Flatness, semicontinuity, and base-change

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Abstract

We give an exposition of various results in algebraic geometry of the interaction between sheaf cohomology and base-change. Applications to Hilbert polynomials, flattening stratifications, and are included.

Let $f : X \to Y$ be a proper morphism of noetherian schemes, $\mathcal{F}$ a coherent sheaf on X. Suppose furthermore that $\mathcal{F}$ is *flat* over Y; intuitively this means that the fibers $\mathcal{F}_y = \mathcal{F} \otimes_Y k(y)$ form a "nice" In this case, we are interested in how the cohomology $H^p(X_y, \mathcal{F}_y) = H^p(X_y, \mathcal{F} \otimes k(y))$ behaves as a function of y. We shall see that it is upper semi-continuous and, under nice circumstances, its constancy can be used to conclude that the higher direct-images are locally free.

1 The setup

1.1 The Grothendieck complex

Let us keep the hypotheses as above, but assume in addition that $Y = \mathrm{Spec} A$ is *affine*, for some noetherian ring A. Consider an open affine cover $\{U_i\}$ of X; we know, as X is separated, that the cohomology of $\mathcal{F}$ on X can be computed using Cech cohomology. That is, there is a cochain complex $C^*(\mathcal{F})$ of A-modules, associated functorially to the sheaf $\mathcal{F}$, such that

$$H^p(X, \mathcal{F}) = H^p(C^*(\mathcal{F})),$$

that is, sheaf cohomology is the cohomology of this cochain complex. Furthermore, since the Cech complex is defined by taking sections over the U_i, we see that each term in $C^*(\mathcal{F})$ is a *flat* A-module as $\mathcal{F}$ is flat.

Thus, we have represented the cohomology of $\mathcal{F}$ in a manageable form. We now want to generalize this to affine base-changes:

Proposition 1.1 *Hypotheses as above, there exists a cochain complex $C^*(\mathcal{F})$ of flat A-modules, associated functorially to $\mathcal{F}$, such that for any A-algebra B with associated morphism $f : \mathrm{Spec} B \to \mathrm{Spec} A$, we have*

$$H^p(X \times_A B, \mathcal{F} \otimes_A B) = H^p(C^*(\mathcal{F}) \otimes_A B).$$

Here, of course, we have abbreviated $X \times_A B$ for the base-change $X \times_{\mathrm{Spec} A} \mathrm{Spec} B$, and $\mathcal{F} \otimes_A B$ for the pull-back sheaf.

Proof $C^*(\mathcal{F})$ will, as before, be the Cech complex. We have already given most of the argument. Now if $\{U_i\}$ is an affine cover of X, then $\{U_i \times_A B\}$ is an affine cover of the scheme $X \times_A B$. Furthermore, we have that

$$\Gamma(U_i \times_A B, \mathcal{F} \otimes_A B) = \Gamma(U_i, \mathcal{F}) \otimes_A B$$

by definition of how the pull-backs are defined. Since taking intersections of the U_i commutes with the base-change $\times_A B$, we see more generally that for any finite set I,

$$\Gamma\left(\bigcap_{i\in I} U_i \times_A B, \mathcal{F} \otimes_A B\right) = \Gamma\left(\bigcap_{i\in I} U_i, \mathcal{F}\right) \otimes_A B.$$

It follows that the Cech complex for $\mathcal{F} \otimes_A B$ over the open cover $\{U_i \times_A B\}$ is just $C^*(\mathcal{F}) \otimes_A B$. The assertion is now clear.

We thus have a reasonably convenient picture of how cohomology behaves with base-change. There is, nevertheless, an objection here: the complex $C^*(\mathcal{F})$ can be poorly behaved as a complex of A-modules. There is nothing to assure that the modules are finitely generated, and they need not be. It will thus be necessary to replace them with finitely generated ones. This we can do by some general homological algebra.

Proposition 1.2 *Let C be a finite cochain complex of flat modules over the noetherian ring A such that each cohomology group $H^p(C)$ is finitely generated. Then there is a finite complex K of finitely generated flat modules and a morphism of complexes $K^* \to C^*$ inducing an isomorphism in cohomology.*

Proof Let us suppose, if necessary by re-indexing, that $C^i \neq 0$ only for $0 \leq i \leq n$. We shall construct a K such that $K^i \neq 0$ only for those i as well. We shall do this by descending induction on i: for i large, we just take $K^i = 0$.

Suppose, inductively, that we have finitely generated free A-modules $K^p, K^{p+1}, K^{p+2}, \ldots$ with boundary maps $K^i \to K^{i+1}$ $(i \geq p)$ which are differentials and morphisms $K^i \to C^i$ $(i \geq p)$ such that the induced morphism $H^i(K) \to H^i(C)$ is an isomorphism for $i \geq p+1$, and furthermore such that the map

$$\ker(K^p \to K^{p+1}) \to H^p(C)$$

is surjective. We can draw a commutative diagram of modules

$$\begin{array}{ccccccc} ? & \dashrightarrow & K^p & \longrightarrow & K^{p+1} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ C^{p-1} & \longrightarrow & C^p & \longrightarrow & C^{p+1} & \longrightarrow & \cdots \end{array}$$

where the problem is to extend this diagram. That is, we need to construct a K^{p-1} with maps $K^{p-1} \to K^p, K^{p-1} \to C^{p-1}$ such that the two initial conditions are also satisfied for $p-1$.

The first condition is that the map $\ker(K^{p-1} \to K^p) \to H^{p-1}(C)$ is surjective. To do this, we simply take a *finite* free module F surjecting onto $H^{p-1}(C)$, and lift this to a map $F \to C^{p-1}$ (or more precisely into the module of $p-1$-cycles). We then define $F \to K^p$ to be zero. This is one part of the definition of K^{p-1}; it is clear that the diagram

$$\begin{array}{ccc} F & \longrightarrow & K^p \\ \downarrow & & \downarrow \\ C^{p-1} & \longrightarrow & C^p \end{array} \tag{1}$$

commutes.

For the other part, we have to kill the kernel of $\ker(K^p \to K^{p+1}) \to H^p(C)$. To do this, note that this kernel is finitely generated, A being noetherian. So we can find a finitely generated free module F' surjecting onto this kernel. Define $F' \to C^{p-1}$ to be zero and $F' \to K^p$ to be the composite of the aforementioned surjection and the imbedding $\ker(K^p \to K^{p+1}) \hookrightarrow K^p$. It is easy to see that we have a similar commutative diagram as in (1).

We then set $K^{p-1} = F \oplus F'$ and define the two maps as in the diagram to be the direct sum of the maps out of F, F'. It is then clear that we have continued the construction of K by another step. We can inductively repeat this until we reach K^0. Here, we have to do something different because we want $K^{-1} = 0$.

Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K^0 & \xrightarrow{f} & K^1 \\ & & \downarrow{\scriptstyle g} & & \downarrow \\ 0 & \longrightarrow & C^0 & \longrightarrow & C^1 \end{array}$$

where we know (by the inductive construction above) that the map on H^1 is an isomorphism while the map $\ker(f) \to \ker(g)$ is surjective. We then replace K^0 by $K^0/(\ker(f) \cap \ker(g))$ and stop the complex here (i.e. define $K^{-p} = 0$ for $p > 0$). Then the cohomology in dimension zero of $0 \to K^0/(\ker(f) \cap \ker(g)) \to K^1$ is $\ker(f)/(\ker(g) \cap \ker(f))$, which is the same as that of C. In total, it is clear that the morphism $K^* \to C^*$ induces an isomorphism in cohomology.

Nonetheless, the proof is still incomplete, as we have not shown that K is a flat complex. We do know that K is flat (even free) in positive dimensions by construction, but in dimension zero the argument took a quotient. Thus, we shall need a lemma.

Lemma 1.3 *Suppose K, C are finite complexes in nonnegative dimensions such that:*

1. *K^i is flat for $i > 0$.*
2. *C is flat (in all dimensions).*
3. *There is a morphism $K^* \xrightarrow{\phi} C^*$ inducing isomorphisms on cohomology.*

Then K is flat in all dimensions.

Proof Let δ^C, δ^K denote the coboundaries of C^* and K^*, respectively. Consider the mapping cylinder L of the morphism $\phi : K^* \to C^*$. By definition, $L^p = C^p \oplus K^{p+1}$, and the coboundary morphism δ^L is defined by $\delta^L(c, k) = (-\delta^C c - \phi(k), \delta^K k)$. There is a short exact sequence of complexes

$$0 \to C \to L \to \widetilde{K} \to 0,$$

where $\widetilde{K}$ denotes K shifted by a degree. In the long exact sequence of chain complexes

$$H^p(L) \to H^p(\widetilde{K}) \to H^{p+1}(C) \to H^{p+1}(L) \to \dots$$

one can easily check that $H^p(\widetilde{K}) \to H^{p+1}(C)$ is identified with $\phi_* : H^{p+1}(K) \to H^{p+1}(C)$. Thus all the connecting homomorphisms are isomorphisms, so exactness implies that L is acyclic. However, we know that L^i is flat for $i \geq 0$, while we need to show that L^{-1} is flat. The exactness of the complex

$$0 \to L^{-1} \to L^0 \to \cdots \to L^N \to 0,$$

where $N \gg 0$, shows that L^{-1} is flat (by induction or "dimension-shifting"). Thus K^0, as a direct factor, is also flat.

Let us return to the algebro-geometric situation of earlier. There, we had associated to each flat, coherent sheaf $\mathcal{F}$ on X a finite[1] complex $C^*(\mathcal{F})$ of flat A-modules whose cohomologies were $H^p(X, \mathcal{F})$. By the proper mapping theorem, these cohomologies are finitely generated A-modules. Thus, by Proposition 1.2, we deduce that there is a finite complex of finitely generated flat modules K^* such that

$$H^p(X, \mathcal{F}) = H^p(K)$$

for all $\mathcal{F}$, and furthermore such that there is a map $K \to C^*(\mathcal{F})$ inducing an isomorphism on cohomology (which is stronger than the displayed statement).

This by itself does not help if we wish to study cohomology and base-change, however, as it does not give information on $H^p(X \times_A B, \mathcal{F} \otimes_A B)$ for an A-algebra B. We would expect and hope that this could be obtained via $H^p(K \otimes_A B)$. The next result will imply that.

Proposition 1.4 *Suppose $K^* \stackrel{\phi}{\to} C^*$ is a morphism of finite flat complexes of A-modules[2] inducing an isomorphism in cohomology. Then for any A-algebra B, the map $K^* \otimes_A B \to C^* \otimes_A B$ induces an isomorphism in cohomology.*

Proof Indeed, consider the mapping cylinder L of ϕ. There is, as before, a short exact sequence $0 \to C \to L \to \widetilde{K} \to 0$, whose long exact sequence shows as earlier that L is acyclic. In addition, L is obviously flat. Since L is flat, we find that $L \otimes_A B$ is also exact. Indeed, we draw the complex

$$0 \to L^{-1} \to L^0 \to \cdots \to L^N \to 0,$$

and note that, by induction, the kernels and cokernels at each step are also flat. So we can split this into several short exact sequences

$$0 \to Z^i \to L^i \to Z^{i-1} \to 0$$

with each Z^i (the group of cycles) flat. Tensoring with any B produces short exact sequences

$$0 \to Z^i \otimes_A B \to L^i \otimes_B \to Z^{i-1} \otimes_A B \to 0,$$

which show that

$$0 \to L^{-1} \otimes_A B \to L^0 \otimes_A B \to \cdots \to L^N \otimes_A B \to 0$$

is exact.

It follows that the mapping cylinder of $\phi \otimes 1_B : K^* \otimes_A B \to C^* \otimes_A B$ is acyclic, so $\phi \otimes 1_B$ induces an isomorphism in cohomology.

[1] The complex is finite since we can choose the affine cover finite.

[2] Here "finite" means that there are finitely many terms; the actual modules need not be finitely generated.

We now see that the finite flat replacement K for the Cech complex $C^*(\mathcal{F})$ has the convenient property that $H^p(X \times_A B, \mathcal{F} \otimes_A B) = H^p(C^*(\mathcal{F}) \otimes_A B) = H^p(K \otimes_A B)$. We have thus proved:

Theorem 1.5 (The Grothendieck Complex) *Let $f : X \to Y$ be a proper morphism of noetherian schemes for $Y = \mathrm{Spec}A$ affine. Let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then there is a finite complex K of finitely generated A-modules such that*

$$H^p(K \otimes_A B) = H^p(X \times_A B, \mathcal{F} \otimes_A B)$$

for any A-algebra B.

1.2 Semicontinuity of the cohomologies

Theorem 1.6 (The Semicontinuity theorem) *Let $f : X \to Y$ be a proper morphism of noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then the function $y \to \dim H^p(X_y, \mathcal{F}_y)$ is upper semi-continuous on Y. Further, the function $y \to \chi(\mathcal{F}_y) = \sum_p (-1)^p \dim H^p(X_y, \mathcal{F}_y)$ is locally constant on Y.*

Proof The assertions are local on $y \in Y$, so we may assume that Y is affine, say $Y = \mathrm{Spec}A$. By Theorem 1.5, there is a finite complex K^* of finite flat A-modules such that $H^p(X_y, \mathcal{F}_y) = H^p(K \otimes_A k(y))$ for $k(y)$ the residue field at y. This is a special case of the base-change formula of Theorem 1.5, in fact.

Let us now prove a lemma that will imply the semicontinuity theorem:

Lemma 1.7 *Let K be a finite complex of finitely generated flat A-modules, for A noetherian. Then:*

1. *For each p, the function $y \to \dim H^p(K \otimes_A k(y))$ is lower semicontinuous on* Spec$A$.
2. *The function $y \to \sum_p (-1)^p \dim H^p(K \otimes_A k(y))$ is locally constant on* Spec$A$.

Proof Over a noetherian ring, a finitely generated flat module is projective, hence locally free. Since the assertion is local on y, we may assume that K consists of free modules.

By elementary homological algebra, one knows that the sum $\sum_p (-1)^p \dim H^p(K^* \otimes_A k(y))$ is equal to the sum $\sum_p (-1)^p \dim(K^p \otimes_A k(y))$; that is, the Euler characteristic does not change upon passage to cohomology. Since by freeness this last sum is constant, the second part of the lemma is clear.

Let us now prove the first part. By definition,

$$H^p(K^* \otimes_A k(y)) = \frac{\ker(K^p \otimes_A k(y) \to K^{p+1} \otimes_A k(y))}{\mathrm{Im}(K^{p-1} \otimes_A k(y) \to K^p \otimes_A k(y))} = Z^p/B^p,$$

where $B^p = \mathrm{Im}(K^{p-1} \otimes_A k(y) \to K^p \otimes_A k(y))$, and $Z^p = \ker(K^p \otimes_A k(y) \to K^{p+1} \otimes_A k(y))$. (These depend on y, of course, though we have suppressed it in the notation.) Consequently, the rank of the cohomology equals

$$\dim_{k(y)} H^p(K \otimes_A k(y)) = \dim Z^p - \dim B^p,$$

where Z^p is the kernel and B^p the image. We have an exact sequence $0 \to Z^p \to K^p \otimes_A k(y) \to B^{p+1} \to 0$, so by linear algebra its dimension equals

$$\dim Z^p = \dim K^p \otimes_A k(y) - \dim B^{p+1}.$$

In total, we find that

$$\dim_{k(y)} H^p(K \otimes_A k(y)) = \dim K^p \otimes_A k(y) - \dim B^p - \dim B^{p+1}. \tag{2}$$

Here $\dim K^p \otimes_A k(y)$ is constant by freeness. Moreover, by the next lemma, $\dim B^p, \dim B^{p+1}$ are lower semicontinuous as functions of y. The difference is thus upper semicontinuous.

The result is now clear in view of Eq. (2).

Lemma 1.8 *Let A be a noetherian ring, and M, N finite free A-modules. Let $\phi : M \to N$ be a homomorphism. Then the function $y \to \dim(\mathrm{Im}(M \otimes_A k(y) \to N \otimes_A k(y))$ is lower semicontinuous.*

Proof Suppose $r \in \mathbb{N}$; we must show that the set of y such that $\dim_{k(y)}(\mathrm{Im}(M \otimes_A k(y) \to N \otimes_A k(y)) < r$ is closed. Now to say that this dimension is less than r is to say that the image of

$$\bigwedge^r M \otimes_A k(y) \to \bigwedge^r N \otimes_A k(y)$$

is zero. This in turn is equivalent to the condition that the morphism $\bigwedge^r M \otimes_A k(y) \to \bigwedge^r N \otimes_A k(y)$ be zero, which is a closed condition: if the morphism of free modules $\bigwedge^r \phi : \bigwedge^r M \to \bigwedge^r N$ is represented by the matrix A, then the condition is equivalent to A having entries lying in the ideal associated to y.

The "semicontinuity" assertion in the semicontinuity theorem actually used considerably less than the full strength of the existence of the Grothendieck complex proved in Theorem 1.5. It would have been sufficient to construct a *bounded above* complex K^* of flat, finitely generated modules and a morphism $K^* \to C^*$ induced isomorphisms in cohomology.

Finally, we note that without flatness, we can still obtain a weaker conclusion.

Theorem 1.9 *Let $p : X \to Y$ be a proper morphism of noetherian schemes, $\mathcal{F}$ a coherent sheaf on X. Then the function F*

$$y \to H^p(X_y, \mathcal{F} \otimes k(y))$$

is constructible for any p.

Recall that a function $Y \to \mathbb{Z}$ is called *constructible* if the inverse image of any finite set is constructible. So even without flatness, the behavior of the cohomologies is not too bad.

Proof By noetherian induction, it suffices to show the following: there is an open dense subset of Y on which $F : y \to H^p(X_y, \mathcal{F} \otimes k(y))$ is constant.[3] We can assume Y reduced, since passing to the reduced scheme does not affect the function F in the statement of the theorem. It will also suffice to prove the result for each of the irreducible components of Y (as constructible sets are closed under finite unions and intersections), so we may assume Y integral.

By generic flatness, there is an open dense subset $U \subset Y$ such that $\mathcal{F}|_{p^{-1}(U)}$ is flat over U. In particular, the assertions of the semicontinuity theorem are valid for $\mathcal{F}|_{p^{-1}(U)}$. Let ξ be the generic point of U, and let $n = F(\xi)$. We claim that there is an open subset $U' \subset U$ containing ξ on which $F \equiv n$. Indeed, F is upper semicontinuous, so $\{y \in U : F(y) \leq n\}$ is open (and thus dense), as it is equivalently $\{y \in U : F(y) < n+1\}$. Similarly, $\{y \in U : F(y) \geq n+1\}$ is closed, and does not contain ξ. This establishes the claim, and thus the theorem.

[3]For then by the noetherian induction hypothesis, the function is constructible on the complement of that open dense set. Splicing the two sets together, we find that the function is constructible.

2 Cohomology and base change

We are now interested in the following question. Let X be a proper scheme over Y, $\mathcal{F}$ a Y-flat coherent sheaf. How does the operation of forming the higher direct images $R^i f_*(\mathcal{F})$ commute with base-extension of Y? In particular, how do the fibers of $R^i f_*(\mathcal{F})$ relate to the fiber cohomologies $H^i(X_y, \mathcal{F}_y)$?

It is known for *flat* base-extension $Y' \to Y$, one has perfect commutation of the two operations (even without $\mathcal{F}$ flat). Unfortunately, the morphisms $\mathrm{Spec} k(y) \to Y, y \in Y$ are almost never flat, so this remark fails in a crucial case. In general, the answer is not so simple. However, in view of Theorem 1.5, it is at least a problem of pure algebra: when do the operations of taking the cohomology of a flat complex and base-change commute?

2.1 General nonsense

Let us consider a A-linear functor $T : A - \mathbf{mod} \to A - \mathbf{mod}$, for $A - \mathbf{mod}$ the category of A-modules. If T is of the form $M \to M \otimes_A N$ for a fixed module N, then T is right-exact.

Suppose that T commutes with (arbitrary) direct sums. Then the converse is true.

Proposition 2.1 *If $T : A - \mathbf{mod} \to A - \mathbf{mod}$ is a right-exact, A-linear functor that commutes with arbitrary direct sums, then there exists an A-module N such that T is naturally isomorphic to the functor $M \to M \otimes_A N$.*

In particular, in this case T commutes with base change!

Proof We can take $N = T(A)$. Indeed, let us define a natural morphism

$$T(A) \otimes M \to T(M) \tag{3}$$

for any A-module M. To do this, note that each $m \in M$ corresponds to a map $A \to M$, and thus induces a map $t_m : T(A) \to T(M)$. There is thus a natural map $M \times T(A) \to T(M)$, which is easily checked to be bilinear. This defines the above natural transformation.

If $M = A$, this simply corresponds to the identity $T(A) \to T(A)$, and Eq. (3) is consequently an isomorphism. Since T commutes with arbitrary direct sums, Eq. (3) is an isomorphism for any free module. The following lemma then implies that it is an isomorphism for any M.

Lemma 2.2 *Let T, U be additive, right-exact functors from the category of A-modules to itself. Suppose and $T \to U$ is a natural transformation inducing an isomorphism on all free A-modules. Then $T \to U$ is a natural isomorphism.*

Proof Fix any A-module M. Pick a free presentation

$$F' \to F \to M \to 0,$$

which we can always do. There is a commutative and exact (by right-exactness) diagram

$$\begin{array}{ccccccc} T(F') & \longrightarrow & T(F) & \longrightarrow & T(M) & \longrightarrow & 0 \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\ U(F') & \longrightarrow & U(F) & \longrightarrow & U(M) & \longrightarrow & 0 \end{array}$$

where the two vertical maps indicated are isomorphisms. A diagram chase shows that $T(M) \to U(M)$ is an isomorphism.

Thus we have a complete description of right-exact (and cocontinuous) functors on the category of A-modules. We are in particular interested in functors arising from the Grothendieck complex, to which we next specialize.

2.2 δ-functors

Consider a δ-functor $\{T^i\}$ on the category of A-modules. In general, each T^i is a *half-exact* functor: a short exact sequence $0 \to M' \to M \to M \to 0$ induces an exact sequence $T^i(M') \to T^i(M) \to T^i(M'')$, but generally zeros cannot be appended on either end. We are interested in conditions when a piece of a δ-functor will be more exact than usual. The following is quite easy:

Proposition 2.3 *T^i is right-exact if and only if T^{i+1} is left-exact.*

Proof Indeed, both are equivalent to the statement that for any short exact sequence $0 \to M' \to M \to M'' \to 0$, the connecting morphism $T^i(M'') \to T^{i+1}(M')$ is zero.

We now consider the case of primary importance to us. Let P^* be a flat cochain complex of A-modules, and consider the functors $T^i(M) = H^i(M \otimes P^*)$. Since P^* is flat, this is in fact a δ-functor in M: a short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a short exact sequence of complexes $0 \to M' \otimes P^* \to M \otimes P^* \to M'' \otimes P^* \to 0$ and thus long exact sequence $T^i(M') \to T^i(M) \to T^i(M'') \to T^{i+1}(M') \to \dots$.

We are interested in understanding when one of the functors T^i turns out to be, for instance, right-exact. (By Proposition 2.3, we may as well understand when a functor like this is left-exact.) This is of interest to us because to say that T^i is right-exact is to say that T^i commutes with base-change, thanks to Proposition 2.1. And, after all, we would like to understand when cohomology of a flat sheaf—that is, cohomology of the Grothendieck complex—commutes with base-change.

Proposition 2.4 *Notation as above, the functor T^i is left-exact if and only if* $\mathrm{coker}(P^{i-1} \to P^i)$ *is flat. In particular, T^i is right-exact if and only if* $\mathrm{coker}(P^i \to P^{i+1})$ *is flat.*

Proof The last statement follows from the rest of the result and Proposition 2.3.

By definition, $T^i(M)$ is the homology of the three-term complex $P^{i-1} \otimes M \stackrel{\delta^{i-1}\otimes M}{\to} P^i \otimes M \stackrel{\delta^i \otimes M}{\to} P^{i+1} \otimes M$, where the $\{\delta^i\}$ are the differentials of P^*. That is, $T^i(M) = \ker(\delta^i \otimes M)/\mathrm{Im}(\delta^{i-1} \otimes M)$.

This implies that we can draw a short exact sequence, functorial in M,

$$0 \to T^i(M) \to \mathrm{coker}(\delta^{i-1} \otimes M) \to P^{i+1} \otimes M,$$

which we can also write as

$$0 \to T^i(M) \to \mathrm{coker}(\delta^{i-1}) \otimes M \to P^{i+1} \otimes M.$$

Suppose now that $\mathrm{coker}(\delta^{i-1})$ is flat, and let $M' \hookrightarrow M$ be a monomorphism of modules. There is then a commutative and exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^i(M') & \longrightarrow & \mathrm{coker}(\delta^{i-1}) \otimes M' & \longrightarrow & P^{i+1} \otimes M' \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^i(M) & \longrightarrow & \mathrm{coker}(\delta^{i-1}) \otimes M & \longrightarrow & P^{i+1} \otimes M \end{array} \qquad (4)$$

Since the middle vertical arrow is a monomorphism by flatness, so is $T^i(M') \to T^i(M)$. This implies left-exactness.

Conversely, suppose that T^i is left-exact. Then draw the diagram as above in (4) for any monomorphism $M' \hookrightarrow M$. In this case, the leftmost and rightmost vertical arrows are monomorphisms (P^{i+1} being flat). An easy diagram chase (or application of the snake lemma, replacing the top-right $P^{i+1} \otimes M$ with the image of $\mathrm{coker}(\delta^{i-1}) \otimes M'$) shows that the middle arrow is a monomorphism, establishing flatness of the cokernel.

This is still unsatisfactory. We may not have a good description of the P^i, let alone the cokernels $\mathrm{coker}(P^{i-1} \to P^i)$, even though we may know the cohomologies well, especially on the fibers. Fortunately, there is a very clean criterion for flatness in terms of the ranks on the fibers. This will enable us to get a more usable version of the above result.

Lemma 2.5 *Let A be a reduced noetherian ring, M a finite A-module. Then M is flat if and only if the rank $M \otimes_A k(\mathfrak{p})$ is locally constant for $\mathfrak{p} \in \mathrm{Spec}A$.*

Here $k(\mathfrak{p})$ is the residue field, i.e. that of $A_{\mathfrak{p}}$.

Proof Recall that for finitely generated modules over noetherian rings, flatness of M is equivalent to the condition that $M_{\mathfrak{p}}$ be free for each $\mathfrak{p}$, or equivalently that the sheaf $\widetilde{M}$ on $\mathrm{Spec}A$ is locally free.

With this in mind, one implication is clear. Let us prove the other. Since the assertion is local, we may assume that the rank is constant throughout $\mathrm{Spec}A$; call k the common rank.

Choose any $\mathfrak{p} \in \mathrm{Spec}A$. There are $m_1, \dots, m_k \in M$ that generate $M_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$-module. There is thus a neighborhood U of $\mathfrak{p}$ in which these elements generate the localizations $M_{\mathfrak{p}}$. Shrinking $\mathrm{Spec}A$, we may assume that $m_1, \dots, m_k$ generate at all the localizations, and hence generate M.

Consider the morphism $A^k \to M$ sending the ith basis vector in A^k to m_i. This is a surjection, and we know furthermore that for each $\mathfrak{p}$,

$$A^k \otimes_A k(\mathfrak{p}) \to M \otimes_A k(\mathfrak{p})$$

is an isomorphism, as it is a surjection of vector spaces of the same dimension. It follows that the kernel is contained in $\mathfrak{p}A^k$ for each $\mathfrak{p} \in \mathrm{Spec}A$, and thus is zero as A is reduced.

Of course, the above fails if A is non-reduced; then $A/\mathrm{rad}(A)$ has constant rank but is not free.

We mention a corollary to be used in the sequel.

Corollary 2.6 *Let A be a noetherian local domain with quotient field K and residue field k. Suppose M is a finitely generated module such that*

$$\dim_k M \otimes k = \dim_K M \otimes K$$

Then M is free.

Proof Let n be the common dimension $\dim_k M \otimes k = \dim_K M \otimes K$.

The function $F : y \to \dim_{k(y)} M \otimes k(y)$ is (as is well-known) upper-semicontinuous on $Y = \mathrm{Spec}A$. It is sufficient to show that F is constant on $\mathrm{Spec}A$ by Lemma 2.5 and the fact that a finitely generated flat module over a noetherian local ring is free.

There is a proper closed subset $X \subset \mathrm{Spec}A$ consisting of the y with $F(y) > n$. If X is nonempty, then it must contain the closed point, a contradiction. So X is empty and all $y \in Y$ satisfy $F(y) \geq n$. Similarly, the set X' of $y \in Y$ with $F(y) < n$ is open; if it is nonempty, it thus contains the generic point. Thus $X = X' = \emptyset$, and $F \equiv n$ on $\mathrm{Spec}A$.

With this, we can get a purely fiberwise criterion for when a δ-functor $\{T^i\}$ of the form above is exact at a given spot.

Proposition 2.7 *Let A be noetherian and reduced. Let P^* be a cochain complex of finitely generated flat A-modules, and define the δ-functor $\{T^i\}$ as before, via $T^i(M) = H^i(P^* \otimes M)$. Suppose*

$$\dim_{k(\mathfrak{p})} T^i(k(\mathfrak{p}))$$

is locally constant for $\mathfrak{p} \in \mathrm{Spec} A$. Then T^i and T^{i-1} commute with base change (i.e. are right-exact). Furthermore, $T^i(A)$ is flat.

Given a $k(\mathfrak{p})$-module N, note that $T^i(N)$ is also a $k(\mathfrak{p})$-module.

Proof Since cohomology commutes with filtered colimits (in particular, arbitrary direct sums), we know by Proposition 2.1 that right-exactness of T^i is equivalent to the assertion that T^i commutes with base-change. We need to show that this is implied by the condition on the fibers.

Let us fix the notation as in the proof of the semicontinuity theorem. We know that

$$H^i(K^* \otimes_A k(y)) = \frac{\ker(K^i \otimes_A k(y) \to K^{i+1} \otimes_A k(y))}{\mathrm{Im}(K^{i-1} \otimes_A k(y) \to K^i \otimes_A k(y))} = Z^i/B^i,$$

where $B^i = \mathrm{Im}(K^{i-1} \otimes_A k(y) \to K^i \otimes_A k(y))$, and $Z^i = \ker(K^i \otimes_A k(y) \to K^{i+1} \otimes_A k(y))$. As before, we find that

$$\dim H^i(K^* \otimes_A k(y)) = \dim K^i \otimes_A k(y) - \dim B^i - \dim B^{i+1}.$$

By assumption, this is locally constant in y. Since on the right we have a sum of upper semicontinuous functions, it follows that $\dim B^i$ and $\dim B^{i+1}$ are locally constant in y.

We are going to show that $\mathrm{coker}(K^i \to K^{i+1})$ is flat. By Proposition 2.4, this will imply that T^i is right-exact, and in particular commutes with base-change. To do this, it suffices by Lemma 2.5 to show that the dimensions $\dim \mathrm{coker}(K^i \to K^{i+1}) \otimes k(y)$ are locally constant. But by linear algebra this is precisely $\dim K^{i+1} \otimes k(y) - \dim B^{i+1}$; we have just seen that this is locally constant in y.

Finally, we must show that T^{i-1} is right-exact as well. That is, we must show that T^i is *left* exact by Proposition 2.3. Since T^i is naturally isomorphic to the functor $M \to M \otimes T(A)$, it suffices to prove $T(A)$ *flat*. By Lemma 2.5, it suffices to show that

$$T^i(A) \otimes k(\mathfrak{p}) = T^i(k(\mathfrak{p}))$$

has locally constant rank; but this was assumed. (Alternatively, we see that $\dim B^i$ is locally constant by the above arguments, so we get as before that T^{i-1} is right-exact.) Thus T^i is even *left-exact*, and since T^i is naturally isomorphic to $N \to N \otimes T^i(A)$, we find that $T^i(A)$ is flat.

For future applications, we note a weaker variant of this result when the ring is not assumed to be reduced.

Proposition 2.8 *Let A be noetherian. Let P^* be a cochain complex of finitely generated flat A-modules, and define the δ-functor $\{T^i\}$ as before, via $T^i(M) = H^i(P^* \otimes M)$. Suppose*

$$\dim_{k(\mathfrak{p})} T^i(k(\mathfrak{p})) = 0$$

for $\mathfrak{p} \in \mathrm{Spec} A$. Then T^{i-1} commutes with base change (i.e. is right-exact).

Proof We will show in fact that T^i is identically the zero functor. T^{i-1} will be forced to be right-exact as a result.

Let $W_i = \text{coker}(K^{i-1} \to K^i)$; we need to show that this is flat. Fix $y \in \text{Spec} A$. We will show that W_i is free in a neighborhood of y. We have an exact sequence

$$K^{i-1} \otimes k(y) \stackrel{d_{i-1}}{\to} K^i \otimes k(y) \stackrel{d_i}{\to} K^{i+1} \otimes k(y).$$

Let $\overline{W}$ be the image of $K^{i-1} \otimes k(y)$ in $K^i \otimes k(y)$. Then there is a complement $\overline{V} \subset K^i \otimes k(y)$ such that $\overline{V} \to K^{I+1} \otimes k(y)$ is exact.

We are now going to lift $\overline{V}$ and $\overline{W}$ to free submodules of K^i. For $\overline{W}$, we lift a basis to elements in the image of d_{i-1}. For $\overline{V}$, we just lift a basis. Then we get submodules $W, V \subset K^i$ such that $\overline{W}$ is the reduction of W (resp. $\overline{V}$ is the reduction of V) Moreover, W is in the image of $d_{i-1} : K^{i-1} \to K^i$. By Nakayama's lemma, we find that $V + W = K^i$.

We know that $V \otimes k(y) \to K^{i+1} \otimes k(y)$ is injective. Thus, if we shrink $\text{Spec} A$ (i.e. localize at some element not in the prime ideal of y), then $V \to K^{i+1}$ is injective as well. In particular, V does not intersect W, since W maps to zero in K^{i+1}. Thus

$$V \oplus W = K^i,$$

and in particular the image W is a split submodule of K^i. In particular, the cokernel of $K^{i-1} \to K^i$, that is K^i/W, is isomorphic to a direct factor of K^i and is consequently projective.

2.3 The base-change theorem

The next theorem states that if the cohomologies along the fibers have the same dimension, then they form a vector bundle.

Theorem 2.9 *Let Y be a reduced noetherian scheme, $f : X \to Y$ a proper morphism, and $\mathcal{F}$ a coherent sheaf on X flat over Y. TFAE:*

1. *The function $y \to \dim H^p(X_y, \mathcal{F}_y)$ is locally constant.*
2. *$R^p f_*(\mathcal{F})$ is locally free on Y and the map $R^p f_*(\mathcal{F}) \otimes k(y) \to H^p(X_y, \mathcal{F}_y)$ is an isomorphism for $y \in Y$.*

It is obvious that 2 implies 1; the real interest is in the reverse implication. It will turn out, though, that this result is just a special case of the general algebra already done, given the interpretation of sheaf cohomology given by the Grothendieck complex.

Proof As usual, we can assume that Y is affine, say $Y = \text{Spec} A$ with A noetherian. Thus by Theorem 1.5 we get a finite complex of flat, finitely generated modules K^* such that $H^p(X_y, \mathcal{F}_y) = H^p(K^* \otimes k(y))$ (and more generally for any affine base-change). In this case, if the dimensions of the cohomologies are locally constant in y, Proposition 2.7 implies that $H^p(K)$ is flat and that taking cohomology for K^* commutes with arbitrary base change in dimension p, that is

$$H^p(K^* \otimes M) \simeq H^p(K^*) \otimes M$$

for any A-module M. If we take $M = k(y)$, we immediately get 2.

Corollary 2.10 *In the situation of Theorem 2.9, if $g : Y' \to Y$ is any morphism and $g' : Y' \times_Y X \to X$ the base-change, so that we have a cartesian square*

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{g'} & X \quad , \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & X \end{array}$$

then for $i = p, p-1$,

$$g^* R^i f_*(\mathcal{F}) \simeq R^i f'_*(g'^*(\mathcal{F})).$$

Thus, in this case, cohomology (in these two dimensions) commutes with arbitrary base-change. Note that this is a generalization of Theorem 2.9.

Proof Indeed, we need only consider the case of $Y' = \mathrm{Spec} B, Y = \mathrm{Spec} A$ both affine. In that case, the conclusion to be proved is that

$$H^i(X \times_A B, \mathcal{F} \otimes_A B) = B \otimes H^i(X, \mathcal{F}),$$

which translates into

$$H^i(K^* \otimes_A B) = B \otimes H^i(K^*),$$

for K^* the Grothendieck complex of $\mathcal{F}$ over Y. But, in the argument for Theorem 2.9, we have seen that cohomology of K^* commutes with base-change (in view of Proposition 2.7). So the corollary is clear.

2.4 The base-change theorem for non-reduced schemes

Theorem 2.11 *Let Y be a noetherian scheme, $f : X \to Y$ a proper morphism, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Suppose $H^p(X_y, \mathcal{F} \otimes k(y)) = 0$ for all $y \in Y$ and for some integer p. Then if $g : Y' \to Y$ is any morphism and $g' : Y' \times_Y X \to X$ the base-change, so that we have a cartesian square*

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{g'} & X \quad , \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & X \end{array}$$

then,

$$g^* R^{p-1} f_*(\mathcal{F}) \simeq R^{p-1} f'_*(g'^*(\mathcal{F})).$$

In particular, it follows that

$$R^{p-1} f_*(\mathcal{F}) \otimes k(y) \to H^{p-1}(X_y, \mathcal{F}_y)$$

is an isomorphism for each $y \in Y$.

Proof In view of the Grothendieck complex, this is immediate from Proposition 2.8.

Corollary 2.12 *Let Y be a noetherian scheme, $f : X \to Y$ a proper morphism, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Suppose $H^1(X_y, \mathcal{F} \otimes k(y)) = 0$ for all $y \in Y$. Then $f_*(\mathcal{F})$ is locally free on Y and there is an isomorphism*

$$f_*(\mathcal{F}) \otimes k(y) \simeq H^0(X_y, \mathcal{F} \otimes k(y)).$$

Proof The only new observation is that $f_*(\mathcal{F})$ is flat (i.e. locally free) on Y. To see this, we reduce to the usual case of $Y = \mathrm{Spec} A$, choose a Grothendieck complex K^* for $\mathcal{F}$, and consider the functor T^0 on A-modules defined by

$$T^0(M) = H^0(K^* \otimes M).$$

Obviously T^0 is left-exact, and by the conclusion about base-change it is right-exact. Thus T^0 is isomorphic to the functor $M \to M \otimes T^0(A)$. Since this is left-exact as well, it follows that $T^0(A) = H^0(K^*)$ is flat. But this is just $\Gamma(X, \mathcal{F})$.

3 Flat sheaves on a projective scheme

Let X be a projective (hence proper) scheme over a noetherian scheme Y. We are interested in the question of when X is flat over Y, and in particular a cohomological answer. Intuitively, flatness ought to mean that the fibers $X_s, s \in S$ "look alike." For instance, their dimensions and degrees (as subschemes of $\mathbf{P}^n_{k(s)}$) should stay constant.

More generally, consider a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_S$. We are interested in when $\mathcal{F}$ is S-flat; intuitively, the fibers $\mathcal{F} \otimes k(s)$ (as sheaves on $\mathbf{P}^n_{k(s)}$) should look similar in some sense. A reasonable invariant of a sheaf on projective space over a field is given by the Hilbert polynomial. Recall that if k is a field and $\mathcal{O}(1)$ is the twisting sheaf on $\mathbf{P}^n_k$, then the **Hilbert polynomial** of a coherent sheaf $\mathcal{G}$ on $\mathbf{P}^n_k$ is given by

$$P_G(m) = \chi(\mathcal{G} \otimes \mathcal{O}(m)).$$

For $m \gg 0$, this is $\dim_k H^0(\mathbf{P}^n_k, \mathcal{G} \otimes \mathcal{O}(m))$, as is well known.

Thus for each $s \in S$, the sheaf $\mathcal{F} \otimes k(s)$ on $\mathbf{P}^n_{k(s)}$ has a Hilbert polynomial that encodes some information about it. We are going to see that if S is integral, then flatness of $\mathcal{F}$ is equivalent to the constancy of these Hilbert polynomials. We will prove this by using the elementary characterization of flatness of a finitely generated module over a reduced noetherian ring by local constancy of the ranks on the fibers. First, however, we need to prove an auxiliary cohomological criterion for flatness.

3.1 The cohomological criterion for flatness

When working with sheaves on a projective scheme, the slogan is "Twist enough and all will be fine."

Theorem 3.1 *Let S be a noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_S$. Then $\mathcal{F}$ is flat over S if and only if there exists m_0 such that $m \geq m_0$ implies that $p_*(\mathcal{F}(m))$ is locally free.*

Here $p : \mathbf{P}^n_S \to S$ is the projection. As usual, $\mathcal{F}(m)$ denotes the twist $\mathcal{F} \otimes \mathcal{O}(m)$.

Proof Everything is local on S, so we assume S affine, say $S = \mathrm{Spec} A$. Suppose that the A-modules $M_m = \Gamma(\mathbf{P}^n_S, \mathcal{F}(m))$ (which are finitely generated by the proper mapping theorem) are locally free for $m \geq m_0$. Then it is easy to see (Lemma 3.2 below) that the coherent sheaf on $\mathbf{P}^n_S = \mathrm{Proj} A[x_0, \dots, x_n]$ associated to the graded $A[x_0, \dots, x_n]$-module $\bigoplus_{m \geq m_0} M_m$ is A-flat. As is well known, this sheaf is precisely $\mathcal{F}$, as dropping the terms of low degree does not change anything. Thus $\mathcal{F}$ is A-flat.

Conversely, suppose $\mathcal{F}$ S-flat. Consider the Cech complex $C^*(\mathcal{F})$ on $\mathbf{P}^n_S$; this consists of flat A-modules as $\mathcal{F}$ is S (i.e. A-) flat. In general, it is not exact at all. However, for $m \gg 0$, the Cech complex $C^*(\mathcal{F}(m))$ is exact except at 0, and it still consists of flat A-modules.

But if we have a finite complex

$$0 \to L_0 \to \dots \to \dots L_N \to 0,$$

which is exact except at L_0 and consists of flat A-modules, then the only nontrivial cohomology (that is, $\ker(L_0 \to L_1)$) is also flat by an easy induction argument. The only nontrivial cohomology of $C^*(\mathcal{F}(m))$ for $m \gg 0$ is $H^0(\mathbf{P}^n_S, \mathcal{F}(m)) = \Gamma(X, p_*(\mathcal{F}(m))$. This is thus A-flat.

Lemma 3.2 *Let A be a ring, R an A-algebra, M a graded R-module which is flat as an A-module. Then the sheaf $\widetilde{M}$ on* $\mathrm{Proj}(R)$ *is flat over* $\mathrm{Spec} A$.

Proof Indeed, the stalk at some point (i.e., homogeneous prime ideal) $\mathfrak{p} \in \mathrm{Proj}(R)$ is given by $M_{(\mathfrak{p})}$, that is, the degree zero component of the localization $M_{\mathfrak{p}}$. But localization is an exact functor, and preserves flatness; thus $M_{\mathfrak{p}}$ is A-flat, so the degree zero component, as a direct factor (as A-module) clearly is.

3.2 Asymptotic base change

We will now see that if one twists enough, push-forwards do commute with *arbitrary* base change on a projective scheme. Again, the slogan about twisting is valid.

Proposition 3.3 *Let S be a noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_S$ and let $g : T \to S$ be a morphism. Draw a cartesian diagram*

$$\begin{array}{ccc} \mathbf{P}^n_T & \xrightarrow{g'} & \mathbf{P}^n_S \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{g} & S \end{array}$$

There exists m_0 such that if $m \geq m_0$, then the natural map

$$g^* p_*(\mathcal{F}(m)) \to p'_*(g^*(\mathcal{F}(m))$$

is an isomorphism.

Of course, it is irrelevant to consider higher direct images here, because then both sides become zero after sufficient twisting!

Proof This is local on T, S, so we can assume that each of them is affine; say $T = \mathrm{Spec} B, S = \mathrm{Spec} A$. Then we know that there is a graded $A[x_0, \ldots, x_n]$-module $M = \bigoplus_m M_m$ such that $\mathcal{F} = \widetilde{M}$ is the sheaf associated to M. Then $g'^*(\mathcal{F})$ is the sheaf on $\mathbf{P}^n_B$ associated to $B \otimes_A M$.

It is known that for m large, we have

$$\Gamma(\mathbf{P}^n_A, \widetilde{M}) = M_m$$

and similarly (as $g'^*(\mathcal{F})$ is the sheaf associated to $\bigoplus B \otimes_A M_m$)

$$\Gamma(\mathbf{P}^n_B, g'^*(\widetilde{M})) = B \otimes_A M_m$$

for $m \gg 0$. REF The result is thus clear.

3.3 The Hilbert polynomial criterion

Suppose $\mathcal{F}$ is S-flat, and by abuse of notation let us use $\mathcal{O}(1)$ to denote the global twisting sheaf on $\mathbf{P}^n_S$ (which restricts to the local ones on each fiber, of course). It follows that $\mathcal{F} \otimes \mathcal{O}(m)$ is S-flat, so we find by Theorem 1.6 that $P_{\mathcal{F} \otimes k(s)}(m) = \chi(\mathcal{F}(m) \otimes k(s)))$ is locally constant in s. On a connected scheme, it follows that the Hilbert polynomials are constant. If $p : X \to S$ is projective and flat, it follows that the Hilbert polynomial of X_s is independent of s; for instance, if the X_s are non-singular curves, then the genus is constant.

Theorem 3.4 *Let S be an integral noetherian scheme. Then a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n_S$ is S-flat if and only if the Hilbert polynomials $P_{\mathcal{F} \otimes k(s)}$ are independent of $s \in S$.*

Proof We have shown one direction. Let us suppose to the contrary that the Hilbert polynomials are independent of s. We may assume $S = \mathrm{Spec} A$, for A a noetherian domain. Consider the A-modules $M_m = \Gamma(X, \mathcal{F}(m))$. We would like to show that these are flat for $m \gg 0$.

It suffices to check this locally, so we may assume that A is local with maximal ideal $\mathfrak{p}$ and generic point (0) (if not, we localize and observe that cohomology commutes with flat base change). If we show that for $m \gg 0$,

$$\dim_{k(\mathfrak{p})} M_m \otimes k(\mathfrak{p}) = \dim_K M_m \otimes K \tag{5}$$

(with K the quotient field), we will be done by Corollary 2.6.

However, we know that for m large, cohomology "asymptotically commutes" with base change. In particular, for $m \gg 0$,

$$\dim_{k(\mathfrak{p})} M_m \otimes k(\mathfrak{p}) = \Gamma(X_{\mathfrak{p}}, \mathcal{F} \otimes k(\mathfrak{p})) = P_{\mathcal{F} \otimes k(\mathfrak{p})}(m)$$

and

$$\dim_K M_m \otimes K = \Gamma(X_\xi, \mathcal{F} \otimes K) = P_{\mathcal{F} \otimes K}(m)$$

for ξ the generic point. Since these are equal, it follows that (5) holds, proving the result.

Lemma 3.5 *For $m \gg 0$ (depending only on $\mathcal{F}$!), and for any $\mathfrak{p} \in \mathrm{Spec} R$, the morphism*

$$\Gamma(X, \mathcal{F}(m))$$

3.4 Flattening stratifications

Let S be a noetherian scheme, $n \in \mathbb{Z}_{\geq 0}$, and $\mathcal{F}$ a coherent sheaf on $\mathbf{P}^n_S$. Even if $\mathcal{F}$ is not flat, it may happen that for some $g : T \to S$, the base-change of $\mathcal{F}$ to $\mathbf{P}^n_T$ is flat. An uninteresting example is given by $T = \operatorname{Spec} k$ for k a field. In fact, if $T \stackrel{g}{\to} S$ makes the pull-back of $\mathcal{F}$ flat, then any composition $T' \to T \stackrel{g}{\to} S$ will work as well, as flatness is preserved under base-change.

Thus, on the category of schemes, we can define a *functor* (in fact, a subfunctor of $\hom(-, S)$) that sends T to the set of morphisms $T \to S$ such that the pull-back of $\mathcal{F}$ to $\mathbf{P}^n_T$ is flat. One might wonder if this is representable.

Theorem 3.6 *There is a S-scheme T such that the pull-back of $\mathcal{F}$ to $\mathbf{P}^n_T$ is representable and, furthermore, any morphism of noetherian schemes $T' \to S$ that pulls $\mathcal{F}$ to a T'-flat sheaf (on $\mathbf{P}^n_T$) factors uniquely through T.*

In fact:

Theorem 3.7 *The universal S-scheme T as above is a* ***stratification*** *of S; that is, it is a disjoint union $\sqcup S_f$ of locally closed, nonintersecting subschemes of S such that $S = \bigcup S_f$. Here f ranges over the polynomials in $\mathbb{Q}[t]$, and set-theoretically S_f consists of all points of s such that $\mathcal{F} \otimes k(s)$ has Hilbert polynomial f.*

This is a fairly difficult theorem, and we shall first consider a simpler case.

(Case where $X = Y$ to be added)

Let us now

4 Various results on line bundles

We shall apply the general cohomological machinery to the study of line bundles on schemes. As we shall see, the semicontinuity theorem and its corollaries will be used primarily in dimension zero.

4.1 The basic tool

Proposition 4.1 *Let Y be a noetherian reduced scheme, $p : X \to Y$ a proper and flat morphism with geometrically integral fibers. Suppose $\mathcal{L}$ is a line bundle on X such that the fibers $\mathcal{L} \otimes k(y)$ are trivial for $y \in Y$ (as line bundles on X_y). Then $\mathcal{L} = p^*(\mathcal{L}_0)$ for some line bundle $\mathcal{L}_0$ on X.*

Proof We take $\mathcal{L}_0 = p_*(\mathcal{L})$. Now $\dim H^0(X_y, \mathcal{L} \otimes k(y))$ is constant (namely, 1) for $y \in X$ as the fibers are trivial and X_y geometrically integral and proper.[4] Thus, we find by Theorem 2.9 that $\mathcal{L}_0$ is locally free of rank one and that the map

$$\mathcal{L}_0 \otimes k(y) \to \Gamma(X_y, \mathcal{L} \otimes k(y))$$

is an isomorphism. There is a natural morphism $p^*(\mathcal{L}_0) \to \mathcal{L}$, and we claim the above isomorphism implies that the maps $p^*(\mathcal{L}_0)(x) \to \mathcal{L}(x)$ for $x \in X$ are surjective, hence isomorphisms. Let us

[4] Indeed, to see this we may base-change and assume $k(y)$ algebraically closed, and X_y integral and proper. Then $\Gamma(X_y, \mathcal{O}_{X_y})$ is a finite-dimensional vector space over $k(y)$ by the proper mapping theorem. In addition, it is an integral domain as X_y is integral. Thus it is equal to $k(y)$.

prove this claim. Choose $x \in X$ lying above $y \in Y$; the triviality of $\mathcal{L} \otimes k(y)$ implies that the morphism

$$\Gamma(X_y, \mathcal{L} \otimes k(y)) \to \mathcal{L} \otimes k(x)$$

is surjective. However, since $\mathcal{L}_0 \otimes k(y) \to \Gamma(X_y, \mathcal{L} \otimes k(y))$ is surjective, we easily find that

$$(p^* \mathcal{L}_0)(x) \to \Gamma(X_y, \mathcal{L} \otimes k(y)) \to \mathcal{L} \otimes k(x)$$

is surjective. It follows that $\mathcal{L} \simeq p^*(\mathcal{L}_0)$ by Nakayama, proving the result.

4.2 Line bundles on projective space

Let Y be a noetherian scheme, and $\mathcal{E}$ a locally free sheaf on Y of finite rank n. We are interested in comparing the Picard groups of Y and the projectivization $\mathbf{P}(\mathcal{E})$.

Let us recall that there are line bundles $\mathcal{O}(m), m \in \mathbb{Z}$ defined on $\mathbf{P}(\mathcal{E})$. (This does not require $\mathcal{E}$ to be anything more than quasi-coherent.) They satisfy $\mathcal{O}(m) \otimes \mathcal{O}(m') \simeq \mathcal{O}(m+m')$. Thus there is a natural morphism $\mathbb{Z} \to \mathrm{Pic}(\mathbf{P}(\mathcal{E}))$. Since the pull-back gives a natural morphism $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbf{P}(\mathcal{E}))$, we get a morphism

$$\phi : \mathbb{Z} \times \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbf{P}(\mathcal{E})).$$

Proposition 4.2 *If X is integral, then ϕ is an injection.*

Proof Intuitively, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbf{P}(\mathcal{E}))$ affects the line bundle on the "horizontal" (parallel to X) direction while the twisting affects it in the vertical direction.

We define a left inverse $\mathrm{Pic}(\mathbf{P}(\mathcal{E})) \to \mathbb{Z}$ as follows. First, we recover the $\mathbb{Z}$ part of it. Pick $x \in X$; then any $\mathcal{L} \in \mathrm{Pic}(\mathbf{P}(\mathcal{E}))$ induces a line bundle on $\mathbf{P}^n_{k(x)}$, so it is isomorphic on this fiber to some $\mathcal{O}(m)$ (on the fiber), in view of the classification of line bundles on projective space. This m will be the integer associated to $\mathcal{L}$. Since pull-back commutes with tensor products, this is a homomorphism, and furthermore it is easy to see that it vanishes on anything of the form $p^*(\mathcal{L}')$ for $\mathcal{L}' \in \mathrm{Pic}(X)$.

We are now reduced to showing that if $\mathcal{L}' \in \mathrm{Pic}(X)$ is such that $p^*(\mathcal{L}')$ is trivial in $\mathrm{Pic}(\mathbf{P}(\mathcal{E}))$, then $\mathcal{L}'$ is itself trivial. Suppose $\{U_\alpha\}$ is an open covering of X on which $\mathcal{L}'$ is trivial. Then $\left\{p^{-1}(U_\alpha)\right\}$ is an open covering of $\mathbf{P}(\mathcal{E})$ on which $p^*(\mathcal{L}')$ is trivial. The transition functions between $p^{-1}(U_\alpha)$ and $p^{-1}(U_\beta)$ must in fact lie in $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ by the Künneth formula. It follows that the cocycle...coboundary...finish this.

More interesting is:

Theorem 4.3 *If X is integral, ϕ is an isomorphism.*

Proof We need to see that ϕ is surjective. In other words, we need to show that if $\mathcal{L}$ is a line bundle on $\mathbf{P}(\mathcal{E})$, then $\mathcal{L}$ is isomorphic to a bundle of the form

$$p^*(\mathcal{L}_0) \otimes \mathcal{O}(m)$$

where $p : \mathbf{P}(\mathcal{E}) \to X$ is the projection, $\mathcal{L}_0$ is a line bundle on X, and $m \in \mathbb{Z}$.

Let the rank of $\mathcal{E}$ be n. For each $x \in X$, there is induced a scheme $\mathbf{P}^n_{k(x)}$, which is the fiber of $\mathbf{P}(\mathcal{E})$ over x. On it is induced a line bundle $\mathcal{L}(x)$ obtained by pulling back $\mathcal{L}$. However, by the classification of line bundles on projective space over a field, we know that $\mathcal{L}(x) = \mathcal{O}(m_x)$ for some $m_x \in \mathbb{Z}$, potentially depending on x.

We claim that m_x is independent of x. For this, we note that as p is flat, $\mathcal{L}$ is flat over X; in particular, the Hilbert polynomial of the fibers $\mathcal{L}(x)$ is independent of x. This implies that the integers m_x are constant, equal to some $m \in \mathbb{Z}$. By twisting $\mathcal{L}$ by $\mathcal{O}(-m)$, we may assume $m = 0$.

In this case, $\mathcal{L}$ is a line bundle whose fibers are all trivial; note also that the projection $p : \mathbf{P}(\mathcal{E}) \to X$ is flat. Then **??** shows that $\mathcal{L}$ lies in the image of $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbf{P}(\mathcal{E}))$, and we are done.

4.3 The seesaw theorem

We shall write *variety* for a finite type, integral scheme over an algebraically closed field k.

Lemma 4.4 *Let X be a proper variety, and $\mathcal{L}$ an invertible sheaf on X. If $\Gamma(X, \mathcal{L}) \neq 0$ and $\Gamma(X, \mathcal{L}^{-1}) \neq 0$, then $\mathcal{L}$ is trivial, and conversely.*

Proof The "conversely" part of the lemma is clear. We prove the other half.

Under the hypotheses, there is a nonzero global section s of $\mathcal{L}$ and a nonzero global section t of $\mathcal{L}^{-1}$. Then s induces a map $\mathcal{O}_X \to \mathcal{L}$ while t induces a map $\mathcal{L} \to \mathcal{O}_X$, both of which are nonzero at the generic points. The composite $\mathcal{O}_X \to \mathcal{L} \to \mathcal{O}_X$ is nonzero at the generic point. However, since X is proper over k, it must be constant. It follows that $\mathcal{O}_X \to \mathcal{L}$ can be zero at no stalk, and in particular, since both are line bundles, it is an isomorphism.

Theorem 4.5 (Seesaw theorem) *Let X be a proper variety over k, and Y a variety. Let $\mathcal{L}$ be a line bundle on $X \times_k Y$, and let $T = \{y \in Y : \mathcal{L} \otimes k(y) \text{ is trivial}\}$. Then T is closed and $\mathcal{L}|_{X \times T}$ is the pull-back of a line bundle on T.*[5]

Proof We only need to check that T is closed. After this, **??** will finish the proof. However, we have seen that T can be described as the set

$$\{y : \Gamma(X_y, \mathcal{L} \otimes k(y)), \Gamma(X_y, \mathcal{L}^\vee \otimes k(y)) \geq 1\},$$

and now the semicontinuity theorem shows that T is closed. (Note that $X \times_k Y \to Y$ is a flat morphism.)

4.4 The theorem of the cube

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[5] Here T is given the reduced subscheme structure.