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Source: *Transactions of the American Mathematical Society*, Vol. 130, No. 2 (Feb., 1968), pp. 208-222

Published by: [American Mathematical Society](#)

Stable URL: <http://www.jstor.org/stable/1994967>

Accessed: 20/01/2011 08:59

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FUNCTORS OF ARTIN RINGS⁽¹⁾

BY

MICHAEL SCHLESSINGER

0. Introduction. In the investigation of functors on the category of preschemes, one is led, by Grothendieck [3], to consider the following situation. Let Λ be a complete noetherian local ring, μ its maximal ideal, and $k = \Lambda/\mu$ the residue field. (In most applications Λ is k itself, or a ring of Witt vectors.) Let \mathcal{C} be the category of Artin local Λ -algebras with residue field k . A covariant functor F from \mathcal{C} to *Sets* is called *pro-representable* if it has the form

$$F(A) \cong \operatorname{Hom}_{\operatorname{local} \Lambda\text{-alg.}}(R, A), \quad A \in \mathcal{C},$$

where R is a *complete* local Λ -algebra such that R/\mathfrak{m}^n is in \mathcal{C} , all n . (\mathfrak{m} is the maximal ideal in R .)

In many cases of interest, F is not pro-representable, but at least one may find an R and a morphism $\operatorname{Hom}(R, \cdot) \rightarrow F$ of functors such that $\operatorname{Hom}(R, A) \rightarrow F(A)$ is surjective for all A in \mathcal{C} . If R is chosen suitably “minimal” then R is called a “hull” of F ; R is then unique up to noncanonical isomorphism. Theorem 2.11, §2, gives a criterion for F to have a hull, and also a simple criterion for pro-representability which avoids the use of Grothendieck’s techniques of nonflat descent [3], in some cases. Grothendieck’s program is carried out by Levelt in [4]. §3 contains a few geometric applications of these results.

To avoid awkward terminology, I have used the word “pro-representable” in a more restrictive sense than Grothendieck [3] has. He considers the category of Λ -algebras of finite length and allows R to be a projective limit of such rings.

The methods of this paper are a simple extension of those used by David Mumford in a proof (unpublished) of the existence of formal moduli for polarized Abelian varieties. I am indebted to Mumford and to John Tate for many valuable suggestions.

1. The category \mathcal{C}_Λ . Let Λ be a complete noetherian local ring, with maximal ideal μ and residue field $k = \Lambda/\mu$. We define $\mathcal{C} = \mathcal{C}_\Lambda$ to be the category of Artinian local Λ -algebras having residue field k . (That is, the “structure morphism” $\Lambda \rightarrow A$ of such a ring A induces a trivial extension of residue fields.) Morphisms in \mathcal{C} are local homomorphisms of Λ -algebras.

Received by the editors March 8, 1966.

⁽¹⁾ The contents of this paper form part of the author’s 1964 Harvard Ph.D. Thesis, which was directed by John Tate. This research was supported in part by a grant from the Air Force Office of Scientific Research.

Let $\hat{\mathcal{C}} = \hat{\mathcal{C}}_\Lambda$ be the category of complete noetherian local Λ -algebras A for which A/\mathfrak{m}^n is in \mathcal{C} , all n . Notice that \mathcal{C} is a full subcategory of $\hat{\mathcal{C}}$.

If $p: A \rightarrow B$, $q: C \rightarrow B$ are morphisms in \mathcal{C} , let $A \times_B C$ denote the ring (in \mathcal{C}) consisting of all pairs (a, c) with $a \in A$, $c \in C$, for which $pa = qc$, with coordinatewise multiplication and addition.

For any A in $\hat{\mathcal{C}}$, we denote by $t_{A/\Lambda}^*$, or just t_A^* , the “Zariski cotangent space” of A over Λ :

$$(1.0) \quad t_A^* = \mathfrak{m}/(\mathfrak{m}^2 + \mu A)$$

where \mathfrak{m} is the maximal ideal of A . A simple calculation shows that the dual vector space, denoted by t_A , may be identified with $\text{Der}_\Lambda(A, k)$, the space of Λ linear derivations of A into k .

LEMMA 1.1. *A morphism $B \rightarrow A$ in $\hat{\mathcal{C}}$ is surjective if and only if the induced map from t_B^* to t_A^* is surjective.*

Proof. First of all, any A in $\hat{\mathcal{C}}$ is generated, as Λ module, by the image of Λ in A and the maximal ideal \mathfrak{m} of A . (For A and Λ have the same residue field k .) Thus the induced map from μ/μ^2 to $\mu A/(\mathfrak{m}^2 \cap \mu A)$ is a surjection. If $B \rightarrow A$ is a morphism in $\hat{\mathcal{C}}$, then denoting the maximal ideal of B by \mathfrak{n} , we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu A/(\mu A \cap \mathfrak{m}^2) & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & t_A^* \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mu B/(\mu B \cap \mathfrak{n}^2) & \longrightarrow & \mathfrak{n}/\mathfrak{n}^2 & \longrightarrow & t_B^* \longrightarrow 0 \end{array}$$

in which the left-hand arrow is a surjection. If the right-hand arrow is also a surjection, then the middle arrow is a surjection, so that the induced map on the graded rings is a surjection. From this it follows that $B \rightarrow A$ is a surjection [1, §2, No. 8, Theorem 1].

Conversely, if $B \rightarrow A$ is a surjection, then the induced map on cotangent spaces is obviously surjective.

Let $p: B \rightarrow A$ be a surjection in \mathcal{C} .

DEFINITION 1.2. p is a *small extension* if kernel p is a nonzero principal ideal (t) such that $\mathfrak{m}t = (0)$, where \mathfrak{m} is the maximal ideal of B .

DEFINITION 1.3. p is *essential* if for any morphism $q: C \rightarrow B$ in \mathcal{C} such that pq is surjective, it follows that q is surjective.

From Lemma 1.1 we obtain easily

LEMMA 1.4. *Let $p: B \rightarrow A$ be a surjection in \mathcal{C} . Then*

- (i) *p is essential if and only if the induced map $p_*: t_B^* \rightarrow t_A^*$ is an isomorphism.*
- (ii) *If p is a small extension, then p is not essential if and only if p has a section $s: A \rightarrow B$, with $ps = 1_A$.*

Proof. (i) If p_* is an isomorphism, then by Lemma 1.1, p is essential. Conversely let $\tilde{t}_1, \dots, \tilde{t}_r$ be a basis of t_A^* , and lift the \tilde{t}_i back to elements t_i in B . Set

$$C = \Lambda[t_1, \dots, t_r] \subseteq B.$$

Then p induces a surjection from C to A , so if p is essential, $C=B$. But then $\dim_k t_B^* \leq r = \dim_k t_A^*$, so $t_B^* \cong t_A^*$.

(ii) If p has a section s , then s is not surjective, so p is not essential. If p is not essential, then the subring C constructed above is a proper subring of B , and hence is isomorphic to A , since $\text{length}(B) = \text{length}(A) + 1$. The isomorphism $C \cong A$ yields the section.

2. Functors on C . We shall consider only *covariant* functors F , from C to *Sets*, such that $F(k)$ contains just one element. By a *couple* for F we mean a pair (A, ξ) where $A \in C$ and $\xi \in F(A)$. A *morphism of couples* $u: (A, \xi) \rightarrow (A', \xi')$ is a morphism $u: A \rightarrow A'$ in C such that $F(u)(\xi) = \xi'$. If we extend F to \hat{C} by the formula $\hat{F}(A) = \text{proj} \lim F(A/\mathfrak{m}^n)$ we may speak analogously of *pro-couples* and morphisms of pro-couples.

For any ring R in \hat{C} , we set $h_R(A) = \text{Hom}(R, A)$ to define a functor h_R on C . Now if F is any functor on C , and R is in \hat{C} , we have a canonical isomorphism

$$\hat{F}(R) \xrightarrow{\sim} \text{Hom}(h_R, F).$$

Namely, let $\xi = \text{proj} \lim \xi_n$ be in $\hat{F}(R)$. Then each $u: R \rightarrow A$ factors through $u_n: R/\mathfrak{m}^n \rightarrow A$ for some n , and we assign to $u \in h_R(A)$ the element $F(u_n)(\xi_n)$ of $F(A)$. This sets up the isomorphism. We therefore say that a pro-couple (R, ξ) for F *pro-represents* F if the morphism $h_R \rightarrow F$ induced by ξ is an isomorphism.

(2.1) *Relation to global functors.* Let G be a *contravariant* functor on the category of preschemes over $\text{Spec } \Lambda$, and pick a fixed $e \in G(\text{Spec } k)$. For A in C , let $F(A) \subseteq G(\text{Spec } A)$ be the set of those $\xi \in G(\text{Spec } A)$ such that $G(i)(\xi) = e$ where i is the inclusion of $\text{Spec } k$ in $\text{Spec } A$. If G is represented by a prescheme X , then e determines a k -rational point $x \in X$, and it is then clear that $F(A)$ is isomorphic to $\text{Hom}_\Lambda(\mathfrak{D}_{X, x}, A)$. Thus the completion of $\mathfrak{D}_{X, x}$ pro-represents F .

Unfortunately, many interesting functors, for example some “formal moduli” functors (§3.7), are not pro-representable. However, one can still look for a “universal object” in some sense, for example in the sense of Definition 2.7 below.

DEFINITION 2.2. A morphism $F \rightarrow G$ of functors is *smooth* if for any *surjection* $B \rightarrow A$ in C , the morphism

$$(*) \quad F(B) \rightarrow F(A) \times_{G(A)} G(B)$$

is surjective.

Part (i) of the *sorités* below will perhaps motivate this definition.

REMARKS. (2.3) It is enough to check surjectivity in $(*)$ for small extensions $B \rightarrow A$.

(2.4) If $F \rightarrow G$ is smooth, then $\hat{F} \rightarrow \hat{G}$ is *surjective*, in the sense that $\hat{F}(A) \rightarrow \hat{G}(A)$ is surjective for all A in $\hat{\mathcal{C}}$ (consider the successive quotients A/\mathfrak{m}^n , $n=1, 2, \dots$).

PROPOSITION 2.5. (i) *Let $R \rightarrow S$ be a morphism in $\hat{\mathcal{C}}$. Then $h_S \rightarrow h_R$ is smooth if and only if S is a power series ring over R .*

(ii) *If $F \rightarrow G$ and $G \rightarrow H$ are smooth morphisms of functors, then the composition $F \rightarrow H$ is smooth.*

(iii) *If $u: F \rightarrow G$ and $v: G \rightarrow H$ are morphisms of functors such that u is surjective and vu is smooth, then v is smooth.*

(iv) *If $F \rightarrow G$ and $H \rightarrow G$ are morphisms of functors such that $F \rightarrow G$ is smooth, then $F \times_G H \rightarrow H$ is smooth.*

Proof. (i) This is more or less well known (see [3, Theorem 3.1]), but we give a proof for the sake of completeness. Suppose $h_S \rightarrow h_R$ is smooth. Let \mathfrak{r} (resp. \mathfrak{s}) be the maximal ideal in R (resp. S), and pick x_1, \dots, x_n in S which induce a basis of $t_{S/R}^* = \mathfrak{s}/(\mathfrak{s}^2 + \mathfrak{r}S)$. If we set $T = R[[X_1, \dots, X_n]]$ and denote the maximal ideal of T by \mathfrak{t} , we get a morphism $u_1: S \rightarrow T/(\mathfrak{t}^2 + \mathfrak{r}T)$ of local R algebras, obtained by mapping x_i on the residue class of X_i . By smoothness u_1 lifts to $u_2: S \rightarrow T/\mathfrak{t}^2$, thence to $u_3: S \rightarrow T/\mathfrak{t}^3, \dots$ etc. Thus we get a $u: S \rightarrow T$ which induces an isomorphism of $t_{S/R}^*$ with $t_{T/R}^*$ (by choice of u_1) so that u is a surjection (1.1). Furthermore, if we choose $y_i \in S$ such that $u y_i = X_i$, we can set $v X_i = y_i$ and produce a local morphism $v: T \rightarrow S$ of R algebras such that $uv = 1_T$; in particular v is an injection. Clearly v induces a bijection on the cotangent spaces, so v is also a surjection (1.1). Hence v is an isomorphism of $T = R[[X_1, \dots, X_n]]$ with S .

Conversely, if S is a power series ring over R , then it is obvious that $h_S \rightarrow h_R$ is smooth.

The proofs of (ii), (iii), (iv) are completely formal and are left to the reader.

(2.6) **NOTATION.** Let $k[\varepsilon]$, where $\varepsilon^2=0$, denote the ring of dual numbers over k . For any functor F , the set $F(k[\varepsilon])$ is called the *tangent space* to F , and is denoted by t_F . It is easy to see that if $F = h_R$, then there is a canonical isomorphism $t_F \cong t_R$:

$$t_R \cong \text{Hom}_\Lambda(R, k[\varepsilon]).$$

Usually t_F will have an intrinsic vector space structure (Lemma 2.10).

DEFINITION 2.7. A pro-couple (R, ξ) for a functor F is called a *pro-representable hull* of F , or just a *hull* of F , if the induced map $h_R \rightarrow F$ is *smooth* (2.2), and if in addition the induced map $t_R \rightarrow t_F$ of tangent spaces is a bijection.

(2.8) Notice that if (R, ξ) pro-represents F then (R, ξ) is a hull of F . In this case (R, ξ) is unique up to canonical isomorphism. In general we have only noncanonical isomorphism:

PROPOSITION 2.9. *Let (R, ξ) and (R', ξ') be hulls of F . Then there exists an isomorphism $u: R \rightarrow R'$ such that $F(u)(\xi) = \xi'$.*

Proof. By (2.4) we have morphisms $u: (R, \xi) \rightarrow (R', \xi')$ and $u': (R', \xi') \rightarrow (R, \xi)$, both inducing an isomorphism on tangent spaces, by the definition of hull. Thus

$u'u$ say, induces an isomorphism on t_R^* , so that $u'u$ is a surjective endomorphism of R , by Lemma 1.1. But an easy argument, which we leave to the reader, shows that a surjective endomorphism of any noetherian ring is an isomorphism. Thus $u'u$ and uu' are isomorphisms and we are done.

REMARK 2.10. Let (R, ξ) be a hull of F . Then R is a power series ring over Λ if and only if F transforms surjections $B \rightarrow A$ in \mathcal{C} into surjections $F(B) \rightarrow F(A)$. In fact the stated condition on F is equivalent to the *smoothness* of the natural morphism $F \rightarrow h_\Lambda$. By applying (2.6), (ii) and (iii) to the diagram

$$\begin{array}{ccc} h_R & \xrightarrow{\quad} & h_\Lambda \\ & \searrow & \nearrow \\ & F & \end{array}$$

we conclude that $h_R \rightarrow h_\Lambda$ is smooth if and only if $F \rightarrow h_\Lambda$ is. Now use 2.5 (i).

LEMMA 2.10. Suppose F is a functor such that

$$F(k[V] \times_k k[W]) \xrightarrow{\sim} F(k[V]) \times F(k[W])$$

for vector spaces V and W over k , where $k[V]$ denotes the ring $k \oplus V$ of \mathcal{C} in which V is a square zero ideal. Then $F(k[V])$, and in particular $t_F = F(k[\varepsilon])$, has a canonical vector space structure, such that $F(k[V]) \cong t_F \otimes V$.

Proof. $k[V]$ is in fact a “vector space object” in the category $\hat{\mathcal{C}}$ (in which k is the final object), for we have a canonical isomorphism

$$\mathrm{Hom}(A, k[V]) \cong \mathrm{Der}_\Lambda(A, V), \quad A \in \hat{\mathcal{C}}.$$

The addition map $k[V] \times_k k[V] \rightarrow k[V]$ is given by $(x, 0) \mapsto x$, $(0, x) \mapsto x$ ($x \in V$), and scalar multiplication by $a \in k$ is given by the endomorphism $x \mapsto ax$ ($x \in V$) of $k[V]$. Thus if F commutes with the necessary products, $F(k[V])$ gets a vector space structure. Finally, we identify V with $\mathrm{Hom}(k[\varepsilon], k[V])$ to get a map

$$t_F \otimes V \rightarrow F(k[V])$$

which is an isomorphism since $k[V]$ is isomorphic to the product of $r = \dim_k V$ copies of $k[\varepsilon]$.

THEOREM 2.11. Let F be a functor from \mathcal{C} to Sets such that $F(k) = (e)$ (= one point). Let $A' \rightarrow A$ and $A'' \rightarrow A$ be morphisms in \mathcal{C} , and consider the map

$$(2.12) \quad F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'').$$

Then

- (1) F has a hull if and only if F has properties (H_1) , (H_2) , (H_3) below:
- (H_1) (2.12) is a surjection whenever $A'' \rightarrow A$ is a small extension (1.2).
- (H_2) (2.12) is a bijection when $A = k$, $A'' = k[\varepsilon]$.
- (H_3) $\dim_k(t_F) < \infty$.

(2) F is pro-representable if and only if F has the additional property (H_4) :

$$(H_4) \quad F(A' \times_A A') \xrightarrow{\sim} F(A') \times_{F(A)} F(A')$$

for any small extension $A' \rightarrow A$.

Notice that if F is isomorphic to some h_R , then (2.12) is an isomorphism for any morphisms $A' \rightarrow A$, $A'' \rightarrow A$; that is, the four conditions are trivially necessary for pro-representability.

REMARKS. (2.13) (H_2) implies that t_F is a vector space by Lemma 2.10. In fact, by induction on $\dim_k W$ we conclude from (H_2) that (2.12) is an isomorphism when $A=k$, $A''=k[W]$; in particular the hypotheses of 2.10 are satisfied.

(2.14) By induction on length A'' -length A it follows from (H_1) that (2.12) is a surjection for any surjection $A'' \rightarrow A$.

(2.15) Condition (H_4) may be usefully viewed as follows. For each A in \mathcal{C} , and each ideal I in A such that $\mathbf{m}_A \cdot I = (0)$, we have an isomorphism

$$(2.16) \quad A \times_{A/I} A \xrightarrow{\sim} A \times_k k[I],$$

induced by the map $(x, y) \mapsto (x, x_0 + y - x)$, where x and y are in A and x_0 is the k residue of x . Now, given a small extension $p: A' \rightarrow A$ with kernel I , we get by (H_2) and (2.16) a map

$$(2.17) \quad F(A') \times (t_F \otimes I) \rightarrow F(A') \times_{F(A)} F(A')$$

which is easily seen to determine, for each $\eta \in F(A)$, a group action of $t_F \otimes I$ on the subset $F(p)^{-1}(\eta)$ of $F(A')$ (provided that subset is not empty). (H_1) implies that this action is "transitive," while (H_4) is precisely the condition that this action makes $F(p)^{-1}(\eta)$ a (formally) principal homogeneous space under $t_F \otimes I$. Thus, in the presence of conditions (H_1) , (H_2) , (H_3) , it is the existence of "fixed points" of $t_F \otimes I$ acting on $F(p)^{-1}(\eta)$ which obstruct the pro-representability of F . In many applications, where the elements of $F(A)$ are isomorphism classes of geometric objects, the existence of such a fixed point $\eta' \in F(p)^{-1}(\eta)$ is equivalent to the existence of an automorphism of an object y in the class of η which cannot be extended to an automorphism of any (or some) object y' in the class of η' .

Proof of 2.11. (1) Suppose F satisfies conditions (H_1) , (H_2) , (H_3) . Let t_1, \dots, t_r be a dual basis of t_F , put $S = \Lambda[[T_1, \dots, T_r]]$, and let \mathfrak{n} be the maximal ideal of S . R will be constructed as the projective limit of successive quotients of S . To begin, let $R_2 = S/(\mathfrak{n}^2 + \mu S) \cong k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]$ (r times). By (H_2) there exists $\xi_2 \in F(R_2)$ which induces a bijection between $t_{R_2} (\cong \text{Hom}(R_2, k[\varepsilon]))$ and t_F . Suppose we have found (R_q, ξ_q) , where $R_q = S/J_q$. We seek an ideal J_{q+1} in S , minimal among those ideals J in S satisfying the conditions (a) $\mathfrak{n}J_q \subseteq J \subseteq J_q$, (b) ξ_q lifts to S/J . Since the set \mathcal{S} of such ideals corresponds to a certain collection of vector subspaces of $J_q/(\mathfrak{n}J_q)$, it suffices to show that \mathcal{S} is stable under pairwise intersection. But if

J and K are in \mathcal{S} , we may enlarge J , say, so that $J+K=J_q$, without changing the intersection $J \cap K$. Then

$$S/J \times_{S/J_q} S/K \cong S/(J \cap K)$$

so that by (H_1) (see (2.14)) we may conclude that $J \cap K$ is in \mathcal{S} . Let J_{q+1} be the intersection of the members of \mathcal{S} , put $R_{q+1}=S/J_{q+1}$, and pick any $\xi_{q+1} \in F(R_{q+1})$ which projects onto $\xi_q \in F(R_q)$.

Now let J be the intersection of all the J_q 's ($q=2, 3, \dots$) and let $R=S/J$. Since $\pi^a \subseteq J_q$, the J_q/J form a base for the topology in R , so that $R=\text{proj } \text{Lim } S/J_q$, and it is legitimate to set $\xi=\text{proj } \text{Lim } \xi_q \in \hat{F}(R)$. Notice that $t_F \cong t_R$, by choice of R_2 .

We claim now that $h_R \rightarrow F$ is smooth. Let $p: (A', \eta') \rightarrow (A, \eta)$ be a morphism of couples, where p is a small extension, $A=A'/I$, and let $u: (R, \xi) \rightarrow (A, \eta)$ be a given morphism. We have to lift u to a morphism $(R, \xi) \rightarrow (A', \eta')$. For this it suffices to find a $u': R \rightarrow A'$ such that $pu'=u$. In fact, we have a transitive action of $t_F \otimes I$ on $F(p)^{-1}(\eta)$ (resp. $h_R(p)^{-1}(\eta)$) by (2.15); thus, given such a u' , there exists $\sigma \in t_F \otimes I$ such that $[F(u')(\xi)]^\sigma = \eta'$, so that $v'=(u')^\sigma$ will satisfy $F(v')(\xi)=\eta'$, $pv'=u$.

Now u factors as $(R, \xi) \rightarrow (R_q, \xi_q) \rightarrow (A, \eta)$ for some q . Thus it suffices to complete the diagram

$$\begin{array}{ccc} R_{q+1} & \dashrightarrow & A' \\ \downarrow & & \downarrow p \\ R_q & \longrightarrow & A \end{array}$$

or equivalently, the diagram

$$\begin{array}{ccc} \Lambda[[T_1, \dots, T_r]] = S & \xrightarrow{w} & R_q \times_A A' \\ \downarrow & \nearrow v & \downarrow pr_1 \\ R_{q+1} & \longrightarrow & R_q \end{array}$$

where w has been chosen so as to make the square commute. If the small extension pr_1 has a section, then v obviously exists. Otherwise, by 1.4(ii), pr_1 is essential, so w is a surjection. By (H_1) , applied to the projections of $R_q \times_A A'$ on its factors, $\xi_q \in F(R_q)$ lifts back to $R_q \times_A A'$, so $\ker w \supseteq J_{q+1}$, by choice of J_{q+1} . Thus w factors through $S/J_{q+1}=R_{q+1}$, and v exists. This completes the proof that (R, ξ) is a hull of F .

Conversely, suppose that a pro-couple (R, ξ) is a hull of F . To verify (H_1) , let $p': (A', \eta') \rightarrow (A, \eta)$ and $p'': (A'', \eta'') \rightarrow (A, \eta)$ be morphisms of couples, where p''

is a surjection. Since $h_R \rightarrow F$ is surjective, there exists a $u': (R, \xi) \rightarrow (A', \eta')$, and hence by smoothness applied to p'' , there exists $u'': (R, \xi) \rightarrow (A'', \eta'')$ rendering the following diagram commutative:

$$\begin{array}{ccccc}
 & & (A' \times_A A'', \zeta) & & \\
 & \nearrow u' \times u'' & \downarrow & \searrow & \\
 (R, \xi) & \xrightarrow{\quad} & (A'', \eta'') & & \\
 & \searrow u' & \downarrow p'' & & \\
 & (A', \eta') & \searrow p' & & (A, \eta)
 \end{array}$$

Therefore $\zeta = F(u' \times u'')(\xi)$ projects onto η' and η'' , so that (H_1) is satisfied.

Now suppose $(A, \eta) = (k, e)$, and $A'' = k[\varepsilon]$. If ζ_1 and ζ_2 in $F(A' \times_k k[\varepsilon])$ have the same projections η' and η'' on the factors, then choosing u' as above we get morphisms

$$u' \times u_i: (R, \xi) \rightarrow (A' \times_k k[\varepsilon], \zeta_i), \quad i = 1, 2,$$

by smoothness applied to the projection of $A' \times_k k[\varepsilon]$ on A' . Since $t_F \cong t_R$ we have $u_1 = u_2$, so that $\zeta_1 = \zeta_2$, which proves (H_2) . The isomorphism $t_R \cong t_F$ also proves (H_3) .

(2) Suppose now that F satisfies conditions (H_1) through (H_4) . By part (1) we know that F has a hull (R, ξ) . We shall prove that $h_R(A) \xrightarrow{\sim} F(A)$ by induction on length A . Consider a small extension $p: A' \rightarrow A = A'/I$, and assume that $h_R(A) \xrightarrow{\sim} F(A)$. For each $\eta \in F(A)$, $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are both formally principal homogeneous spaces under $t_F \otimes I$ (2.15); since $h_R(A')$ maps *onto* $F(A')$, we have $h_R(A') \xrightarrow{\sim} F(A')$, which proves the induction step.

The necessity of the four conditions has already been noted.

3. Examples.

(3.1) *The Picard functor.* If X is a prescheme, we define $\text{Pic}(X) = H^1(X, \mathfrak{D}_X^*)$, the group of isomorphism classes of invertible (i.e., locally free of rank one) sheaves on X . Recall that the group of automorphisms of an invertible sheaf is canonically isomorphic to $H^0(X, \mathfrak{D}_X^*)$.

Now suppose X is a prescheme over $\text{Spec } \Lambda$. We let X_A abbreviate $X \times_{\text{Spec } \Lambda} \text{Spec } A$ for A in \mathcal{C} , and set $X_0 = X_k$. If η (resp. L) is an element of $\text{Pic}(X_A)$ (resp. an invertible sheaf on X_A) and $A \rightarrow B$ is a morphism in \mathcal{C} , let $\eta \otimes_A B$ (resp. $L \otimes_A B$) denote the induced element of $\text{Pic}(X_B)$ (resp. induced invertible sheaf on X_B). Let ξ_0 be an element of $\text{Pic}(X_0)$ *fixed once and for all* in this discussion, and let

$P(A)$ be the subset of $\text{Pic}(X_A)$ consisting of those η such that $\eta \otimes_A k = \xi_0$. We claim that P is pro-representable under suitable conditions, namely:

PROPOSITION 3.2. *Assume*

- (i) X is flat over Λ ,
- (ii) $A \xrightarrow{\sim} H^0(X_A, \mathfrak{D}_{X_A})$ for each $A \in C$,
- (iii) $\dim_k H^1(X_0, \mathfrak{D}_{X_0}) < \infty$.

Then P is pro-representable by a pro-couple (R, ξ) ; furthermore $t_R \cong H^1(X_0, \mathfrak{D}_{X_0})$.

Notice that condition (ii) is equivalent to the condition $k \xrightarrow{\sim} H^0(X_0, \mathfrak{D}_{X_0})$, in view of (i). In fact, by flatness, the functor $M \mapsto T(M) = H^0(X, \mathfrak{D}_X \otimes M)$ of Λ modules is left exact. A standard five lemma type of argument then shows that the natural map $M \rightarrow T(M)$ is an isomorphism for all M of finite length.

For the proof of 3.2 we need two simple lemmas on flatness.

LEMMA 3.3. *Let A be a ring, J a nilpotent ideal in A , and $u: M \rightarrow N$ a homomorphism of A modules, with N flat over A . If $\bar{u}: M/JM \rightarrow N/JN$ is an isomorphism, then u is an isomorphism.*

Proof. Let $K = \text{coker } u$ and tensor the exact sequence

$$M \rightarrow N \rightarrow K \rightarrow 0$$

with A/J . Then we find $K/JK = 0$, which implies $K = 0$, since J is nilpotent. Thus, if $K' = \ker u$, we get an exact sequence

$$0 \rightarrow K'/JK' \rightarrow M/JM \rightarrow N/JN \rightarrow 0$$

by the flatness of N . Hence $K' = 0$, so that u is an isomorphism.

LEMMA 3.4. *Consider a commutative diagram*

$$\begin{array}{ccccc}
 N & \xrightarrow{p''} & M'' & & \\
 \downarrow p' & & \downarrow u'' & & \\
 M' & \xrightarrow{u'} & M & & \\
 \downarrow & & \downarrow & & \\
 B & \xrightarrow{\quad} & A'' & & \\
 \downarrow & & \downarrow & & \\
 A' & \xrightarrow{\quad} & A & &
 \end{array}$$

of compatible ring and module homomorphisms, where $B = A' \times_A A''$, $N = M' \times_M M''$ and M' (resp. M'') is a flat A' (resp. A'') module. Suppose

- (i) $A''/J \xrightarrow{\sim} A$, where J is a nilpotent ideal in A'' ,
- (ii) u' (resp. u'') induces $M' \otimes_{A'} A \xrightarrow{\sim} M$ (resp. $M'' \otimes_{A''} A \xrightarrow{\sim} M$).

Then N is flat over B and p' (resp. p'') induces $N \otimes_B A' \xrightarrow{\sim} M'$ (resp. $N \otimes_B A'' \xrightarrow{\sim} M''$).

Proof. We shall consider only the case where M' is actually a free A' module. (This case actually suffices for our purposes, since a simple application of Lemma 3.3 shows that a flat module over an Artin local ring is free.) Choose a basis $(x'_i)_{i \in I}$ for M' . Then by (ii) we find that M is the free module on generators $u'(x'_i)$. Choosing $x''_i \in M''$ such that $u''(x''_i) = u'(x'_i)$, we get a map $\sum A'' x''_i \rightarrow M''$ of A'' modules, whose reduction modulo the ideal J is an isomorphism. Therefore M'' is free on generators x''_i (Lemma 3.3) and it follows easily that $N = M' \times_M M''$ is free on generators $x'_i \times x''_i$, and that the projections on the factors induce isomorphisms

$$N \otimes_B A' \xrightarrow{\sim} M', \quad N \otimes_B A'' \xrightarrow{\sim} M''$$

as desired. (A similar argument for the case of general M' is given in [4, §1, Proposition 2].)

COROLLARY 3.6. *With the notations as above, let L be a B module which may be inserted in a commutative diagram*

$$\begin{array}{ccc} & L & \\ q' \swarrow & & \searrow q'' \\ M' & & M'' \\ u' \searrow & & \swarrow u'' \\ & M & \end{array}$$

where q' induces $L \otimes_B A' \xrightarrow{\sim} M'$. Then the canonical morphism $q' \times q'': L \rightarrow N = M' \times_M M''$ is an isomorphism.

Proof. Apply Lemma 3.3 to the morphism $u = q' \times q''$.

REMARK. Lemma 3.4 is false, in general, if neither $A'' \rightarrow A$ nor $A' \rightarrow A$ is assumed surjective. For example, let A' be a sublocal ring of the local ring A , and map $A_1 = A''$ into A by inclusion. Let a be a unit of A such that the ideal $(aA') \cap A'$ of A' is not flat (=free) over A' . (In C_Δ one could take $A = k[t]/(t^3)$, $A' = k[t^2]$, $a = 1 + t$.) Let $M' = M'' = A'$, $M = A$, u' = inclusion, u'' = multiplication by a^{-1} . Then $B \cong A'$, while $N \cong (aA') \cap A'$ is not flat over B .

Proof of Proposition 3.2. Let $u': (A', \eta') \rightarrow (A, \eta)$, $u'': (A'', \eta'') \rightarrow (A, \eta)$ be morphisms of couples, where u'' is a surjection. Let L', L, L'' be corresponding invertible sheaves on $X' = X_{A'}$, $Y = X_A$, and $X'' = X_{A''}$. Then we have morphisms $p': L' \rightarrow L$, $p'': L'' \rightarrow L$ (of sheaves on the topological space $|X_0|$, compatible with $\mathfrak{D}_{X'} \rightarrow \mathfrak{D}_Y$, $\mathfrak{D}_{X''} \rightarrow \mathfrak{D}_Y$) which induce isomorphisms $L' \otimes_{A'} A \xrightarrow{\sim} L$, $L'' \otimes_{A''} A \xrightarrow{\sim} L$.

Let $B = A' \times_A A''$, and let $Z = X_B$. Then we have a commutative diagram

$$\begin{array}{ccc}
 & \mathfrak{D}_Z & \\
 \swarrow & & \searrow \\
 \mathfrak{D}_{X'} & & \mathfrak{D}_{X''} \\
 \searrow & & \swarrow \\
 & \mathfrak{D}_Y &
 \end{array}$$

of sheaves on $|X_0|$; thus by Corollary 3.6 there is a canonical isomorphism $\mathfrak{D}_Z \xrightarrow{\sim} \mathfrak{D}_{X'} \times_{\mathfrak{D}_Y} \mathfrak{D}_{X''}$, where $\mathfrak{D}_{X'} \times_{\mathfrak{D}_Y} \mathfrak{D}_{X''}$ is the sheaf of B -algebras whose sections over an open U in $|X_0|$ are given by

$$\mathfrak{D}_{X'} \times_{\mathfrak{D}_Y} \mathfrak{D}_{X''}(U) = \mathfrak{D}_{X'}(U) \times_{\mathfrak{D}_Y(U)} \mathfrak{D}_{X''}(U).$$

Hence $N = L' \times_L L''$ is a sheaf on Z , obviously invertible, and the projections of N on L' and L'' induce isomorphisms $N \otimes_B A' \xrightarrow{\sim} L'$, $N \otimes_B A'' \xrightarrow{\sim} L''$ by Lemma 3.4.

If M is another invertible sheaf on Z for which there exist isomorphisms

$$M \otimes A' \xrightarrow{\sim} L', \quad M \otimes A'' \xrightarrow{\sim} L'',$$

we have morphisms $q': M \rightarrow L'$, $q'': M \rightarrow L''$ which induce these isomorphisms, and thus a commutative diagram

$$\begin{array}{ccccc}
 & & M & & \\
 & \swarrow q' & & \searrow q'' & \\
 L' & & & & L'' \\
 & \searrow u' & & \swarrow u'' & \\
 & L & \xrightarrow{\theta} & L &
 \end{array}$$

Here θ is the automorphism of L given by the composition

$$L \xrightarrow{\sim} L' \otimes_{A'} A \xrightarrow{\sim} M \otimes_B A \xrightarrow{\sim} L'' \otimes_{A''} A \xrightarrow{\sim} L.$$

By hypothesis (ii) of 3.2, θ is multiplication by some unit $a \in A$. Lifting a back to a'' in A'' , we can change q'' to $a''q''$; thus we may assume that $u'q' = u''q''$. It follows from Corollary 3.6 that $M \xrightarrow{\sim} N$. We have therefore proved that

$$P(A' \times_A A'') \xrightarrow{\sim} P(A') \times_{P(A)} P(A'')$$

for any surjection $A'' \rightarrow A$ in \mathcal{C} .

Finally, letting $Y = X_{k[\varepsilon]}$, we have $\mathfrak{D}_Y = \mathfrak{D}_{X_0} \oplus \varepsilon \mathfrak{D}_{X_0}$, so there is a split exact sequence

$$0 \longrightarrow \mathfrak{D}_{X_0} \xrightarrow{\exp} \mathfrak{D}_Y^* \longrightarrow \mathfrak{D}_{X_0}^* \longrightarrow 1$$

where \exp maps the (additive) sheaf \mathfrak{D}_{X_0} into \mathfrak{D}_Y^* by $\exp(f) = 1 + \varepsilon f$. Hence

$$F(k[\varepsilon]) \cong \ker \{H^1(X_0, \mathfrak{D}_Y^*) \rightarrow H^1(X_0, \mathfrak{D}_{X_0}^*)\} \cong H^1(X_0, \mathfrak{D}_{X_0})$$

which has finite dimension, by assumption. This completes the proof of Proposition 3.2.

(3.7) *Formal moduli.* Let X be a fixed prescheme over k , and $A \in \mathcal{C}$. By an (infinitesimal) deformation of X/k to A we mean a *product* diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ \text{Spec } k & \rightarrow & \text{Spec } A \end{array} \quad X \xrightarrow{\sim} Y \times_{\text{Spec } A} \text{Spec } k$$

where Y is flat over $\text{Spec } A$ and i is (necessarily) a closed immersion. We will suppress the i and refer to Y as a deformation, if no confusion is possible. If Y' is another deformation to A then Y and Y' are *isomorphic* if there exists a morphism $f: Y \rightarrow Y'$ over A which induces the identity on the closed fibre X . (f must then be an isomorphism of preschemes, by Lemma 3.3.) Given the deformation Y over A and a morphism $A \rightarrow B$ in \mathcal{C} , one has evidently an induced deformation $Y \otimes_A B$ over B ; and if Z is a deformation over B , one can define the notion of morphism $Z \rightarrow Y$ of deformations. (Notice that there is a one-to-one correspondence between such morphisms and the isomorphisms $Z \xrightarrow{\sim} Y \otimes_A B$ which they induce.)

Define the deformation functor $\mathbf{D} = \mathbf{D}_{X/k}$ by setting

$$\mathbf{D}(A) = \text{Set of isomorphism classes of deformations of } X/k \text{ to } A.$$

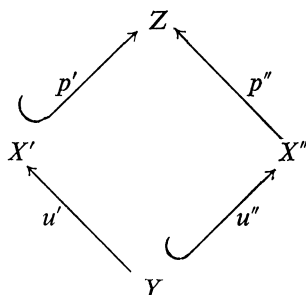
We shall find that, in general, \mathbf{D} is not pro-representable, but that with rather weak finiteness restrictions on X , \mathbf{D} will have a hull.

Suppose that $(A', \eta') \rightarrow (A, \eta)$ and $(A'', \eta'') \rightarrow (A, \eta)$ are morphisms of couples, where $A'' \rightarrow A$ is a surjection. Letting X', Y, X'' denote deformations in the class of η', η, η'' respectively, we have a diagram

$$\begin{array}{ccc} X' & & X'' \\ & \swarrow u' \quad \searrow u'' & \\ & Y & \end{array}$$

of deformations. Therefore we can construct, as in the proof of 3.2 the sheaf $\mathfrak{D}_{X'} \times_{\mathfrak{D}_Y} \mathfrak{D}_{X''}$ of $A' \times_A A''$ algebras, and $(|X|, \mathfrak{D}_{X'} \times_{\mathfrak{D}_Y} \mathfrak{D}_{X''})$ defines a prescheme Z flat over $A' \times_A A''$. (The fact that Z is actually a prescheme consists of straightforward checking; in fact it is the *sum* of X' and X'' in the category of preschemes

under Y , homeomorphic to Y . Z is flat over $A' \times_A A''$ by Lemma 3.4.) Furthermore the closed immersions $X \rightarrow Y \rightarrow Z$ give Z a structure of deformation of X/k to $A' \times_A A''$ such that

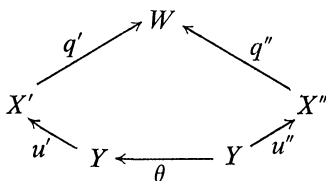


is a commutative diagram of deformations. In particular this shows that

$$D(A' \times_A A') \rightarrow D(A') \times_{D(A)} D(A')$$

is surjective, for every surjection $A'' \rightarrow A$. That is, condition (H_1) of 2.11 is satisfied.

Suppose now that W is another deformation over B , inducing the deformations



X' and X'' . Then there is a commutative diagram of deformations, where θ is the composition

$$Y \xrightarrow{\sim} X' \otimes_{A'} A \xrightarrow{\sim} W \otimes_B A \longrightarrow X'' \otimes_{A''} A \xrightarrow{\sim} Y.$$

If θ can be lifted to an automorphism θ' of X' , such that $\theta'u' = u'\theta$, then we can replace q' with $q'\theta'$; then we would have an isomorphism $W \xrightarrow{\sim} Z$ by Corollary 3.6. Now if $A=k$ (so that $Y=X$, $\theta=\text{id}$) θ' certainly exists, so condition (H_2) is satisfied.

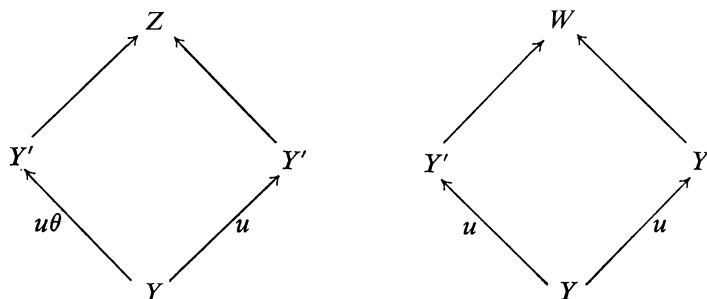
To consider the condition (H_4) , let $p: (A', \eta') \rightarrow (A, \eta)$ be a morphism of couples, where p is a small extension. For each morphism $B \rightarrow A$, let $D_\eta(B)$ denote as usual the set of $\zeta \in D(B)$ such that $\zeta \otimes_B A = \eta$. Pick a deformation Y' in the class of η' ; then

LEMMA 3.8. *The following are equivalent*

- (i) $D_\eta(A' \times_A A') \xrightarrow{\sim} D_\eta(A') \times D_\eta(A')$,
- (ii) Every automorphism of the deformation $Y = Y' \otimes_{A'} A$ is induced by an automorphism of the deformation Y' .

Proof. (i) \Rightarrow (ii). Let $u: Y \rightarrow Y'$ be the induced morphism of deformations.

If θ is an automorphism of Y , then one can construct deformations Z, W over $A' \times_A A'$ to yield "sum diagrams"



of deformations. Since Z and W have isomorphic projections on both factors, there is an isomorphism $\rho: Z \xrightarrow{\sim} W$. ρ induces automorphisms θ_1 and θ_2 of Y' , and an automorphism ϕ of Y such that

$$\theta_1 u \theta = u \phi, \quad \theta_2 u = u \phi.$$

Therefore $u \theta = \theta_1^{-1} \theta_2 u$ and $\theta_1^{-1} \theta_2$ induces θ .

(ii) \Rightarrow (i). In a similar manner, it follows from (ii) that $t_F \otimes I$ ($I = \ker p$) acts freely on η' (i.e., $(\eta')^\sigma = \eta'$ implies $\sigma = 0$). Since the action of $t_F \otimes I$ on $\mathbf{D}_\eta(A')$ is transitive, it follows that $\mathbf{D}_\eta(A')$ is a principal homogeneous space under $t_F \otimes I$, which is equivalent to (i).

It should be remarked that the obstruction to lifting θ lies in $t_F \otimes I$ and is often nonzero (see e.g., [4, §4]).

Finally it remains to consider the finiteness condition (H_3) . If X is smooth over k (in ancient terminology *absolutely simple*), then Grothendieck has shown in S.G.A. III, Theorem 6.3, that

$$t_D \cong H^1(X, \Theta)$$

where Θ is the tangent sheaf of X over k . Thus t_D has finite dimension if X is smooth and proper over k . In general, it is shown in [4] that for any scheme X locally of finite type over k , there is an exact sequence

$$(3.9) \quad 0 \rightarrow H^1(X, T^0) \rightarrow t_D \rightarrow H^0(X, T^1) \rightarrow H^2(X, T^0)$$

where T^0 is the sheaf of derivations of \mathfrak{D}_X , and T^1 is a (coherent) sheaf isomorphic to the sheaf of germs of deformations of X/k to $k[\varepsilon]$. If X is smooth over k , then $T^0 = \Theta$, $T^1 = 0$. Thus, in summary

PROPOSITION 3.10. *If X is either*

(a) *proper over k or*

(b) *affine with only isolated singularities,*

then \mathbf{D} has a hull (R, ξ) . (R, ξ) pro-represents \mathbf{D} if and only if for each small extension $A' \rightarrow A$, and each deformation Y' of X/k to A' , every automorphism of the deformation $Y' \otimes_{A'} A$ is induced by an automorphism of Y' .

(3.11) *The automorphism functor.* One can formalize the obstructions to pro-representing \mathbf{D} as follows. Let X be a prescheme *proper* over k , and let (R, ξ) be a hull of the deformation functor \mathbf{D} . ξ is represented by a formal prescheme $\mathfrak{X} = \text{inj} \lim X_n$ over R , where X_n is a deformation of X/k to R/\mathfrak{m}^n . For each morphism $R \rightarrow A$ in \mathbf{C}_Δ , we get a deformation $\mathfrak{X}_A = \mathfrak{X} \times_{\text{Spec } R} \text{Spec } A$ of X/k to A . We can therefore define a group functor \mathcal{A} on the category \mathbf{C}_R of Artin local R -algebras:

$$\mathcal{A}: A \mapsto \text{group of automorphisms of the deformation } \mathfrak{X}_A.$$

If $A' \rightarrow A$ and $A'' \rightarrow A$ are morphisms in \mathbf{C}_R with $A'' \rightarrow A$ a surjection, and if we put $B = A' \times_A A''$ then we have a canonical isomorphism, respecting the structures as deformations:

$$\mathfrak{D}_{\mathfrak{X}_B} \cong \mathfrak{D}_{\mathfrak{X}_A} \times_{\mathfrak{D}_{\mathfrak{X}_A}} \mathfrak{D}_{\mathfrak{X}_{A''}}$$

by Corollary 3.6. It follows easily that (2.12) is an isomorphism, so that (H_1) , (H_2) and (H_4) of Theorem 2.11 are satisfied. Finally the computations of Grothendieck in S.G.A. III, §6, show that the tangent space of \mathcal{A} is given by

$$t_{\mathcal{A}/R} \cong H^0(X_0, T^0)$$

where T^0 is, again, the (coherent) sheaf of derivations of \mathfrak{D}_X over k . Thus $t_{\mathcal{A}}$ has finite dimension, and we find:

PROPOSITION 3.12. *If X is proper over k , the functor \mathcal{A} is pro-represented by a complete local R algebra, S , which is a group object in the category dual to $\hat{\mathbf{C}}_R$ (i.e., S is a formal Lie group over R). The deformation functor \mathbf{D} is pro-representable (by R) if and only if S is a power series ring over R .*

The last statement follows from Lemma 3.8 and the smoothness criterion of Remark 2.10.

In a future paper I will discuss the deformation functor in more detail, with particular attention to the contribution of singular points on X .

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