_{ab}]$ by the relations

$$G_{ab} = (-1)^m G_{ba} \tag{2}$$

$$(G_{ab})^2 = 0 \tag{3}$$

$$G_{ab}G_{ac} + G_{bc}G_{ba} + G_{ca}G_{cb} = 0 \quad \text{for } a, b, c \text{ distinct} \tag{4}$$

$$p_a^*(x)\, G_{ab} = p_b^*(x)\, G_{ab} \quad \text{for } a \neq b, x \in H^*(X) \tag{5}$$

where $H^i(X^n;\mathbb{Z})$ has degree $(i, 0)$ and the G_{ab} are generators of degree $(0, m-1)$ for $1 \leq a, b \leq n$ $(a \neq b)$. The differential d is given by $dG_{ab} = p_{ab}^*(\Delta)$.

Sketch proof. Let $f : F_n(X) \hookrightarrow X^n$ be the inclusion. The Leray spectral sequence provides an E_2 page with

$$E_2^{p,q} = H^p(X^n; R^q f_* \mathbb{Z})$$

where $R^q f_* \mathbb{Z}$ is the sheaf on X^n defined by $U \mapsto H^q\left(F_n(X) \cap U; \mathbb{Z}\right)$. Looking at the stalks of these sheaves, we use the local euclidean structure of X^n in the following way. The coordinates of x define a partition I of n into s non-empty sets of size $i_1, \ldots, i_s$. Then, for a small neighbourhood $x \in U$ we have that

$$F_n(X) \cap U \cong \prod_{l=1}^{s} F_{i_l}\left(\mathbb{R}^m\right) \implies \left(R^q f_* \mathbb{Z}\right)_x \cong H^q\left(\prod_{l=1}^{s} F_{i_l}\left(\mathbb{R}^m\right); \mathbb{Z}\right)$$

and following [Arn69], we compute $H^*\left(F_n\left(\mathbb{R}^m\right); \mathbb{Z}\right)$ by using the fibration $F_n\left(\mathbb{R}^m\right) \to F_{n-1}\left(\mathbb{R}^m\right)$ with fiber homotopic to $\bigvee^{n-1} S^{m-1}$. This gives

$$H^*\left(F_n\left(\mathbb{R}^m\right); \mathbb{Z}\right) \cong \Lambda^*[G_{ab}] / \left\langle R_{a,b,c}\right\rangle \qquad \text{for distinct } 1 \leq a, b, c \leq n$$

where the G_{ab} are generators in degree $m-1$, satisfying the relations (2)-(4) in theorem 3.2. This shows that $E_2^{p,q}$ can be non-zero only for $q = r(m-1)$ $(r = 0, 1, \ldots, n-1)$. Using the additive basis of $H^*\left(F_n\left(\mathbb{R}^m\right); \mathbb{Z}\right)$ and applying the Kunneth formula gives isomorphisms

$$H^{r(m-1)}\left(\prod_{l=1}^{s} F_{i_l}\left(\mathbb{R}^m\right); \mathbb{Z}\right) \cong \bigoplus_{|J|=n-r} H^{r(m-1)}\left(\prod_{l=1}^{n-r} F_{j_l}\left(\mathbb{R}^m\right); \mathbb{Z}\right)$$

where J runs over refinements of I into $n-r$ sets. The inclusion $F_{j_1}\left(\mathbb{R}^m\right) \times \cdots \times F_{j_{n-r}}\left(\mathbb{R}^m\right) \to X^n$ has an associated Leray spectral sequence whose sheaf is

$$\bigoplus_{|J|=n-r} \mathbb{Z}^{c_J}_{X_J^{n-r}} \cong R^{r(m-1)} f_* \mathbb{Z}$$

where $c_J = (j_1 - 1)! \cdots (j_{n-r} - 1)!$ is the product of the top degrees of $F_{j_l}\left(\mathbb{R}^m\right)$. Hence

$$E_2^{i,r(m-1)} \cong \bigoplus_{|J|=n-r} H^i\left(X_J^{n-r}; \mathbb{Z}\right) \otimes \mathbb{Z}^{c_J}$$

□

Now, it is clear that the first possible non-zero differential is d_m. Nevertheless, Totaro proves that for a smooth complex projective variety X, a weight filtration can be associated to the spectral sequence. In particular, $E_2^{i,r(m-1)}$ has pure weight $i + rm$, and differentials respect pure weight, so for $M > m$, the differentials d_M must vanish, and thus $E_{m+1} \cong E_\infty$. Hence, we obtain the following theorem.

Theorem 3.3 ([Tot96])**.** *Let X be a smooth complex projective variety of real dimension m. Then d_m is the only possible non-zero differential, and $H^*\left(F_n(X); \mathbb{Q}\right)$ is given by the homology of the algebra E_2 determined in Theorem 3.2.*

3.1 Configuration space of an oriented genus g surface

Now we apply Theorem 3.2 to $H^*\left(F_n\left(S_g\right); \mathbb{Q}\right)$ for S_g an orientable surface of genus g. We know that

$$H^i\left(S_g; \mathbb{Z}\right) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & \text{else} \end{cases}$$

with generators $a_1, \ldots, a_g, b_1, \ldots, b_g \in H^1$ and $w \in H^2$ satisfying

$$-b_j a_i = a_i b_j = \delta_{ij} w \quad \text{and} \quad a_i a_j = b_i b_j = 0$$

for all $1 \leq i, j \leq g$. Moreover, the diagonal class is given by

$$\Delta = 1 \otimes w + w \otimes 1 + \sum_{i=1}^{g} b_i \otimes a_i - a_i \otimes b_i \in H^2\left(S_g^2; \mathbb{Z}\right)$$

Thus, by the Kunneth formula, the rank of $H^i\left(S_g^n; \mathbb{Z}\right)$ is given by the coefficient of x^i of the polynomial $\left(1 + 2gx + x^2\right)^n$. Moreover, we have that

$$\text{rank}\left(\bigoplus_{|J|=n-k} H^i\left(\left(S_g\right)_J^{n-k}; \mathbb{Z}\right) \otimes \mathbb{Z}^{c_J}\right) = \left(\text{rank } H^i\left(S_g^{n-k}; \mathbb{Z}\right)\right)\left(\sum_{|J|=n-k} c_J\right).$$

Inductively, we show that the numbers $\sum_{|J|=n-k} c_J$ are precisely the stirling numbers of the first kind, which are defined as follows:

Definition 3.4. Let $k \leq n$ be positive integers. The **unsigned Stirling number of the first kind** $c(n, k)$ is defined as the coefficient of x^k in the polynomial $x(x+1)\cdots(x+n-1)$. Namely,

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} c(n,k)\, x^k$$

Proposition 3.5. For all positive integers $k \leq n$ and partitions J of $\{1, \ldots, n\}$, we have

$$\sum_{|J|=n-r} c_J = c(n,k).$$

Proof. We proceed inductively on n. The statement holds trivially for $n = 1$. Now, suppose the statement is true for n. Then

$$\begin{aligned} x(x+1)\cdots(x+n) &= \left(\sum_{k=1}^{n}\left(\sum_{|J|=n-k} c_J\right) x^k\right)(x+n) \\ &= \sum_{k=1}^{n+1}\left(\left(\sum_{|J|=n+1-k} c_J\right) + n\left(\sum_{|J|=n-k} c_J\right)\right) x^k \end{aligned}$$

so it suffices to prove that

$$\left(\sum_{|J|=n+1-k} c_J\right) + n\left(\sum_{|J|=n-k} c_J\right) = \sum_{|J'|=n+1-k} c_{J'}$$

where J' is a partition of $\{1, \ldots, n+1\}$. This can be seen in a combinatorial way. From every partition J of $\{1, \ldots, n\}$, we can build a partition J' of $\{1, \ldots, n+1\}$ in two manners:

- Adding a singleton $\{n+1\}$ to J.
- Adding $n+1$ to an already existing set of J of size j_i.

Note this process exhausts the partitions J' of $\{1, \ldots, n+1\}$. Thus, a partition J' into $n+1-k$ sets must be obtained from this process from a partition J into $n-k$ sets (in the first case) or from a partition J into $n+1-k$ sets (in the second case). In the first case we have $c_{J'} = c_J$ and in the second case we have $c_{J'} = j_i c_J$. Since in the second case J gives rise to $n+1-k$ partitons $J'_1, \ldots, J'_{n+1-k}$, we have

$$\sum_{i=1}^{n+1-k} c_{J'_i} = \sum_{i=1}^{n+1-k} j_i c_J = \left(\sum_{i=1}^{n+1-k} j_i\right) c_J = n c_J$$

and thus the equality above holds and our induction is complete. $\square$

With Proposition 3.5 we can now compute the ranks of the terms in our E_2 pages for any n. To illustrate this, we provide the calculation for $n = 2, 3$. We turn to rational coefficients, in order to ignore torsion, and realize our elements in the E_2 page as $\mathbb{Q}S_n$-modules.

- $\boxed{n = 2}$ The E_2 page, with rational cohomology, has the following structure

$$\begin{tikzcd}
1 & \mathbb{Q} \arrow[drr] & \mathbb{Q}^{2g} \arrow[drr] & \mathbb{Q} \arrow[drr] & & \\
0 & \mathbb{Q} & \mathbb{Q}^{4g} & \mathbb{Q}^{4g^2+2} & \mathbb{Q}^{4g} & \mathbb{Q} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

  and here we have that $p_{12}^* = \mathrm{Id}$ so $dG_{21} = \Delta$. This implies that

$$\begin{aligned}
(a_i \otimes 1)\, G_{12} &\mapsto (a_i \otimes 1)\, \Delta = w \otimes a_i + a_i \otimes w \\
(b_i \otimes 1)\, G_{12} &\mapsto (b_i \otimes 1)\, \Delta = w \otimes b_i + b_i \otimes w \\
(w \otimes 1)\, G_{12} &\mapsto (w \otimes 1)\, \Delta = w \otimes w
\end{aligned}$$

  so all differentials are injective in this case, thus showing that

$$H^k\left(F_2\left(S_g\right); \mathbb{Q}\right) = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q}^{4g} & k = 1 \\ \mathbb{Q}^{4g^2+1} & k = 2 \\ \mathbb{Q}^{2g} & k = 3 \\ 0 & \text{else} \end{cases}$$

- $\boxed{n = 3}$ The corresponding E_2 page has the following structure

$$\begin{tikzcd}
2 & \mathbb{Q}^2 \arrow[drr] & \mathbb{Q}^{4g} \arrow[drr] & \mathbb{Q}^2 \arrow[drr] & & & & \\
1 & \mathbb{Q}^3 \arrow[drr] & \mathbb{Q}^{12g} \arrow[drr] & \mathbb{Q}^{12g^2+6} \arrow[drr] & \mathbb{Q}^{12g} \arrow[drr] & \mathbb{Q}^3 \arrow[drr] & & \\
0 & \mathbb{Q} & \mathbb{Q}^{6g} & \mathbb{Q}^{12g^2+3} & \mathbb{Q}^{8g^3+12g} & \mathbb{Q}^{12g^2+3} & \mathbb{Q}^{6g} & \mathbb{Q} \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}$$

  where now we have three generators G_{21}, G_{31}, G_{32} in degree $(0, 1)$. With computations similar as the ones for $n = 2$, it can be shown this sepectral sequence degenerates to

$$\begin{tikzcd}
1 & & & \mathbb{Q}^{2g^2+g+1} & \mathbb{Q}^{2g} & \\
0 & \mathbb{Q} & \mathbb{Q}^{6g} & \mathbb{Q}^{12g^2} & \mathbb{Q}^{8g^3} & \mathbb{Q}^{2g^2+g} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

and hence we conclude

$$H^k\left(F_3\left(S_g\right);\mathbb{Q}\right)=\begin{cases}\mathbb{Q} & k=0\\ \mathbb{Q}^{6g} & k=1\\ \mathbb{Q}^{12g^2} & k=2\\ \mathbb{Q}^{8g^3+2g^2+g+1} & k=3\\ \mathbb{Q}^{2g^2+3g} & k=4\\ 0 & \text{else}\end{cases}$$

4 Representation Stability

As we noted previously in Section 2, the cohomology of P_n is not stable. Nevertheless, we also noted that $H^1\left(F_n\left(\mathbb{C}\right);\mathbb{Q}\right)$ decomposes in a determined matter as a $\mathbb{Q}S_n$-representation. We now introduce a notion of stability for this phenomenon, developed by B. Farb and T. Church, called *representation stability.*

First, we recall the representation theory of S_n over a field k of characteristic 0 (such as $\mathbb{Q}$) is well known, and from now on, we deal only with such representations. By Maschke's Theorem, any finite-dimensional S_n-representation can be decomposed into a direct sum of irreducible ones. Moreover, the irreducible S_n-representations are classified by partitions of n, which in turn correspond to Young diagrams with n blocks. For a given partition $\lambda \vdash n$, the irreducible S_n-representation V_λ associated to λ is given as a subrepresentation of the regular representation of S_n, by the $\mathbb{Q}S_n$-span of an element $c_\lambda \in \mathbb{Q}S_n$, the Young symmetrizer associated to λ.

Hence to we can identify partitions $n \vdash \lambda$ with irreducible S_n-representations. We can also assign an S_n-irreducible representation to an irreducible S_m representation, for $n \geq m$, in the following way: Let λ be a partition of m by $a_0+\cdots+a_r$ with $a_0 \geq \cdots \geq a_r$. Then, let

$$V\left(\lambda\right)_n = V\left(a_1,\ldots,a_r\right)_n$$

be the irreducible S_n-representation given by the partition of n of the form $(n-\sum_{i=1}^r a_i)+a_1+\cdots+a_r$. This corresponds to the partition obtained by adding $n-m$ blocks to the uppermost row of the Young diagram of λ. This identification of S_m-representations leads to the notion of representation stability.

Definition 4.1. A sequence $\{V_n\}_{n\geq 0}$ of S_n-representations with maps $\phi_n : V_n \to V_{n+1}$ is **consistent** if ϕ_n is compatible with the S_n action, making the following diagram commute:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

where g' is image of g under the natural inclusion $S_n \hookrightarrow S_{n+1}$.

Definition 4.2. A consistent sequence $\{V_n\}_{n\geq 0}$ of S_n-representations is **representation stable** if it satisfies the following three properties:

1. **Injectivity.** The maps ϕ_n are injective for sufficiently large n.

2. **Surjectivity.** The space V_{n+1} is spanned by the S_{n+1}-orbit of $\phi_n\left(V_n\right)$, for n sufficiently large.

3. **Multiplicities.** In the decomposition of V_n into irreducible representations

$$V_n = \bigoplus_\lambda c_{\lambda,n} V\left(\lambda\right)_n$$

for each λ, the coefficient $c_{\lambda,n}$ is independent of n, for sufficiently large n.

Moreover, a representation stable sequence $\{V_n\}_{n\geq 0}$ is **uniformly** representation stable if the multiplicities $c_{\lambda,n}$ stabilize for some $N \geq 0$ not depending on λ.

4.1 FI-Modules

The notion of representation stability can be translated in the context of FI-modules, a tool that permits working with sequences of S_n-representations at once, using category theory.

Definition 4.3. Fix a noetherian ring k. The category FI has as objects finite sets S, and its morphisms are injections $S \hookrightarrow T$. An FI-module V is a functor FI $\to$ k-mod.

The endomorphims of $V(\{1, \ldots, n\}) = V_n$ provide V_n an S_n action, thus making it an S_n representation. Moreover, the inclusions $\{1, \ldots, n\} \hookrightarrow \{1, \ldots, n+1\}$ induce k-module homomorphisms $V_n \to V_{n+1}$. Thus, one obtains a consistent sequence $\{V_n\}_{n \geq 0}$ from an FI-module V. One may think of V as an usual module, since the notions such as of a quotient and submodule are inherited from k-mod in a pointwise manner. For example, $W \subset V$ is a sub FI-module if for every n, $W_n \subset V_n$ is a k-submodule.

The next definition will prove very useful, as it gives us an equivalent condition of $\{V_n\}_{n \geq 0}$ being representation stable, as a property of V.

Definition 4.4. An FI-module V is **finitely generated** if there is a finite set of elements $S \subset \coprod_n V_n$ so that no proper sub-FI-module $W \subset V$ contains S.

The next theorem, due to Church-Ellenberg-Farb, provides the desired equivalence.

Theorem 4.5 ([CEF15])**.** Let V be an FI-module over a field k of characteristic 0. Then V is finitely generated if and only if $\{V_n\}$ is a uniformly representation stable sequence of S_n-representations with $\dim_k V_n < \infty$ for all n.

4.2 Character Polynomials

Fix a field k of characteristic 0. The *character* χ_V associated to a representation $\rho : G \to GL(V)$ is the function $\chi_V(g) = \text{Trace}\,\rho(g)$. Characters are constant on conjugacy classes, and thus are class functions. A fundamental result in representation theory of finite groups is that two representations V, W are isomorphic if their characters χ_V, χ_W coincide. Now, we will restrict ourselves to $\mathbb{Q}$ for the following definition.

Definition 4.6. Let $X_i : \coprod_n S_n \to \mathbb{Z}$ be the functions defined by

$$X_i(\sigma) = \#\text{ of } i\text{-cycles in the cycle decomposition of } \sigma.$$

A **character polynomial** is a polynomial in $\mathbb{Q}[X_1, X_2, \ldots]$.

To illustrate the idea, we look at two examples. The permutation representation $\mathbb{Q}^n$ of S_n satisfies that the trace of any element $\sigma \in S_n$ equals its number of fix points, or 1-cycles. We previously saw $H^1(P_n; \mathbb{Q})$ is generated by the ω_{ij} for $i < j$, with S_n action given by $\sigma \cdot \omega_{ij} = \omega_{\sigma(i)\sigma(j)}$. The trace of $\sigma \in S_n$ is then given by the number of pairs of fixed points and the number of 2-cycles. Thus, for all $n \geq 1$ we have

$$\chi_{\mathbb{Q}^n} = X_1 \quad \text{and} \quad \chi_{H^1(P_n;\mathbb{Q})} = \binom{X_1}{2} + X_2$$

Theorem 4.7. Let V be a finitely generated FI-module over a field k of characteristic 0. Then the sequence of characters χ_{V_n} of the S_n-representations V_n is eventually polynomial. Namely, there exists $N \geq 0$ and a polynomial $P(X_1, \ldots, X_r)$ for some $r > 0$ so that

$$\chi_{V_n} = P(X_1, \ldots, X_r) \quad \text{for all } n \geq N$$

In particular, $\dim_k V_n$ is eventually polynomial, since $\chi_{V_n}(\text{Id}) = P(n, 0, \ldots, 0)$.

Proof. By Theorem 4.5, we have $V_n = \bigoplus_\lambda c_\lambda V(\lambda)_n$ for large n. Each $V(\lambda)_n$ has an associated character polynomial P_λ, and thus we simply take the polynomial $P_n = \sum_\lambda c_\lambda P_\lambda$. $\square$

4.3 Representation Stability of the Cohomology of P_n

Recall the maps of configuration spaces $f_n : F_{n+1}(\mathbb{C}) \to F_n(\mathbb{C})$ which forget the last coordinate induce maps $\phi_n : H^i(P_n;\mathbb{Q}) \to H^i(P_n;\mathbb{Q})$ compatible with the S_n action on $H^i(P_n;\mathbb{Q})$, for each $i \geq 0$. This gives a consistent sequence of S_n-representations $\left\{H^i(P_n;\mathbb{Q})\right\}$. We present a sketch proof of the uniform representation stability of these sequences, due to Church-Farb. Let us recall some important representation-theoretic constructions we will employ.

Definition 4.8. Let H be a subgroup of a finite group G, and let V be an H-representation. We define the G-representation $\mathrm{Ind}_H^G V = \bigoplus_{\sigma \in G/H} \sigma V$ as the **induced representation**.

Let H be a subgroup of S_k and let V be an H-representation. For $n \geq k$ we can extend the action of H on V to $H \times S_{n-k}$ by letting S_{n-k} act trivially on V, thus giving the $(H \times S_{n-k})$-representation $V \boxtimes \mathbb{Q}$. We will use the following result due to Hemmer.

Theorem 4.9 ([Hem11])**.** Fix $k \geq 1$, a subgroup $H < S_k$, and a H-representation V. Then the sequence

$$\left\{\mathrm{Ind}_{H \times S_{n-k}}^{S_n} V \boxtimes \mathbb{Q}\right\}$$

of S_n-representations is uniformly representation stable, and its decomposition stabilizes for $n \geq 2k$.

Theorem 4.10 ([CF13])**.** For each fixed $i \geq 0$, the sequence of S_n-representations $\left\{H^i(P_n;\mathbb{Q})\right\}$ is uniformly representation stable, and in fact stabilizes once $n \geq 4i$.

Sketch proof. We check the three properties for uniform representation stability. Injectivity and surjectivity for $n \geq 2i$ follow from the additive basis of $H^i(P_n;\mathbb{Q})$ described in the proof of Theorem 2.1, so only uniform stability of multiplicities is left, for which we use Theorem 4.9. We need one more combinatorial notion, along with an observation by Orlik-Solomon.

A partition S of $\{1,\ldots,n\}$ defines a partition $\bar{S} \vdash n$ by the size of its components. Conversely, a partition $\lambda = \bar{S_\lambda} \vdash n$ has a representative S_λ, which generates all partitions S of $\{1,\ldots,n\}$ such that $\bar{S} = \lambda$ via its S_n orbit. Partitions S of $\{1,\ldots,n\}$ into blocks of size $j_1,\ldots,j_s$ induce **Young subgroups** $P_S \cong P_{j_1} \times \cdots \times P_{j_s}$ of P_n. In [OS80] it is proved that as an S_n-module

$$H^*(P_n;\mathbb{Q}) \cong \bigoplus_S H^S(P_n;\mathbb{Q})$$

where $H^S(P_n;\mathbb{Q})$ are the images of top-degree cohomology by the maps induced by the projections $P_n \to P_S$. In particular, degree i images come exactly from partitions into $n-i$ blocks, so for a fixed degree i, we have

$$H^i(P_n;\mathbb{Q}) \cong \bigoplus_{|S|=n-i} H^S(P_n;\mathbb{Q}) \cong \bigoplus_{\lambda \vdash n, |\lambda|=n-i} \mathrm{Ind}_{\mathrm{Stab}(S_\lambda)}^{S_n} H^{S_\lambda}(P_n;\mathbb{Q})$$

For $m > n \geq 2i$, every partition S_μ of $\{1,\ldots,m\}$ into $m-i$ blocks must contain a singleton. Inductively, S_μ must contain $m-n$ singletons, so S_μ is equivalent to partition S_λ with $\lambda \vdash n$ along with $m-n$ singletons. Since λ is uniquely determined, we can write $\mu = \lambda\langle m\rangle$. Since $H^{S_\lambda}(P_n;\mathbb{Q}) \cong H^{S_{\lambda\langle m\rangle}}(P_m;\mathbb{Q})$ this gives

$$\begin{aligned} H^i(P_m;\mathbb{Q}) &\cong \bigoplus_{\lambda \vdash n, |\lambda|=n-i} \mathrm{Ind}_{\mathrm{Stab}\left(S_{\lambda\langle m\rangle}\right)}^{S_m} H^{S_\lambda\langle m\rangle}(P_m;\mathbb{Q}) \\ &\cong \bigoplus_{\lambda \vdash n, |\lambda|=n-i} \mathrm{Ind}_{\mathrm{Stab}(S_\lambda)\times S_{m-n}}^{S_m} H^{S_\lambda}(P_n;\mathbb{Q}) \boxtimes \mathbb{Q} \end{aligned}$$

and thus applying Theorem 4.9 to each summand the theorem follows. ∎

For low degree i, the stable irreducible decomposition of $H^i(P_n;\mathbb{Q})$ has been computed. For instance

$$H^1(P_n;\mathbb{Q}) = V(0)_n \oplus V(1)_n \oplus V(2)_n \qquad \text{for } n \geq 4$$
$$H^2(P_n;\mathbb{Q}) = V(1)_n^{\oplus 2} \oplus V(1,1)_n^{\oplus 2} \oplus V(2)_n^{\oplus 2} \oplus V(2,1)_n^{\oplus 2} \oplus V(3)_n \oplus V(3,1)_n \qquad \text{for } n \geq 7$$

and their associated character polynomials are given by

$$\chi_{H^1(P_n;\mathbb{Q})} = \binom{X_1}{2} + X_2$$
$$\chi_{H^2(P_n;\mathbb{Q})} = 2\binom{X_1}{3} + 3\binom{X_1}{4} + \binom{X_1}{2}X_2 - \binom{X_2}{2} - X_3 - X_4.$$

In general, there are not many known examples of character polynomials, and their computation is an interesting research problem.

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