# COURSE NOTES (ROUGH) ON MATH 36501, PERFECTOID SPACES

ABSTRACT. Rough notes (to be updated frequently) for my topics course in fall 2018. Comments and corrections are very welcome!

## 1. INTRODUCTION

Let $X/\mathbb{C}$ be a smooth projective algebraic variety. Then we have the classical *Hodge decomposition*: for each $n$, we have an isomorphism of $\mathbb{C}$-vector spaces

$$H^n(X,\mathbb{C}) \simeq \bigoplus_{i+j=n} H^i(X,\Omega_X^j). \tag{1}$$

The datum of the cohomology $H^n(X,\mathbb{Q})$ and the above decomposition of $H^n(X,\mathbb{C}) \simeq H^n(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ forms what is known as a *Hodge structure,* which is a basic invariant of smooth projective varieties over $\mathbb{C}$.

**Definition 1.** A **Hodge structure of weight** $n$ is the datum of a $\mathbb{Q}$-vector space $V$ together with a decomposition $V \otimes_{\mathbb{Q}} \mathbb{C} \simeq \bigoplus_{p+q=n} V^{p,q}$ such that $V^{p,q} = \overline{V^{q,p}}$.

The Hodge decomposition is proved using transcendental methods (such as the use of a Kähler metric). More generally, such a decomposition holds for compact Kähler manifolds.

One of the goals of *$p$-adic Hodge theory* is to prove similar results in the case where $\mathbb{C}$ is replaced instead by a $p$-adic field $F$. Let's first try to formulate an appropriate setting for this.

For instance, instead of a variety over $\mathbb{C}$, let us fix a smooth projective variety $Y/F$ for $F$ a finite extension of $\mathbb{Q}_p$. Our goal is to formulate an analog of (1) for $Y$. The RHS of (1) is defined purely algebraically, so it makes sense for $Y$ and now produces a $F$-vector space. For the left-hand-side, we no longer have a theory of singular cohomology, but should instead use Grothendieck's theory of *étale cohomology.* Specifically, we work with $\mathbb{Q}_p$-étale cohomology. Then the idea is that both sides of (1) are types of cohomology theories, and we seek to compare them, together with certain natural structures on both sides.

### 1.1. Reminders on étale cohomology.

Recall that *$l$-adic étale cohomology* is a construction that for any algebraic variety $Z$ over an algebraically closed field $K$ of characteristic $\neq l$, produces cohomology groups $H^i_{\text{et}}(Z,\mathbb{Q}_l)$. These behave as a purely algebraic substitute for the singular cohomology groups $H^i(\cdot,\mathbb{Q})$ for an algebraic variety over $\mathbb{C}$. Some things to know:

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- The groups $H^i_{\text{et}}(Z, \mathbb{Q}_l)$ are finite-dimensional $\mathbb{Q}_l$-vector spaces, which vanish for $i > 2\dim(Z)$.
- If $K = \mathbb{C}$ (so $Z$ is a complex variety), then $H^i_{\text{et}}(Z, \mathbb{Q}_l) \simeq H^i_{\text{sing}}(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l$ where the RHS denotes singular cohomology. (**Remark:** For a general field of characteristic $> 0$, one doesn't even know if the dimensions of $H^i_{\text{et}}(Z, \mathbb{Q}_l)$ are independent of $l$! It is known in the proper smooth case, but as a consequence of the Weil conjectures. In characteristic zero, it holds by the comparison with singular cohomology.)
- The construction $Z \mapsto H^i_{\text{et}}(Z, \mathbb{Q}_l)$ does not depend on the structure map $Z \to \operatorname{Spec}(K)$: in fact, it makes sense for all schemes. (It might not be quite as well-behaved; in this course, we will later need to discuss étale cohomology in significantly more general settings.)
- As a consequence of the last bullet point, if $K = \overline{F}$ and $Z = Z' \otimes_F \overline{F}$ is obtained as the base-change of an $F$-variety $Z'$, then the cohomology groups $H^*_{\text{et}}(Z, \mathbb{Q}_l)$ are naturally continuous representations of the Galois group $\operatorname{Gal}(F)$ in finite-dimensional $\mathbb{Q}_l$-vector spaces.

**Remark 2.** The last bullet point is crucial and pretty amazing: it means that when $Y$ is an algebraic variety defined over $\mathbb{Q}$, then the $l$-adic cohomology of the complex points $Y(\mathbb{C})$ as a topological space inherits an action of the Galois group $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Of course, there is no continuous action of $\operatorname{Gal}(\mathbb{Q})$ on $Y(\mathbb{C})$.

**Example 3** (Tate twists)**.** Let $F$ be any field of characteristic $\neq l$. Then the Galois group $\operatorname{Gal}(F)$ acts on the module $\mu_{l^\infty}(\overline{F})$ of $l$-power roots of unity in the algebraic closure $\overline{F}$. Abstractly, $\mu_{l^\infty}(\overline{F}) \simeq \mathbb{Q}_l/\mathbb{Z}_l$ and the Galois group acts on this by automorphisms; note that automorphisms of $\mathbb{Q}_l/\mathbb{Z}_l$ are $\mathbb{Z}_l^\times$.

If we consider the *Tate module*

$$\mathbb{Z}_l(1) = \varprojlim_{z \mapsto z^l} \mu_{l^n}(\overline{F}),$$

then we get a free $\mathbb{Z}_l$-module of rank 1 together with a continuous action of $\operatorname{Gal}(F)$. We set $\mathbb{Q}_l(1) \simeq \mathbb{Z}_l(1)[1/l]$; this is a continuous $\operatorname{Gal}(F)$-representation.

**Example 4** (The projective line)**.** For a field $F$ of characteristic $\neq l$, we have that $H^2(\mathbb{P}^1_{\overline{F}}; \mathbb{Q}_l) \simeq \mathbb{Q}_l(-1)$.

**Example 5** (Elliptic curves)**.** The standard example of this construction is when $E$ is an elliptic curve over a field $F$ of characteristic $\neq l$. Then, the $\overline{F}$-valued points $E(\overline{F})$ are naturally equipped with an action of the Galois group $\operatorname{Gal}(F)$, as are the submodule of $l$-power torsion points $E(\overline{F})[l^\infty] \subset E(\overline{F})$; as an abelian group this is isomorphic to $(\mathbb{Q}_l/\mathbb{Z}_l)^2$. Then we have an isomorphism of $\operatorname{Gal}(F)$-representations

$$H^1_{\text{et}}(E_{\overline{F}}, \mathbb{Q}_l) = \operatorname{Hom}(E(\overline{F})[l^\infty], \mathbb{Q}_l/\mathbb{Z}_l)[1/l].$$

1.2. **Statements of results.** We're primarily going to be interested in the case where $F$ is a finite extension of $\mathbb{Q}_p$. Recall that $F$ is equipped with an absolute value extending the $p$-adic absolute value $|\cdot| : \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ and that $F$ is complete with respect to it.

Given an algebraic variety $X/F$, we obtain $\mathrm{Gal}(F)$-representations $H^*_{\mathrm{et}}(X_{\overline{F}}, \mathbb{Q}_p)$; in general, these are quite complicated. The Hodge-Tate decomposition will relate them to the Hodge cohomology groups $H^*(X, \Omega^*_{X/F})$. First, however, we have to extend scalars as follows.

**Definition 6.** We let $\mathbb{C}_p$, the field of $p$-adic complex numbers, be the completed algebraic closure of $\mathbb{Q}_p$. The field $\mathbb{C}_p$ is equipped with a nonarchimedean absolute value

$$|\cdot| : \mathbb{C}_p \to \mathbb{R}_{\geq 0}$$

with respect to which it is complete, and a continuous action of the Galois group $\mathrm{Gal}(\mathbb{Q}_p)$ (in particular, $\mathrm{Gal}(F)$).

Let $V$ be a continuous representation of the Galois group $\mathrm{Gal}(F)$ on a finite-dimensional $\mathbb{Q}_p$-vector space. Then $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ is a finite-dimensional $\mathbb{C}_p$-vector space with a *diagonal* action (in $\mathbb{Q}_p$-vector spaces) of $\mathrm{Gal}(F)$.

One of the goals of this course is to understand the proof of the following theorem, the *Hodge-Tate decomposition*. This was originally proved by Tate for abelian varieties with good reduction (and more generally a result for $p$-divisible groups), and has since been generalized and extended by many other authors.

**Theorem 7** (Faltings). *Let $F$ be a finite extension of $\mathbb{Q}_p$. Let $X/F$ be a smooth proper variety. Then we have an isomorphism of $\mathbb{C}_p$-representations with $\mathrm{Gal}(F)$-action*

$$H^n(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/F}) \otimes_F \mathbb{C}_p(-j). \tag{2}$$

A consequence is that the Galois representation $H^n(X_{\overline{F}}, \mathbb{Q}_p)$ determines the Hodge numbers, in view of the following result.

**Proposition 8** (Tate-Sen). *Let $F$ be a finite extension of the $p$-adic numbers. We have that $\mathbb{C}_p(i)^{\mathrm{Gal}(F)} = 0$ for $i \neq 0$ and $F$ for $i = 0$.*

Let $X/\mathbb{C}$ be a smooth projective variety. In particular, $X$ defines a compact complex manifold. The proof of the Hodge decomposition for $X$ relies on working locally on $X$ in the *analytic* topology (rather than the Zariski topology), i.e., on thinking about $X$ as a manifold rather than an algebraic variety. One could hope for a $p$-adic "analytic" proof of the Hodge-Tate decomposition.

### 1.3. **The Hodge-Tate spectral sequence.**

In fact, the aim of this course is to describe such an "analytic" approach. Here is the main result.

**Theorem 9** (Scholze). *Let $X/\mathbb{C}_p$ be a proper smooth rigid analytic space (e.g., a proper smooth algebraic variety). Then there exists a natural spectral sequence $H^i(X, \Omega^j)(-j) \implies H^{i+j}_{\mathrm{et}}(X, \mathbb{C}_p)$ which degenerates.*

**Remark 10.** When $X$ is defined over a a finite extension of $\mathbb{Q}_p$, then the spectral sequence canonically degenerates, i.e., differentials and extensions are zero. This is by the Tate-Sen results.

The Hodge-Tate spectral sequence is constructed via a procedure local in the sense of *analytic* rather than algebraic geometry. The language of *rigid analytic spaces* let us do this. A rigid analytic space is a special type of locally ringed space (with a slight caveat) such as:

**Example 11.** The basic example of a rigid analytic space is the unit $n$-disk, which corresponds to those $n$-tuples $\{(z_1, \ldots, z_n) : |z_i| \leq 1, \text{each } i\}$. This corresponds to the *Tate algebra*

$$T_n = \mathbb{C}_p \langle X_1, \ldots, X_n \rangle = \left\{ f = \sum_{i_1, \ldots, i_n \geq 0} f_{i_1, \ldots, i_n} X^{i_1} \ldots X^{i_n} | f_{i_1, \ldots, i_n} \to 0, \quad i_1, \ldots, i_n \to \infty \right\},$$

which is the associated ring of functions.

**Definition 12.** An *affinoid algebra* $A$ over $\mathbb{C}_p$ is an algebra which is a quotient of $T_n$ (for some $n$). The algebra $A$ is then automatically a noetherian ring, and we associate to $A$ its set $\mathrm{Sp}(A)$ of maximal ideals. In the case of $T_n$ itself, this recovers the set of $n$-tuples $\left\{(z_1, \ldots, z_n) \in \mathbb{C}_p^n : |z_i| \leq 1, \text{each } i\right\}$.

With some work, to an affinoid algebra one can associate a locally ringed space, and more general rigid analytic spaces are defined by glueing such spaces. There is a theory of étale cohomology for rigid analytic spaces. Any smooth proper variety $X/\mathbb{C}_p$ determines a quasi-compact rigid analytic space (which is locally of the form $\mathrm{Sp}(A)$, and covered by finitely many such).

Then one has the following result. Note that it implies the Hodge-Tate spectral sequence, by considering the hypercohomology spectral sequence.

**Theorem 13** (Scholze)**.** *Let $A/\mathbb{C}_p$ be a smooth affinoid algebra. Then there exists a complex $\mathcal{F}(A)$ of $A$-modules (defined up to quasi-isomorphism, so in the derived category of $A$) with the following properties:*

(1) *We have isomorphisms $H^*(\mathcal{F}(A)) \simeq \Omega^*_{A/\mathbb{C}_p}$. That is, the cohomology of $\mathcal{F}(A)$ is just differential forms on $A$.*
(2) *For every étale map $A \to B$ of smooth affinoid algebras, we have a natural isomorphism $\mathcal{F}(A) \otimes_B \simeq \mathcal{F}(B)$.*
(3) *(The primitive comparison theorem.) If $X/\mathbb{C}_p$ is a proper smooth rigid analytic space, then we have an equivalence*

$$R\Gamma_{\text{et}}(X, \mathbb{C}_p) \simeq R\Gamma(X, \mathcal{F}(\mathcal{O}_X)).$$

**Remark 14.** In fact, the construction $\mathcal{F}$ is defined for every Banach algebra over $\mathbb{Q}_p$. The construction of $\mathcal{F}$ goes through the theory of perfectoid spaces. In particular, $\mathcal{F}$ is the identity functor on perfectoid algebras, and then is determined by the condition that it should be a *pro-étale sheaf.* The description of $\mathcal{F}(A)$ for $A$ smooth affinoid becomes an explicit calculation.

## 2. Perfect rings and tilting

Today, we will start getting into some of the details. We discuss the theory of perfect rings. Many of the constructions here are toy analogs of constructions for perfectoid rings.

**Definition 15.** An $\mathbb{F}_p$-algebra $R$ is *perfect* if the Frobenius map $\phi : R \to R$ given by $\phi(z) = z^p$ is an isomorphism. Let $\mathrm{Perf}_{\mathbb{F}_p}$ denote the category of perfect rings.

**Remark 16.** This agrees with the usual notion of a perfect field, when $R$ is a field.

**Example 17** (Perfect polynomial rings)**.** The ring $R = \mathbb{F}_p[x^{1/p^\infty}]$, obtained as the union of the polynomial rings $\mathbb{F}_p[x^{1/p^n}]$ as $n \to \infty$, is a perfect ring. For any perfect ring $S$, we have that

$$\mathrm{Hom}_{\mathrm{Perf}_{\mathbb{F}_p}}(R, S) \simeq S,$$

i.e., $R$ is the free perfect ring on one generator.

**Definition 18** (Perfections)**.** Let $A$ be any $\mathbb{F}_p$-algebra. There are two ways we can form a perfect ring out of $A$:

(1) The *direct limit perfection* $A_{\mathrm{perf}}$ is the directed limit of the system $A_{\mathrm{perf}} = \varinjlim A \to A \to A \to \dots$ where all the maps are the Frobenius. When we just say *perfection*, it will refer to this construction.
(2) The *inverse limit perfection* $A^{\mathrm{perf}}$ is the similar inverse limit $A^{\mathrm{perf}} = \varprojlim \cdots \to A \to A \to A$ where the maps are the Frobenius.

**Remark 19.** The constructions $A_{\mathrm{perf}}, A^{\mathrm{perf}}$ have universal properties: any map $A \to B$, where $B$ is perfect, factors through $A_{\mathrm{perf}}$ uniquely, so the construction $(\cdot)_{\mathrm{perf}}$ is the left adjoint of the inclusion of perfect rings in all $\mathbb{F}_p$-algebras. Similarly, $(\cdot)^{\mathrm{perf}}$ is the right adjoint.

**Remark 20.** For instance, Example 17 is the direct limit perfection of the polynomial ring $\mathbb{F}_p[x]$. By contrast, the inverse limit perfection of a finite type algebra is uninteresting; the construction is most useful for an $\mathbb{F}_p$-algebra which is *semiperfect*, i.e., where the Frobenius is surjective. This will be useful for the tilting correspondence. Let's do an example of such. Consider the ring

$$A = \mathbb{F}_p[x^{1/p^\infty}]/x.$$

This is a semiperfect ring. The perfection $A^{\mathrm{perf}} = \widehat{\mathbb{F}_p[x^{1/p^\infty}]}$ is the $x$-adic completion of the perfect polynomial ring.

Perfect rings have a number of surprisingly pleasant homological properties. See the paper of Bhatt-Scholze ("Projectivity of the Witt vector affine Grassmannian") for a detailed treatment. In particular, it uses the following observation.

**Proposition 21** (Bhatt-Scholze)**.** *Let $A \to B, A \to C$ be two maps in $\mathrm{Perf}_{\mathbb{F}_p}$. Then $\mathrm{Tor}_i^A(B, C) = 0$ for $i > 0$.*

*Proof.* There are (at least) two ways to prove this.

(1) (Sketch). Use the theory of topological (or simplicial) commutative rings. The Tor-groups $\mathrm{Tor}_i^A(B, C)$ are the homotopy groups of a topological $\mathbb{F}_p$-algebra $B \otimes_A^L C$. Since $A$ and $B$ are perfect, the Frobenius on $B \otimes_A^L C$ is a weak homotopy equivalence.

However, the Frobenius is always *zero* on the higher homotopy groups of a topological $\mathbb{F}_p$-algebra. To see this, let $R$ be a topological $\mathbb{F}_p$-algebra. Then the map

$$m : R^p \to R$$

given by multiplying together the factors is a map of topological spaces (but not of rings), and it induces a map on the homotopy groups $\pi_i$ for $i > 0$. Since $m$ is the zero map on each factor of the form $R^{p-1} \times \{0\}$, and since $m$ induces a *homomorphism* on $\pi_i$ (as does any map of topological spaces), it follows that $m$ necessarily induces zero on $\pi_i, i > 0$. The Frobenius is the composite $R \stackrel{\Delta}{\to} R^p \stackrel{m}{\to} R$ for $\Delta$ the diagonal; it follows that the Frobenius too is zero on higher homotopy.

(2) One first reduces to the case where $A \to B$ is surjective since $A \to B$ is the composite of a faithfully flat map of perfect rings together with a surjection (take a large perfect polynomial ring). Let $I = \ker(A \to B)$. Without loss of generality (with a filtered colimit argument), $I = \mathrm{rad}(f_1, \dots, f_n)$ for $f_1, \dots, f_n \in I$. By an induction on $n$, we can assume that $n = 1$ and $f = f_1$. Then we claim that $I = (f^{1/p^\infty})$ is flat. In fact, one shows that

$$I = \varinjlim A \stackrel{f^{1-1/p}}{\to} A \stackrel{f^{1/p-1/p^2}}{\to} A \to \dots$$

using the perfectness of $A$.

Similarly, we get

$$IC = \varinjlim C \stackrel{f^{1-1/p}}{\to} C \stackrel{f^{1/p-1/p^2}}{\to} C \to \dots = I \otimes_A C.$$

Thus, we get from long exact sequences that $\mathrm{Tor}_i^A(B, C)$ vanishes.

□

A basic fact about perfect $\mathbb{F}_p$-algebras is that they have a unique lift to characteristic zero, in the following sense.

**Definition 22.** Given an $\mathbb{F}_p$-algebra $R$, a *lift to characteristic zero* of $R$ is a $p$-complete, $p$-torsion-free ring $\widetilde{R}$ such that $\widetilde{R}/p \simeq R$.

**Remark 23.** A smooth $\mathbb{F}_p$-algebra always has a lift to characteristic zero, and the lift is unique up to *non-unique* isomorphism. In general, an $\mathbb{F}_p$-algebra need not admit a lift, and the lift need not be unique if it exists.

**Definition 24** (Strict $p$-rings)**.** A *strict $p$-ring* is a ring $A$ which is $p$-adically complete, $p$-torsion-free, and such that $A/p$ is a perfect $\mathbb{F}_p$-algebra: in particular, it is a lift to characteristic zero of the perfect ring $A/p$.

**Example 25.** Let $E$ be a finite extension of $\mathbb{Q}_p$ which is unramified. Then the ring of integers $\mathcal{O}_E$ is an example of a strict $p$-ring (in fact, $\mathcal{O}_E/p$ is a finite field).

**Remark 26.** A strict $p$-ring is the mixed characteristic analog of the construction $A[[t]]$, for $A$ a ring (i.e., the power series ring construction), except $t$ is replaced by $p$.

In order to proceed further, we will use the following fact.

**Proposition 27.** *Let $A \to B, A \to C$ be maps of strict $p$-rings. Then $B \otimes_A C$ is $p$-torsion-free and its $p$-completion is a strict $p$-ring.*

*Proof.* It suffices to show that $B \otimes_A C$ is $p$-torsion-free, since mod $p$ it is clearly perfect. Form the derived tensor product $B \otimes_A^L C$ in the derived category. It suffices to show that the derived tensor product $(B \otimes_A^L C) \otimes_{\mathbb{Z}}^L \mathbb{Z}/p$ is concentrated in degree zero; any $p$-torsion would show up as higher homology groups. But this is $(B/p) \otimes_{(A/p)}^L (C/p)$, which we have seen is discrete. $\square$

**Construction 28.** Given a strict $p$-ring $A$, we construct a multiplicative (but not additive!) map

$$\sharp : A/p \to A.$$

The map $\sharp$ is defined as follows. Fix $\overline{a} \in A/p$. For each $n$, choose $b_n \in A$ such that $b_n \equiv \overline{a}^{1/p^n}$ in $A/p$. Then $\sharp(\overline{a}) = \lim_{n\to\infty} b_n^{p^n}$.

The fact that this limit in the $p$-adic topology exists (and is independent of choices) is a consequence of the following fact proved via the binomial theorem: if $b, b' \in A$ are congruent mod $p^n$, then $b^p, b'^p$ are congruent modulo $p^{n+1}$. In addition, it's easy to see that the composite map

$$A/p \xrightarrow{\sharp} A \to A/p$$

is just the identity. Finally, the construction $\sharp$ is clearly natural in maps of strict $p$-rings, just by construction.

Using the above, it is easy to see that every $a \in A$ has a unique expansion

$$a = \sum_{i=0}^{\infty} (a_i^\sharp) p^i \tag{3}$$

where $a_i \in A/p, i \geq 0$.

**Proposition 29.** *Given a strict $p$-ring $A$, we have an isomorphism of multiplicative monoids*

$$A/p \simeq \varprojlim_{x \mapsto x^p} A.$$

*In particular, the image of the sharp map $\sharp : A/p \to A$ is precisely those elements of $A$ which admit $p$-power roots of arbitrary order.*

*Proof.* Note that the construction $a \mapsto a^\sharp, A/p \to A$ is multiplicative and since $A/p$ is perfect, then we get a map $A/p \to \varprojlim_{x\mapsto x^p} A$, given by $a \mapsto (a^\sharp, a^{1/p\sharp}, \dots)$. Conversely, given a sequence $(a_0, a_1, a_2, \dots,)$ in $A$ such that each $a_i = a_{i+1}^p$, it suffices to see that $a_i = \overline{a_0}^{1/p^i \sharp}$. This follows from the definition of the sharp construction. $\square$

For future reference, let's state this as a lemma.

**Lemma 30.** *Let $A$ be any $p$-complete ring. Then the natural map*

$$\varprojlim_{x \mapsto x^p} A \to \varprojlim_{x \mapsto x^p} A/p$$

*is an isomorphism.*

**Theorem 31.** *The construction $A \mapsto A/p$ establishes an equivalence of categories between strict $p$-rings and perfect $\mathbb{F}_p$-algebras. That is, if $A, A'$ are strict $p$-rings, then*

$$\mathrm{Hom}(A, A') \simeq \mathrm{Hom}(A/p, A'/p) \tag{4}$$

*and every perfect $\mathbb{F}_p$-algebra can be lifted to a strict $p$-ring. More generally, the above isomorphism holds if $A'$ is only assumed $p$-complete.*

**Remark 32.** This is a toy analog of the *tilting equivalence* in the context of perfectoid algebras. As such, we'll go through it in some detail.

**Example 33** (Perfect polynomial rings)**.** Consider the ring $\widehat{\mathbb{Z}_p[X^{1/p^\infty}]}$ (hat denotes $p$-adic completion). This is a strict $p$-ring, whose reduction modulo $p$ is the perfect polynomial ring $P_0$ mapping surjectively to $A_0$.

Let $B$ be any $p$-complete ring. Then

$$\mathrm{Hom}(\widehat{\mathbb{Z}_p[X^{1/p^\infty}]}, B) \simeq \varprojlim_{x \mapsto x^p} B \simeq \varprojlim_{x \mapsto x^p} B/p \simeq \mathrm{Hom}(\mathbb{F}_p[X^{1/p^\infty}], B/p).$$

**Example 34.** Let $k$ be a perfect field. In this case, the associated strict $p$-ring is a complete discrete valuation ring with uniformizer $p$ and residue field $k$.

*Proof of Theorem 31.* Let $A_0$ be a perfect $\mathbb{F}_p$-algebra. Then we can find a perfect polynomial ring $P_0$ with a surjection $P_0 \twoheadrightarrow A_0$. We can then choose a presentation

$$A_0 = P_0 \otimes_{P_1} \mathbb{F}_p$$

where $P_0, P_1$ are perfect polynomial rings. Now we know that the formula (4) works fine when $A$ is $\widehat{\mathbb{Z}_p[X^{1/p^\infty}]}$ and hence for any perfect (completed) polynomial ring over $\mathbb{Z}_p$. Thus, we can choose canonical lifts $\widetilde{P}_0, \widetilde{P}_1$ and maps $\widetilde{P}_1 \to \widetilde{P}_0$ and the relative tensor product $\widetilde{A} = \widetilde{P_0} \otimes_{\widetilde{P_1}} \mathbb{Z}_p$ has $p$-completion a strict $p$-ring which is a lift of $A$. Moreover, since the formula (4) works for $\widetilde{P}_0, \widetilde{P}_1$, it must work for $\widetilde{A}$. It follows easily that this construction exhausts all strict $p$-rings, and that everything claimed holds. $\square$

**Remark 35.** Given a strict $p$-ring $A$, we saw earlier that we can expand an element $a \in A$ as an infinite sum $\sum_{i=0}^{\infty} (a_i)^\sharp p^i$ for elements $a_i \in A/p$. Since the construction $\sharp$ is multiplicative, this tells you how to multiply elements if you know how to add them. That is, one needs the formula for $a^\sharp + b^\sharp$. There is a universal formula for this. For instance,

$$a^\sharp + b^\sharp = (a+b)^\sharp + p(-\sum_{0<i<p} \frac{1}{p} a^{i/p} b^{1-i/p})^\sharp + \dots .$$

These formulas are already determined by what happens in the ring $\widehat{\mathbb{Z}_p[x^{1/p^\infty}, y^{1/p^\infty}]}$.

**Remark 36.** The inverse to the above equivalence of categories is implemented explicitly by the $p$-*typical Witt vector* functor $W(\cdot)$. $W(\cdot)$ is a functor on all rings, but on perfect rings it produces the lift to characteristic zero above.

In fact, the above result (4) tells you how to map *out* of a strict $p$-ring $A$. There is also a formula of how to map *into* a strict $p$-ring. Namely, $A$ has a unique endomorphism $\phi_A : A \to A$ which lifts the Frobenius on $A/p$ (by (4)). If $B$ is any $p$-torsion-free, $p$-complete ring with an endomorphism $\phi_B$ lifting the Frobenius modulo $p$ (e.g., $B = \mathbb{Z}_p[X]$ with $\phi_B(X) = X^p$) then any map $B \to A/p$ lifts uniquely to a $\phi$-compatible map $B \to A$. This is a special case of the universal property of the Witt vectors.

The above constructions can be extended to an adjunction on all $p$-complete rings.

**Definition 37** (The tilting construction)**.** Let $A$ be a $p$-complete ring. The *tilt* $A^\flat$ of $A$ is defined to be the inverse limit perfection $(A/p)^{\mathrm{perf}}$. Explicitly, an element of $A^\flat$ is a system of elements $(x_0, x_1, \ldots, )$ such that $x_{i+1}^p = x_i$ in $A/p$. In particular, this is an $\mathbb{F}_p$-algebra. As before, we can also write as multiplicative monoids

$$A^\flat = \varprojlim_{x \mapsto x^p} A.$$

**Construction 38** (The sharp map)**.** We have a multiplicative but not additive map

$$A^\flat \to A$$

for any $p$-complete ring $A$.

**Proposition 39.** *Let $A$ be a $p$-complete ring. We have an isomorphism of multiplicative monoids (not additive!)*

$$A^\flat \simeq \varprojlim_{x \mapsto x^p} A. \tag{5}$$

*Proof.* See Lemma 30. □

**Corollary 40.** *Let $A$ be a $p$-complete ring. Then if $A$ is a domain (resp. valuation ring, resp. field), then so is $A^\flat$.*

*Proof.* This follows easily from the description (5), since these conditions are all in terms of the multiplicative monoid. □

**Theorem 41** (Tilting adjunction)**.** *We have an adjunction*

$$\mathrm{Perf}_{\mathbb{F}_p} \to p - \text{complete rings},$$

*where the left adjoint sends $A \mapsto W(A)$ and the right adjoint sends a $p$-complete ring $R$ to $R^\flat$.*

*Proof.* This is just a restatement of the previous theorem. □

**Construction 42** (The sharp map)**.** Given $R$ $p$-complete, we have a sharp map

$$\sharp : R^\flat \to R.$$

**Construction 43** (The Fontaine map $\theta$)**.** Using the above adjunction, we get a natural map for any $p$-complete ring $R$

$$\theta : W(R^\flat) \to R.$$

As we'll see later, for *perfectoid* rings the map $\theta$ is a surjection, and exhibits $R$ as the quotient of $W(R)$ (a strict $p$-ring) by a nonzerodivisor (which also happens to be a so-called "primitive element").

## 3. Topological rings and nonarchimedean fields

We start with some preliminaries on topological rings. A topological ring is just a ring with a topology such that the ring operations are continuous.

**Example 44.** The real numbers $\mathbb{R}$ is a topological ring.

Here are examples more relevant for the course.

**Definition 45.** A topological ring $R$ is *non-archimedean* if there is a neighborhood basis at the origin consisting of subgroups. The phrase *linearly topologized* is also used. A map of topological rings is a continuous homomorphism.

**Remark 46** (Construction of NA topological rings)**.** Let $R$ be a ring. To make $R$ into a non-archimedean topological ring, it suffices to give a collection of subgroups $\{S_\alpha\}_{\alpha \in A}$ of $R$ such that:

(1) (Subbasis property): Given $\alpha, \beta$, there exists $\gamma$ such that $S_\alpha \cap S_\beta \subset S_\gamma$.
(2) (Product property): Given $\delta$, there exist $\mu, \nu$ such that $S_\mu S_\nu \subset S_\delta$.
(3) (Product property, II): Given $S_\epsilon$ and given $x \in R$, there exists $S_\rho$ with $x S_\rho \subset S_\epsilon$.

Then we can make $R$ into a topological ring declaring $\{S_\alpha\}$ to be a neighborhood basis at zero.

**Example 47** (Adic rings)**.** Let $R$ be a ring, and let $I \subset R$ be an ideal. We can make $R$ into a topological ring by declaring the powers $\{I^n\}$ to be a neighborhood basis at the origin; then $R$ is called an *adic* topological ring, and $I$ an *ideal of definition*.

**Example 48** (Tate rings)**.** Let $R_0$ be a ring. Let $\omega \in R_0$ be a nonzerodivisor. Then $R_0[1/\omega]$ is a topological ring such that:

(1) $R_0$ is an open subring.
(2) A neighborhood basis at zero is given by the ideals $(\omega^n) \subset R_0$.

These topological rings are called *Tate* rings.

A major class of examples that we will be interested is the following.

**Definition 49** (Nonarchimedean fields)**.** A *nonarchimedean field* is a field $K$, equipped with an absolute value $|\cdot| : K \to \mathbb{R}_{\geq 0}$ which satisfies the following conditions:

(1) $|x| = 0$ precisely when $x = 0$.

(2) $|x+y| \le \max(|x|, |y|)$.
(3) $|xy| = |x|\,|y|$.

This makes $K$ into a metric space via the metric $d(x,y) = |x-y|$, and thus into a topological ring.

We say that a nonarchimedean field $K$ is *complete* if it is complete as a metric space.

**Example 50.** (1) The field of $p$-adic numbers $\mathbb{Q}_p$ with the usual $p$-adic norm.
(2) The field $\mathbb{F}_p((t))$, with the absolute value that

$$\left| \sum_{i \gg -\infty}^{\infty} a_i t^i \right| = p^j$$

for $j = -\min_{i : a_i \neq 0} i$. Note that both of these examples are complete.

**Example 51.** Right now, we are also allowing the trivial absolute value $|\cdot| : K \to \mathbb{R}_{\ge 0}$ given by $|x| = 1$ if $x \neq 0$ and $|0| = 0$.

**Remark 52.** Given a field $K$ with an absolute value $|\cdot| : K \to \mathbb{R}_{\ge 0}$ which satisfies the above conditions (1) through (3), we can form the (metric) completion which is then a complete nonarchimedean field.

The actual absolute value $|\cdot| : K \to \mathbb{R}_{\ge 0}$ itself is not so important, since it can be replaced by $|\cdot|^r$ for any $r \in \mathbb{R}_{>0}$, and it induces precisely the same topology. In fact, the structure of $K$ as a *topological field* determines the absolute value (up to scaling).

**Definition 53** (Bounded subsets)**.** Let $R$ be a topological ring. A subset $S \subset R$ is *bounded* if for every neighborhood $V$ of zero, there is a neighborhood $U$ of zero such that $US \subset V$.

**Definition 54** (Powerbounded and topologically nilpotent elements)**.** Given a topological ring $R$, we let $R^\circ$ denote the collection of $x \in R$ such that the system of powers $\left\{x^i, i \ge 0\right\}$ is a bounded set. These are the *powerbounded elements.* Similarly, let $R^{\circ\circ} \subset R^\circ$ denote the subset of elements $x \in R$ whose powers $x^n \to 0, n \to \infty$. These are the *topologically nilpotent* elements.

**Example 55.** Let $K$ be a nonarchimedean field with absolute value $|\cdot|$. Then $K^\circ = \{x : |x| \le 1\}$. This is a ring by the nonarchimedean property. Similarly, $K^{\circ\circ} = \{x \in K : |x| < 1\}$.

**Remark 56** (Powerbounded elements need not be a ring)**.** If $R = \mathbb{R}$ is the real numbers, then the powerbounded elements in $R$ are given by $\{x \in \mathbb{R} : |x| \le 1\}$.

**Proposition 57.** *Let $R$ be a nonarchimedean topological ring. Then the powerbounded elements $R^\circ \subset R$ form a ring, and the topologically nilpotent elements $R^{\circ\circ} \subset R^\circ$ form an ideal.*

*Proof.* Let $V$ be a neighborhood of zero; wlog $V$ is a subgroup. There exists a neighborhood $U$ of zero such that $Ux^{\mathbb{Z}_{\ge 0}}, Uy^{\mathbb{Z}_{\ge 0}} \subset V$. Using the binomial theorem, it follows that $U(x \pm y)^{\mathbb{Z}_{\ge 0}}$ too since $V$ is a subgroup. This shows that $x+y$ is powerbounded.

The case of $xy$ is similar (and easier, since we don't need the nonarchimedean property). Fix a neighborhood $V$ of zero. Choose a neighborhood $V'$ such that $V'y^{\mathbb{Z}_{\ge 0}} \subset V$. Choose $U$ such that

$Ux^{\mathbb{Z}_{\geq 0}} \subset V'$. Then

$$U(xy)^{\mathbb{Z}_{\geq 0}} \subset V'y^{\mathbb{Z}_{\geq 0}} \subset V,$$

as desired.

We omit the (similar) proof for topologically nilpotent elements. □

**Example 58** (Tate rings, II)**.** We continue Example 48. Given a Tate ring $R = R_0[1/x]$ (with the $x$-adic topology on $R_0$), we note that the element $x$ is topologically nilpotent and a unit. Moreover, the subring $R_0 \subset R$ is open and bounded, so $R_0 \subset R^\circ$. However, we need not have equality. Moreover, the subring $R_0$ is generally not determined by $R$: any two will be "commensurable" by a power of $x$.

The primary examples of topological rings we are interested in will be Tate rings. In this case, we observe that $R^\circ$ is an open subring. It is generally not itself bounded; if so $R$ is said to be *uniform.* (The notion of "perfectoid algebra" can be developed in the category of uniform Tate rings.) We claim that $R^\circ$ is the union of all open bounded subrings. In fact, if $a \in R^\circ$, then the subring of $R$ generated by $R_0, a$ is bounded and contains $x$.

**Proposition 59** (Powerbounded elements are integrally closed)**.** *Let $R$ be a nonarchimedean topological ring. Then $R^\circ \subset R$ is integrally closed.*

*Proof.* Indeed, suppose $x \in R$ satisfies an integral equation $f(x) = 0$ over $R^\circ$. The coefficients of $f$ generate a bounded subring of $R$ (by assumption), and all powers of $x$ are contained in a finitely generated module over this bounded subring inside of $R$. It follows that $x$ is powerbounded. □

Let $K$ be a nonarchimedean field. Let's look at the ring $K^\circ = \{x : |x| \leq 1\}$. It has the following properties:

(1) If $x, y \in K^\circ$, then $x \mid y$ if and only if $|y| \leq |x|$.
(2) It is a local ring; the maximal ideal is given by $K^{\circ\circ} = \{x \in K : |x| < 1\}$. Any element not in here is a unit.
(3) Given two elements $x, y \in K^{\circ\circ}$, not both zero, there exists $n$ such that $x \mid y^n$ and $y \mid x^n$.

**Remark 60.** Suppose that $K$ has a nontrivial nonarchimedean valuation and the image of $|\cdot| : K^\times \to \mathbb{R}_{>0}$ is a discrete subgroup. Then, there exists $\pi \in K^{\circ\circ}$ which maximizes $|\pi|$ (over $K^{\circ\circ}$), so $\pi$ generates the ideal $K^{\circ\circ}$ is a discrete valuation ring. In general, it is not true that one has this discreteness, and $K^\circ$ is a rank one valuation ring.

**Definition 61** (Valuation rings)**.** A *valuation ring* is an integral domain $V$ such that for any $x, y \in V$, either $x \mid y$ or $y \mid x$.

**Remark 62.** In a valuation ring, all the ideals form a partially ordered set under inclusion. In addition, any radical ideal is prime. Let $I \subset V$ be a radical ideal in the valuation ring $V$. Suppose $x, y \notin I$ but $xy \in I$. Without loss of generality, $x = yr$ and we have $y^2 r \in I$, hence $y^2 r^2 \in I$ and $x \in I$.

**Remark 63.** A valuation ring $V$ is necessarily local: the non-units form an ideal. In fact, suppose $x, y$ are non-units and $z(x+y) = 1$. Without loss of generality $y \mid x$, so $z(x+y) \subset (x)$ which is a proper ideal by hypothesis, a contradiction.

**Definition 64.** A valuation ring $V$ is said to be *rank* 1 if:

(1) $V$ is not a field.
(2) Given nonzero non-units $x, y \in V$, there exists $n > 0$ such that $x \mid y^n$.

We say that a nonzero nonunit $x \in V$ is a *pseudouniformizer.* For a pseudouniformizer $x$, we can give $V$ the $x$-adic topology; this is independent of $x$ by our assumptions.

Given a nonarchimedean field $K$, it follows that $K^\circ$ is a rank 1 valuation ring. Conversely, if $V$ is a rank 1 valuation ring, then we claim that the fraction field $K$ is a nonarchimedean field with $K^\circ = V$.

*Proof.* Fix a nonzero nonunit $x$. We define a function $|\cdot| : V \to \mathbb{R}_{\geq 0}$ via

$$v(y) = \inf_{r,s \in \mathbb{Z}_{>0} : x^r | y^s} 2^{-r/s}.$$

One checks that this defines a multiplicative function $V \to \mathbb{R}_{\geq 0}$ which satisfies the nonarchimedean property, and we can pass to the fraction field to obtain an absolute value on $K$.

□

Unwinding the above, we find:

**Proposition 65.** *The datum of a nontrivial nonarchimedean field $K$ (with absolute value determined up to scaling) is determined by (and equivalent to) the datum of a rank 1 valuation ring $V$ (given by $K^\circ$). Moreover, $K$ is complete if and only if $V$ is $x$-adically complete for any pseudouniformizer $x \in V$.*

This is a more algebraic (rather than topological) way of thinking of what a nonarchimedean field is.

Explicitly, what's happening here is that we consider the group $K^\times/V^\times$. Since $V$ is a valuation ring, this becomes a totally ordered group under divisibility. Moreover, given a pseudouniformizer $x \in V$, we find that any element of the totally ordered group $K^\times/V^\times$ is dominated by some power of $x$.

**Definition 66.** A totally ordered group $\Gamma$ is called *archimedean* if there exists an element $x \in \Gamma$ such that for any $y \in \Gamma$, $y < x^n$ for some $n$.

**Proposition 67.** *Given a totally ordered group $\Gamma$ which is archimedean, there exists an injective homomorphism of ordered groups $\Gamma \to \mathbb{R}$.*

Next, we need some examples of nonarchimedean fields.

**Proposition 68** (Extensions of absolute values)**.** *Let $K$ be a* complete *nonarchimedean field, and let $L$ be a finite extension of $K$. Then $L$ also acquires the structure of a nonarchimedean field: there is a unique absolute value on $L$ extending the one on $K$. Moreover, $L$ is complete.*

*Proof.* □

**Example 69.** Let $E$ be any finite extension of $\mathbb{Q}_p$. Then $E$ is a nonarchimedean field, necessarily complete, in a unique fashion as above. The filtered colimit $\overline{\mathbb{Q}_p}$ is a nonarchimedean field but it is no longer complete. We let $\mathbb{C}_p$ be the completion. Then $\mathbb{C}_p$ is a complete nonarchimedean field with a continuous action of the Galois group $\mathrm{Gal}(\mathbb{Q}_p)$.

## 4. Perfectoid fields

**Definition 70** (Perfectoid fields)**.** Throughout, we work with complete nonarchimedean fields $K$ where $|p| < 1$ (e.g., characteristic $p$). This means that $K$ is either characteristic $p$ or contains a copy of $\mathbb{Q}_p$.

The nonarchimedean field $K$ is called *perfectoid* if $K$ is complete and:

(1) The Frobenius map $K^\circ/p \to K^\circ/p$ is surjective.
(2) The absolute value on $K$ is nondiscrete. That is, the image of $|\cdot| : K \to \mathbb{R}_{\geq 0}$ contains elements arbitrarily close to (but not equal to) 1.

**Remark 71** (Translation in terms of valuation rings)**.** Note that $K^\circ$ is a rank 1 (necessarily nondiscrete) valuation ring, and it is complete with respect to any pseudouniformizer. So an equivalent datum to a perfectoid field $K$ is the datum of a nondiscrete complete rank 1 valuation ring $V$ whose residue field has characteristic $p$, and such that the Frobenius is surjective on $V/p$. Thus, again, we can phrase the definition of a perfectoid field purely algebraically, without referring to topologies or absolute values.

**Definition 72** (Perfectoid rank 1 valuation rings)**.** A *perfectoid rank* 1 *valuation ring* is a rank 1 valuation ring $V$ (of residue characteristic $p$) which is not a DVR, $x$-adically complete with respect to any pseudouniformizer $x \in V$, and such that Frobenius is surjective on $V/p$.

It follows that the datum of a perfectoid field is *equivalent* to that of a perfectoid rank 1 valuation ring. Note that any such is also $p$-adically complete (and $p$ is a pseudouniformizer if $V$ has mixed characteristic). Let's give several examples of perfectoid fields (and the associated valuation rings).

**Example 73.** Any complete perfect nonarchimedean field of characteristic $p > 0$. For example, $\mathbb{F}_p((t^{1/p^\infty}))$ (i.e., the completed perfection of $\mathbb{F}_p((t))$). In this case, the associated valuation ring is the $t$-adic completion of $\mathbb{F}_p[t^{1/p^\infty}]$, the perfect polynomial ring on one generator. A complete nonarchimedean field of characteristic $p$ is perfectoid if and only if it is perfect.

In order to get perfectoid fields of characteristic zero, we need a lot of ramification.

**Example 74.** $\mathbb{Q}_p$ is *not* an example of a perfectoid field. In fact, while the residue field is perfect, the value group is discrete. In fact, the assumption of non-discreteness rules out any finite extension of $\mathbb{Q}_p$.

**Example 75.** Any algebraically closed nonarchimedean field $K$ of characteristic zero is perfectoid. For instance, $\mathbb{C}_p$ is perfectoid. This is because taking $p$th powers, $K^\circ \to K^\circ$ is surjective.

To proceed further (and give smaller examples of perfectoid fields), we need a well-known lemma. Recall that given a complete nonarchimedean field $K$, any finite extension $L$ of $K$ also inherits the structure of a nonarchimedean field: the absolute value extends uniquely to $L$. There is then the question of determining $L^\circ$, in terms of $K^\circ$. First, $L^\circ$ is the integral closure of $K^\circ$ in $K$.

**Lemma 76.** *Let $E$ be a complete, discretely valued field. Let $E^\circ \subset E$ be the powerbounded elements. Consider an* Eisenstein polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, *which means that all the $a_i \in E^{\circ\circ}$ and $a_0$ is a uniformizer. Then: $f(x)$ is irreducible, and $E^\circ[x]/f(x)$ is a complete DVR which gives the ring of powerbounded elements in the field $E[x]/f(x)$. Furthermore, the class of $x$ defines a uniformizer in $E^\circ$.*

**Example 77.** The fields $K_1 = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $K_2 = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ are perfectoid. The associated valuation rings are $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$ and $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$.

For $K_1$, the ring of integers in the former is $K_1^\circ = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$, which is the completed direct limit of the rings $\mathbb{Z}_p[x_n]/(x_n^{p^n} = p)$. Note that $K_1^\circ/p \simeq \mathbb{F}_p[x^{1/p^\infty}]/x$ is semiperfect, so $K_1$ is perfectoid.

For $K_2$, we have that $K_2$ is obtained by adjoining iteratively $p$-power roots of $\zeta_p$ in $\mathbb{Q}_p(\zeta_p)$, whose ring of integers is $\mathbb{Z}_p[\zeta_p]$ (and then completing). It follows that $K_2^\circ/p = (\mathbb{Z}_p[\zeta_p]/p)[x^{1/p^\infty}]/(x = \zeta_p)$. The ring $\mathbb{Z}_p[\zeta_p]/p$ is isomorphic to $\mathbb{F}_p[y]/y^{p-1}$ where $y$ is the image of the class $1 - \zeta_p$. So, $K_2^\circ/p \simeq \mathbb{F}_p[y, x^{1/p^\infty}]/(x = y, y^{p-1})$ which is a quotient of the ring $\mathbb{F}_p[x^{1/p^\infty}]$ and therefore semiperfect, as desired.

**Proposition 78.** *The following are equivalent for a complete nonarchimedean field $K$ with $|p| < 1$:*

1. *$K$ is perfectoid.*
2. *There exists $\omega \in K^\circ$ with $|\omega^p| = |p|$ (so $\omega^p$ is $p$ times a unit), and the Frobenius induces an isomorphism $K^\circ/\omega \simeq K^\circ/\omega^p$. Moreover, we can assume that $\omega$ admits a compatible system of $p$-power roots.*

*Proof.* Suppose $K$ perfectoid. Choose an element $x \in K^\circ$ with $|p| < |x| < 1$. The Frobenius is surjective on $K^\circ/p$, so we can write $x = y^p + pz$ for $z \in K^\circ$. Using the nonarchimedean property, it follows that $|y|^p = |x|$. In particular, any element in the value group in $\mathbb{R}^\times$ between $|p|$ and 1 has a $p$th root. Now applying this to $x, px^{-1}$ and multiplying, we get an element $\omega_0 \in K^\circ$ with $|\omega_0|^p = |p|$. The Frobenius map $K^\circ/\omega_0 \to K^\circ/\omega_0^p = K^\circ/p$ is surjective by assumption, and clearly injective too and hence an isomorphism.

The only issue is that $\omega_0$ need not admit a compatible system of $p$-power roots. Now let's look at $\overline{\omega_0} \in K^\circ/p$. We know that $K^\circ/p$ has surjective Frobenius, so we can find a sequence of liftings $\overline{\omega_i} \in K^\circ/p$ such that $\overline{\omega_{i+1}}^p = \overline{\omega_i}$ (this of course depends on some choices). The sequence $\{\overline{\omega_i}\}$ lives in $\varprojlim_{x \mapsto x^p} K^\circ/p$, which is the same as $\varprojlim_{x \mapsto x^p} K^\circ$ and therefore determines an element $\omega \in K^\circ$ with $\omega \equiv \omega_0 \mod p$. It follows easily that $\omega$ has the right valuation and therefore is $\omega_0$ times a unit. This completes the proof that (1) implies (2). The converse direction is easy: note that the hypothesis that $\omega$ admits all $p$-power roots forces the valuation to be non-discrete if $K$ has characteristic zero. □

**Definition 79.** Let $K$ be a perfectoid field. We say that a *perfectoid pseudo-uniformizer* is a pseudouniformizer $\omega$ of $K^\circ$ together with a chosen system of compatible $p$-power roots $\omega^{1/p^n}, n \geq 0$. We will also assume that $\omega \mid p$. (Previously, we had $|\omega| = |p|^{1/p}$, but we will also want to consider $|\omega| = |p|$ for instance.)

**Proposition 80.** *Let $K$ be a perfectoid field of characteristic zero, and let $\omega \in K^\circ$ be such that $|\omega|^p = |p|$. Then every element of the ring $K^\circ/\omega p$ is a pth power. (Note that this ring is not characteristic p!)*

*Proof.* Let $x \in K^\circ$, and write $x = y^p + \omega^p z$ for some $y, z \in K^\circ$ (which we can by assumption: $\omega^p$ is $p$ times a unit). Furthermore, we can write $z = z_1^p + \omega^p z_2$ for the same reason, so that

$$x = y^p + \omega^p z_1^p + \omega^{2p} z_2 \equiv (y + \omega z_1)^p \mod p\omega.$$

□

The main basic construction involving perfectoid fields is that there is a canonical way of associating to them a perfect(oid) characteristic $p$ field. First, we do this at the level of valuation rings.

**Construction 81** (The tilt $K^{\circ\flat}$)**.** Let $K$ be a perfectoid field. Let $K^\circ$ be the subring of power-bounded elements. By assumption, $K^\circ$ is a rank $1$ valuation ring which is $x$-adically complete with respect to any pseudouniformizer $x \in K^\circ$ (cf. Remark 71). This means that $K^\circ$ in particular is $p$-adically complete (it's possible that $p = 0$, so is not a pseudouniformizer, but then everything is trivially $p$-complete).

Therefore, we form the tilt $K^{\circ\flat}$, which is a perfect ring of characteristic $p$. Recall that as rings

$$K^{\circ\flat} = \varprojlim_{x \mapsto x^p} K^\circ/p$$

and as monoids

$$K^{\circ\flat} = \varprojlim_{x \mapsto x^p} K^\circ.$$

(We already used the equivalence of these constructions in producing the perfectoid pseudo-uniformizer $\omega$.)

**Remark 82.** If $K$ is already of characteristic $p$, then canonically $K^\circ = K^{\circ\flat}$, i.e, tilting doesn't do anything.

**Theorem 83.** (1) *The tilt $K^{\circ\flat}$ is a rank 1 valuation ring, perfect of characteristic $p$, which is complete for any pseudouniformizer.*
(2) *Suppose $K$ has characteristic zero. Choose a perfectoid pseudo-uniformizer $\omega \in K^\circ$ such that $|\omega| = |p|$. The choice of $p$-power roots $\omega^{1/p^n} \in K^\circ$ determines element $\omega^\flat \in K^{\circ\flat}$: then $\omega^\flat$ is a pseudouniformizer.*
(3) *We have an isomorphism*

$$K^\circ/\omega \simeq K^{\circ\flat}/\omega^\flat.$$

*Proof.* Recall that $K^{\circ\flat} = \varprojlim_{x\mapsto x^p} K^\circ/p$, and as a multiplicative monoid, $K^{\circ\flat} = \varprojlim_{x\mapsto x^p} K^\circ$. Since $K^\circ$ is preordered under divisibility (as a valuation ring), and is an integral domain, it is easy to see that $K^{\circ\flat}$ is too. Moreover, given elements $x, y \in K^{\circ\flat}$, which we represent as $p$-power compatible sequences $\{x_n\}, \{y_n\} \in K^\circ$, then we have $x \mid y$ if and only if $x_0 \mid y_0$ (which forces $x_n \mid y_n$ for each $n$). It follows easily that if $x, y$ are two nonzero nounits in $K^{\circ\flat}$, then $x$ divides some power of $y$. Thus, $K^{\circ\flat}$ has rank 1. Let $\omega^\flat$ be the element as in (2), depending on the choice of $\left\{\omega^{1/p^n}\right\}$. It then follows that $\omega^\flat$ is a pseudouniformizer in $K^{\circ\flat}$.

We have a natural forgetful map

$$K^{\circ\flat} \to K^\circ/p$$

since the former is the inverse limit perfection of the former. It is surjective, since the ring $K^\circ/p$ is semiperfect. Moreover, it carries the class $\omega^\flat$ to zero, since $\omega$ is $p$ times a unit. Conversely, let's say we have a class $x$ in $K^{\circ\flat}$ which maps to zero in $K^\circ/p$. Explicitly, the class determines a sequence of elements $\{x_n\}_{n\geq 0}$ in $K^\circ$ compatible under $p$th powers. The condition that $x \mapsto 0$ in $K^\circ/p$ means that $p \mid x_0$, or equivalently that $\omega \mid x_0$ and then that $\omega^{1/p^n} \mid x_n$ for each $n$. In view of this, we get that $\omega^\flat \mid x$. □

**Definition 84** (Tilting of perfectoid fields)**.** Let $K$ be a perfectoid field. We let $K^\flat$ be the quotient field of the perfectoid rank 1 valuation ring $K^{\circ\flat}$.

**Remark 85.** As a multiplicative monoid, we still have

$$K^\flat = \varprojlim_{x\mapsto x^p} K, \tag{6}$$

which we derive from the formula $K^{\circ\flat} = \varprojlim_{x\mapsto x^p} K^\circ$ by inverting pseudouniformizers on both sides.

**Example 86.** What are the tilts of the perfectoid fields $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}, \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$?

Let's start with the second one. The valuation ring is the $p$-completion $\mathbb{Z}_p[x^{1/p^\infty}]/(x = p)$. When we reduce modulo $p$, that's the ring $\mathbb{F}_p[x^{1/p^\infty}]/x$ and when we take the perfection of that, it is the ring $\widehat{\mathbb{F}_p[x^{1/p^\infty}]}$. It follows that the tilt is precisely $\mathbb{F}_p\widehat{((x^{1/p^\infty}))}$. Here we can take as $x$ the sequence $(p, p^{1/p}, p^{1/p^2}, \dots,) \in \varprojlim_{x\mapsto x^p} \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$.

What about the first one? We saw that the ring of integers is $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$. As a ring, this is the completion of

$$\mathbb{Z}_p[x^{1/p^\infty}]/((x^p-1)/(x-1)).$$

The quotient modulo $p$ is the ring $\mathbb{F}_p[y^{1/p^\infty}]/(y-1)^{p-1}$ where $y = \zeta_p - 1$. When we form the inverse limit perfection, quotienting by a nilpotent ideal does not affect anything so we can quotient by $y$. Unwinding the definitions, we get exactly the same tilt $\mathbb{F}_p\widehat{((y^{1/p^\infty}))}$. Here explicitly $y$ can be chosen such that $y+1$ is the element $(\zeta_p, \zeta_{p^2}, \dots)$.

In particular, the process of tilting is *lossy*: two different perfectoid fields in characteristic zero can have the same tilt to characteristic $p$. When we fix a perfectoid base, we will see that the tilt becomes an equivalence.

**Proposition 87.** *Let $K$ be a perfectoid field. As a multiplicative monoid, we have $K^\flat = \varprojlim_{x \mapsto x^p} K$ as in* (6)*. This defines a map*

$$\sharp : K^\flat \to K$$

*Moreover, the absolute value on $K^\flat$ is given by $|x|_{K^\flat} = \left|x^\sharp\right|_K$ for $x \in K^\flat$.*

*Proof.* Let $\omega$ be a perfectoid pseudo-uniformizer for $K$, with tilt $\omega^\flat$. To understand the absolute value on $K^\flat$, note that the construction $x \mapsto \left|x^\sharp\right|_K, K^\flat \to \mathbb{R}_{\geq 0}$ is multiplicative and sends $t$ to $|\omega| \in (0,1)$. To complete the proof, we need to show that this construction satisfies the nonarchimedean property. That is, we need that for $x, y \in K^\flat$,

$$\left|(x+y)^\sharp\right|_K \leq \sup(\left|x^\sharp\right|_K, \left|y^\sharp\right|_K).$$

For this, multiplying by a (fractional) power of $t$, we reduce to the case where $x, y$ have absolute value $\leq 1$, and it suffices to show that $x+y$ have absolute value $\leq 1$. This is the assertion that $K^{\circ\flat}$ is closed under sums, but we know it is a ring. □

Thus, there is a procedure for taking a perfectoid field and producing a perfectoid field of characteristic $p$. The main theorem is the following.

**Theorem 88** (Fontaine-Wintenberger, Scholze, Kedlaya-Liu)**.** *Let $K$ be a perfectoid field. Then there is a canonical isomorphism of Galois groups* $\mathrm{Gal}(K) \simeq \mathrm{Gal}(K^\flat)$.

*Explicitly, there is an equivalence of categories between finite extensions of $K$ and $K^\flat$. The functor is as follows: a finite extension $L$ of $K$ is perfectoid, and one considers $L^\flat$, which is a finite extension of $K^\flat$.*

One of the first main goals of the course is to prove the above result, and its generalization to perfectoid $K$-algebras.

**Example 89.** There is an isomorphism between the Galois groups of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))$. This uses two facts:

(1) The Galois group of $\mathbb{Q}_p(p^{1/p^\infty})$ (which is not complete as a valued field but is *henselian*) is the same as that of its completion $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$.
(2) The Galois group of $\mathbb{F}_p((t))$ is the same as that of its perfection (which is henselian but not complete) and its completed perfection $\mathbb{F}_p\widehat{((t^{1/p^\infty}))}$

## 5. Primitive elements

First, let's explain the functor in the other direction (inverse to tilting), and review some facts from the first week.

Recall that for a $p$-complete ring $R$, we have the Fontaine map

$$\theta : W(R^\flat) \to R.$$

What does it do? First, recall that $W(R^\flat)$ is supposed to be "power series in $p$" over $R^\flat$: more precisely, we have a multiplicative map

$$[\cdot] : R^\flat \to W(R^\flat),$$

and every element $v$ of $W(R^\flat)$ admits a unique expansion of the form

$$v = \sum_{i \geq 0} p^i [x_i], \quad x_i \in R^\flat.$$

Moreover, we have the multiplicative *sharp map*

$$\sharp : R^\flat \to R$$

which uses the identification $R^\flat \simeq \varprojlim_{x \mapsto x^p} R$ and is then projection on the first factor. Unwinding the definitions, the Fontaine map $\theta$ is defined via

$$\theta(\sum_{i \geq 0} p^i [x_i]) = \sum_{i \geq 0} p^i x_i^\sharp.$$

The following result is crucial.

**Proposition 90.** *Let $K$ be a perfectoid field. The map*

$$\theta : W(K^{\circ\flat}) \to K^\circ$$

*is surjective, and its kernel is generated by a nonzerodivisor.*

*Proof.* To see that the map is surjective, it suffices to check mod $p$ (since everything is $p$-complete). Mod $p$, we obtain the map

$$(K^\circ/p)^{\mathrm{perf}} \to K^\circ/p$$

and since $K^\circ/p$ is *semiperfect* (i.e., Frobenius surjective) this map is also surjective.

Let's consider the kernel of $\theta$. Since the characteristic $p$ case is straightforward, we may assume $K$ has characteristic zero. Consider an element $\omega \in K^\circ$ admitting a system of $p$-power roots, so that $\omega = t^\sharp$ for $t \in K^{\circ\flat}$ and such that $\omega = pu$ for $u$ a unit. In particular, $\omega$ is a perfectoid pseudo-uniformizer.

Choose an element $\widetilde{u} \in W(K^{\circ\flat})$ such that $\theta(\widetilde{u}) = u$; $\widetilde{u}$ is also forced to be a unit. Then the element

$$[t] - p\widetilde{u} \in W(K^{\circ\flat})$$

has the property that it is in the kernel of $\theta$. I claim that it generates the kernel, i.e., that the map $W(K^{\circ\flat})/([t] - p\widetilde{u}) \to K^\circ$ is an isomorphism. In fact, when one reduces mod $p$, one gets the map $K^{\circ\flat}/t \to K^\circ/p$ which one knows is an isomorphism. So one needs to know that $W(K^{\circ\flat})/([t] - p\widetilde{u})$ is $p$-torsion-free. (Except in the case $t = 0$, in which case $K$ has characteristic $p$; this case is straightforward.) In fact, if we have an equation in $W(K^{\circ\flat})$,

$$px = ([t] - p\widetilde{u})z,$$

then it is easy to see that $z$ has itself to be divisible by $p$. Then $x = ([t] - p\widetilde{u})(z/p)$, so $x$ also maps to zero modulo $([t] - p\widetilde{u})$. This argument easily shows that $W(K^{\circ\flat})/([t] - p\widetilde{u})$ is $p$-torsion-free, as desired. □

**Definition 91.** Let $V$ be a perfectoid rank 1 valuation ring of characteristic $p$. An element of $W(K^{\circ\flat})$ of the form $\sum_{i\geq 0} p^i[x_i]$ is called *primitive of degree* 1 if:

(1) $x_0$ is a nonunit (in particular, topologically nilpotent) in $K^{\circ\flat}$.
(2) $x_1$ is a unit.

We just saw that if $K$ is a perfectoid field, then $K^{\circ\flat}$ is a perfectoid valuation ring of rank 1, and the map $W(K^{\circ\flat}) \to K^\circ$ has kernel generated by a primitive element of degree one.

**Remark 92.** Let $V$ be a perfectoid valuation ring of characteristic $p$, and let $x = \sum_{i\geq 0} p^i[x_i]$ be an element of $W(K^{\circ\flat})$. Let $k$ be the residue field of $V$. Then $x$ is primitive of degree one if and only if $x$ maps to $p$ times a unit in $W(k)$.

Let's do two examples.

**Example 93.** Consider the case where $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. In this case, in the ring $K^{\circ\flat}$, we have the element $p^\flat \in K^{\circ\flat}$ identified with the sequence $(p, p^{1/p}, p^{1/p^2}, \dots)$. The element $p - [p^\flat] \in W(K^{\circ\flat})$ and clearly maps to zero .

**Example 94.** Consider the case where $K = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$. We have an element $\epsilon \in K^\flat$ given by the sequence $(1, \zeta_p, \zeta_{p^2}, \dots)$. Consider the element

$$\xi = \frac{[\epsilon] - 1}{[\epsilon^{1/p}] - 1} = 1 + [\epsilon^{1/p}] + [\epsilon^{2/p}] + \cdots + [\epsilon^{(p-1)/p}].$$

Note that under the map $\theta : W(K^{\circ\flat}) \to K^\circ$, $[\epsilon^{1/p}] \mapsto \zeta_p$. Therefore, $\xi$ is in the kernel of $\theta$. The residue field of $K^{\circ\flat}$ is $\mathbb{F}_p$, and $\epsilon$ maps to 1 in $\mathbb{F}_p$. Thus, $\xi$ maps to $p$.

**Theorem 95.** *Let $V$ a perfectoid rank $1$ valuation ring of characteristic $p$. Suppose that $v = \sum_{i\geq 0} p^i[v_i]$ is a primitive element in $W(V)$. Then $V^\sharp = W(V)/v$ is a perfectoid rank $1$ valuation ring, corresponding to a perfectoid field, whose tilt is given by $V$.*

*Proof.* First, we see directly by hand that $p$ is a nonzerodivisor (assuming the primitive element is nonzero mod $p$) as earlier. Note that the class of $[v_0]$ mod $v$ is equal to $p$ times a unit. The image of $[v_0]$ in $V^\sharp = W(V)/v$ has all $p$-power roots, call them $\omega^{1/p^n}, n \geq 0$. Note that $V/v_0 = V^\sharp/\omega$.

Finally, suppose given $x, y \in V^\sharp$. We want to say that $x \mid y$ or $y \mid x$. To do this, we will show that we can write $x = [x']u$ for $x' \in V^\flat, u$ a unit. Similarly for $u$. Then we will get the valuation ring property. In fact, without loss of generality we can assume $x$ not divisible by $p$, since $p$ is $[v_0]$ times a unit. Write $x = \sum_{i\geq 0}[x_i]p^i$ where $x_0$ must not be divisible by $v_0$. Then $x$ is divisible by $x_0$, and the quotient is a unit (1 plus a topologically nilpotent). □

A consequence of this is that given a perfectoid valuation ring of rank 1, the *untilts* correspond to ideals in $W(V)$ which are generated by a primitive element of degree one.

**Corollary 96.** *Let $K$ be a perfectoid field. There is an equivalence of categories between perfectoid fields over $K$ and perfectoid fields containing $K^\flat$. The equivalence sends a perfectoid field $L/K$ to $L^\flat$. In the reverse direction, if $E/K^\flat$ is a perfectoid field, we form $E^\sharp := \mathrm{Frac}(W(E^\circ \otimes_{W(K^{\circ\flat})} K^\circ))$.*

This doesn't get us that far: note that part of the theorem we want to show is that any finite extension of $K$ is perfectoid. This is easy in characteristic $p$, but more subtle in characteristic zero.

## 6. Banach algebras

Let $K$ be a complete nonarchimedean field. (Often, it will be convenient to assume $K$ perfectoid, or at least nondiscretely valued.)

**Definition 97.** A *$K$-Banach algebra* is a $K$-algebra $A$, equipped with a norm $|\cdot| : A \to \mathbb{R}_{\geq 0}$ such that:

(1) $|x + y| \leq \sup(|x|, |y|)$ for $x, y \in A$. Moreover, $|x| = 0$ if and only if $x = 0$, and $|1| = 1$.
(2) For $x, y \in A$, we have $|xy| \leq |x|\,|y|$. Moreover, we have *equality* if $x \in K$.
(3) $A$ is complete as a metric space with respect to the distance function $d(x, y) = |x - y|$.

Note in particular that $A$ becomes a topological ring (and it is linearly topologized). A basis of neighborhoods near zero is given by $\{x \in A : |x| \leq \epsilon\}$ for $\epsilon > 0$.

A morphism of Banach $K$-algebras $A \to B$ is a map of $K$-algebras which is continuous; equivalently, bounded with respect to the norms on $A, B$.

Where do Banach $K$-algebras come from? Here's the basic construction.

**Construction 98.** Let $K^\circ \subset K$ be the ring of powerbounded elements. Let $A^0$ be a $K^\circ$-algebra which has the property that for some (hence any) pseudouniformizer $x \in K^\circ$, $A^0$ is $x$-adically complete and $x$-torsion-free. Then $A := A^0[1/x] = A^0 \otimes_{K^\circ} K$ becomes a $K$-Banach algebra under the norm

$$|a| = \inf_{t \in K^\times : ta \in A^0} \frac{1}{|t|}.$$

That is, $A^0 \subset A$ is determined as the subring of elements of norm $\leq 1$.

Conversely, given a $K$-Banach algebra $B$, we can recover $B$ via this construction by setting $B^0 = \{x \in B : |x| \leq 1\}$. In this case, note that $B^0$ is automatically $x$-adically complete and $x$-torsion-free.

**Remark 99.** Let $A^0 \to B^0$ be a map of $x$-adically complete, $x$-torsion-free $K^\circ$-algebras. Then we obtain a map of $K$-Banach algebras $A \to B$ from the above construction. However, it is not true that a map $A \to B$ necessarily gives rise to a map $A^0 \to B^0$.

**Example 100.** Let $A^0 \subset B^0$ be an inclusion of $x$-adically complete, $x$-torsion-free $K^\circ$-algebras ($x$ a pseudouniformizer). Suppose that there exists $N$ such that $x^N B^0 \subset A^0$. Then the associated Banach algebras $A, B$ are identified by the induced map $A \to B$ which is continuous with a continuous inverse (but not norm-preserving).

**Example 101** (Topologically finite type algebras)**.** This is the example of interest for *rigid analytic geometry* over $K$. The *Tate algebra* $T_1$ is the $K$-Banach algebra $T_1 = K\langle X\rangle$ obtained from the completion $\widehat{K^\circ[X]}$ by inverting a pseudouniformizer, as in Construction 98. This is to be interpreted as the ring of all power series with coefficients in $K$, say

$$\sum_{i \geq 0} a_i X^i, \quad \text{such that } a_i \to 0 \text{ as } i \to \infty,$$

which means that it converges on the unit disk of $K$, i.e., $K^\circ$. The Banach norm on $K\langle X\rangle$ can also be described via

$$\left|\sum_{i \geq 0} a_i X^i\right| = \sup_{i \geq 0} |a_i|.$$

More generally, we have the Tate algebra $T_n$ for each $n$, which should be intepreted as functions on the $n$-disk over $K$. A $K$-algebra is said to be *topologically finite type* if it is a quotient of some $T_n$. Any tft $K$-algebra has the structure of a $K$-Banach algebra (the norm is not canonical, but the Banach algebra structure is), and any morphism of $K$-algebras is automatically a morphism of $K$-Banach algebras. These are important foundational results for rigid analytic geometry.

Let $A$ be a $K$-Banach algebra. Since $A$ is a linearly topologized topological ring, we have notions such as powerbounded and topologically nilpotent elements, i.e., we have a subring $A^\circ \subset A$, and an ideal $A^{\circ\circ} \subset A^\circ$. We see easily that

$$\{x \in A : |x| \leq 1\} \subset A^\circ, \{x \in A : |x| < 1\} \subset A^{\circ\circ}$$

but in general these inclusions are not strict.

**Example 102.** Suppose $A$ is a $K$-Banach algebra with a nonzero nilpotent element $u$. Then, any $K$-multiple of $u$ is powerbounded (and topologically nilpotent), but the absolute value $|\cdot|$ can be arranged to be as large as possible.

A crucial property of $K$-Banach algebras that prevents this type of pathology is *uniformity.*

**Definition 103.** A $K$-Banach algebra $A$ is called *uniform* if the subring $A^\circ \subset A$ of powerbounded elements is bounded.

**Remark 104.** If $A$ is uniform, then $A$ has no nonzero nilpotent elements. This is because of the issue observed in Example 102. However, this is not an if and only if. Consider the polynomial ring $R = \mathbb{Z}[X_1, X_2, \dots]$ in infinitely many variables; we consider this as a graded ring with $|X_i| = 1$ for all $i$. Consider the (graded) subring $R' \subset R$ generated by all elements of degree $\geq 2$ as well as the elements $p^i X_i, i \geq 1$. Then $\hat{R}'_p[1/p]$ has the structure of a $\mathbb{Q}_p$-Banach algebra with $\hat{R}'$ an open bounded subring. Note however that all the $X_i \in \hat{R}'[1/p]$ are power-bounded, but the set $\{X_i\}$ is not bounded by construction. Thus, $\hat{R}'_p[1/p]$ is not uniform.

Let $A$ be a $K$-Banach algebra. Note that $A^\circ \subset A$ always contains the unit disk at the origin. So, $A^\circ$ is an open subring. If $A^\circ$ is also bounded (i.e., $A$ uniform), then it follows that $A^\circ$ is, as $K^\circ$-algebra, $x$-adically complete for any pseudouniformizer $x \in K^\circ$, and we have $A = A^\circ[1/x]$. In other words, we are in the situation of Example 100. In particular, we can define a new Banach norm on $A$ in terms of $A^\circ$, such that $A^\circ$ is exactly those elements of norm $\leq 1$. Now we give another description of it.

**Definition 105** (The spectral norm)**.** Let $A$ be a uniform Banach algebra over a perfectoid field $K$, with some norm $|\cdot|$. We define the *spectral norm* $|\cdot|_{\mathrm{sp}}$ via

$$|x|_{\mathrm{sp}} = \lim_{n\to\infty} |x^n|^{1/n} = \inf_n |x^n|^{1/n}.$$

This limit exists (and agrees with the infimum) as in the standard theory of Banach algebras, and it is zero precisely if $x$ is topologically nilpotent.

**Proposition 106.** *Let $A$ be a uniform Banach algebra over a perfectoid field $K$. Fix a pseudouniformizer $\omega \in K$ with $p$-power roots $\omega^{1/p^n}$.*

*Then given $x \in A$, the following are equivalent:*

(1) *$x \in A^\circ$ (i.e., $x$ is powerbounded).*
(2) $|x|_{\mathrm{sp}} \leq 1$.
(3) *For each $n$, $\omega^{1/p^n} x$ is topologically nilpotent.*

*Then $x \in A^\circ$ if and only if $|x|_{\mathrm{sp}} = \lim_{n\to\infty} |x^n|^{1/n} \leq 1$: that is, if the spectral norm is at most* 1.

*Proof.* Clearly (1) implies (2) implies (3), so it suffices to show that (3) implies (1).

By assumption, for each $n > 0$, $\omega^{1/p^n} x$ is topologically nilpotent and therefore is in $A^\circ$. Thus for any $i \leq p^n$, we have (raising to the $i$th power) $\omega^{i/p^n} x^i \in A^\circ$, so $x^i \in \omega^{-1} A^\circ$. Since was arbitrary, it follows that $x$ was actually powerbounded: all of its powers are in $\omega^{-1} A^\circ$, and hence $x \in A^\circ$ as desired. □

By Proposition 106, an element of $A$ has norm $\leq 1$ with respect to the spectral norm if and only if it is powerbounded.

**Corollary 107.** *If $A$ is a uniform Banach algebra over the perfectoid field $K$, then $|\cdot|_{\mathrm{sp}}$ is an equivalent Banach norm for $A$.*

*Proof.* This follows from the fact that $\left\{x : |x|_{\mathrm{sp}} \leq 1\right\}$ has just been shown to be an open bounded subring. □

**Remark 108.** Note that for any continuous $f : A \to B$ of uniform $K$-Banach algebras, $f$ has norm $\leq 1$ with respect to the spectral norm. This follows because $f$ preserves powerbounded elements.

The key advantage of uniform Banach $K$-algebras $A$ is that they admit this canonical norm $|\cdot|_{\mathrm{sp}}$, or equivalently the canonical subring $A^\circ \subset A$. In fact, it follows that uniform $K$-Banach algebras admit a purely algebraic description. This is explained in detail in Bhatt's lecture notes.

Let $K$ be a perfectoid field with a perfectoid pseudo-uniformizer $\omega$ and roots $\omega^{1/p^n}, n \geq 0$. Let $R$ be a $K^\circ$-algebra. We ask when $R$ arises as the powerbounded elements $A^\circ$ in a uniform $K$-Banach algebra $A$.

**Proposition 109** (Characterization of uniform Banach algebras)**.** *Suppose $R = A^\circ$ for $A$ a uniform Banach algebra over $K$. Then:*

(1) *$R$ is $\omega$-adically complete and $\omega$-torsion-free.*
(2) *$R$ is* saturated *in the sense that if $x \in R[1/\omega]$ and $\omega^{1/p^n} x \in R$ for all $n > 0$, then $x \in R$.*
(3) *$R$ is* $p$-root closed*: if $x \in R[1/\omega]$ and $x^p \in R$, then $x \in R$.*

*Conversely, if $R$ is a $K^\circ$-algebra as above, then the associated $K$-Banach algebra $A = R[1/\omega]$ is uniform, and $R = A^\circ$.*

*Proof.* It's easy to see that if $R = A^\circ$ for $A$ a uniform Banach algebra, then $R$ satisfies the above three properties. Now suppose $R$ is a $K^\circ$-algebra which satisfies the above conditions. Set $A = R[1/\omega]$ with the induced Banach structure (such that $R$ yields the elements of norm $\leq 1$). The claim is that $R$ is precisely the powerbounded elements $A^\circ$. Clearly $R \subset A^\circ$, but we need to go the other way.

In fact, suppose $a \in A^\circ$; we need to show that $a \in R$. Since $R$ is saturated, it suffices to show that for each $n$, $\omega^{1/p^n} a \in R$. In fact, we know this element is topologically nilpotent, so $(\omega^{1/p^n} a)^{p^N} \in R$ for $N \gg 0$. Since $R$ is assumed $p$-root closed, we find that $\omega^{1/p^n} a \in R$ as desired; since $n$ was arbitrary then we get that $a \in R$. □

Next, we want to define perfectoid algebras. Fix a perfectoid field $K$. Let $\omega \in K$ be a perfectoid pseudo-uniformizer (with $p$-power roots $\omega^{1/p^n}$) such that $p$ is divisible by $\omega$ in $K^\circ$ (in particular, $K^\circ/\omega$ is an $\mathbb{F}_p$-algebra).

**Definition 110** (Perfectoid $K$-algebras)**.** A Banach $K$-algebra $A$ is *perfectoid* if:

(1) $A$ is uniform.
(2) The Frobenius on $A^\circ/\omega$ is surjective.

**Remark 111.** The second condition does not depend on the choice of $\omega$. In fact, it is equivalent to state that $A^\circ/p$ is semiperfect. The reason is that if $A^\circ/\omega$ has surjective Frobenius, then for any $a \in A^\circ$, we can write $a = a_1^p + \omega b$ for $b \in A^\circ$. Continuing, we write $b = b_1^p + \omega c$ and

$$a = a_1^p + (\omega^{1/p} b_1)^p + \omega^2 c.$$

Continuing in this way, since everything is $\omega$-adically complete, we can write $a$ is an infinite sum of $p$th powers in $A^\circ$ (which are tending to zero). Of course, mod $p$, that means that $a$ itself is a $p$th power.

The first main theorem in this course that we want to prove is the following.

**Theorem 112** (Tilting equivalence – Scholze, Kedlaya-Liu )**.** *Let $K$ be a perfectoid field. There is a tilting equivalence between the categories of perfectoid $K$-algebras and perfectoid $K^\flat$-algebras. Given a perfectoid $K$-algebra $R$, the étale fundamental group (and étale cohomology) of $R, R^\flat$ are identified.*

Before going further, we want to give some examples of perfectoid algebras, and give the purely algebraic characterization.

**Proposition 113.** *Let $K$ be a perfectoid field, and let $\omega \in K^\circ$ be a perfectoid pseudo-uniformizer as above, so that $\omega \mid p$. Let $A$ be a perfectoid $K$-algebra. Then the Frobenius $A^\circ/\omega^{1/p} \to A^\circ/\omega$ is an isomorphism.*

*Proof.* By assumption it is surjective. Let $z \in A^\circ$; suppose $z^p \in \omega A^\circ$. It follows that $z/\omega^{1/p} \in A$ has the property that its $p$th power is powerbounded, and hence it must be powerbounded itself. Thus, $z \in \omega^{1/p} A^\circ$ as desired. $\square$

Fix a perfectoid field $K$, and let $\omega$ be a perfectoid pseudo-uniformizer in $K^\circ$ with $\omega \mid p$, as above. We can axiomatize the above condition.

**Definition 114** (Integral perfectoid $K^\circ$-algebras)**.** Let $R$ be a $K^\circ$-algebra which is $\omega$-torsion-free and $\omega$-adically complete. Suppose that the map $R/\omega^{1/p} \to R/\omega$ is an isomorphism. Then $R$ is called *integral perfectoid.*

**Example 115** (The perfectoid unit disc)**.** Consider the completed algebra $K^\circ \left\langle X^{1/p^\infty} \right\rangle$ obtained as the $\omega$-adic completion of $K^\circ[X^{1/p^\infty}] = \bigcup_n K^\circ[X^{1/p^n}]$. Then it's easy to see that this algebra is integral perfectoid.

Before going further, we want to see how integral perfectoid algebras give rise to perfectoid $K$-algebras.

**Proposition 116.** *If $R$ is integral perfectoid, then $R$ is $p$-root closed in $R[1/\omega]$.*

*Proof.* Suppose $x \in R[1/\omega]$ and $x^p \in R$. We have $\omega^{i/p} x \in R$ for some $i > 0$. Now $(\omega^{i/p} x)^p \in \omega R$ (because $x^p \in R$), so $\omega^{i/p} x \in \omega^{1/p} R$ by integral perfectoidness. Thus, $\omega^{(i-1)/p} x \in R$ too. Inducting downwards, we get that $x \in R$. $\square$

Thus if $R$ is integral perfectoid, $R$ satisfies most of the conditions to arise as the powerbounded elements in a uniform $K$-algebra. However, saturatedness is not automatic.

**Example 117.** Let $K$ be a perfectoid field as above, and let $k$ be the residue field of $K$, i.e., $k = K^\circ / K^{\circ\circ}$. Then the fiber product

$$K^\circ \times_k K^\circ$$

has the property that it is integral perfectoid (exercise!). However, it is not saturated.

**Definition 118** (The saturation)**.** Let $R$ be an $\omega$-torsion-free $K^\circ$-algebra. We let $R_*$, the ring of *almost elements* of $R$, denote the subring of $R[1/\omega]$ consisting of those elements $x$ such that $\omega^{1/p^n} x \in R$ for all $n > 0$. (Note that this is a ring!) Saturation is an idempotent procedure on $\omega$-torsion-free $K^\circ$-algebras.

**Proposition 119.** *If $R$ is integral perfectoid, then the subring $R_* \subset R[1/\omega]$ is integral perfectoid as well.*

*Proof.* First, note that $R_* \subset R[1/\omega]$ is $p$-root closed. This is easy to see. If $x \in R[1/\omega]$ and $x^p \in R_*$, then $\omega^{1/p^n} x^p = (\omega^{1/p^{n-1}})^p \in R$; since $R$ is $p$-root closed we get that $\omega^{1/p^{n-1}} x \in R$; since $n$ was arbitrary we get $x \in R_*$.

Suppose $x \in R_*$. For each $n$, we can write $\omega^{1/p^n} x = y^p + \omega z$ for $y, z \in R$. Dividing, we get that $x = (y/\omega^{1/p^{n+1}})^p + \omega^{1-1/p^n} z$. Necessarily, we have $y/\omega^{1/p^{n+1}} \in R_*$ since we just saw that $R_*$ is $p$-root closed. Thus, the $p$th power map $R_*/\omega^{1/p} \to R_*/\omega^{1-1/p}$ is surjective. This implies that $R_*/\omega$ is semiperfect by a successive approximation argument.

One checks easily that if $x \in R_*$ and $x^p \in \omega R_*$, then $x \in \omega^{1/p} R_*$, as desired. Finally, $R_*$ is between $R$ and $(1/\omega)R$, so it is $\omega$-adically complete. This proves $R_*$ is integral perfectoid. $\square$

The following is the main result relating integral perfectoid to perfectoid algebras.

**Proposition 120.** *Let $R$ be an integral perfectoid $K^\circ$-algebra. Consider $A = R[1/\omega]$ with the Banach structure inherited from R; then $A$ is a perfectoid $K$-algebra. Moreover, $A^\circ = R_*$.*

*Proof.* Without loss of generality, we can assume $R = R_*$ is saturated. Then $R$ is also $p$-root closed, and we can apply the characterization of uniform Banach algebras (Proposition 109) to see that $R = A^\circ$ for $A = R[1/\omega]$ as desired. $\square$

Thus, we have an *equivalence of categories* between perfectoid $K$-algebras and saturated integral perfectoid $K^\circ$-algebras.

**Example 121** (The characteristic $p$ case)**.** Suppose $K$ has characteristic $p$. Then:

(1) A $K^\circ$-algebra $R$ is integral perfectoid if and only if $R$ is $\omega$-complete, $\omega$-torsion-free, and perfect. In fact, we already showed that if $R$ is integral perfectoid, then $R/p = R$ is semiperfect. So it suffices to show that $R$ is reduced, which follows because it sits inside a uniform Banach algebra. But it's very explicit: if $x \in R$ with $x^p = 0$, then $x \in \omega^{1/p}R$ by assumption. But by torsion-freeness, we get $x/\omega^{1/p} \in \omega^{1/p}R$ as well — continuing, we get that $x$ is arbitrarily divisible by $\omega$, and hence $x = 0$.

    Conversely, if $R$ is perfect and $\omega$-adically complete, $\omega$-torsion-free, then it's easy to see that $R/\omega^{1/p} \to R/\omega$ is both injective and surjective, as desired.

(2) (André): Let $A$ be a $K$-Banach algebra. Then $A$ is perfectoid if and only if $A$ is perfect. In particular, uniformity is automatic. This relies on an argument with Banach's open mapping theorem.

    The argument is as follows. Let $A_0 \subset A$ be an open bounded subring. The Frobenius $\phi : A \to A$ is a continuous isomorphism, so it is open (by the open mapping theorem). Thus $\phi(A_0) \supset \omega^N A_0$ for some $N \gg 0$. In particular, $A_0 \supset \omega^{N/p}\phi^{-1}(A_0)$, or equivalently $\phi^{-1}(A_0) \subset \omega^{-N/p}A_0$. For each $i$, we get by induction
$$\phi^{-i}(A_0) \subset \omega^{-(N/p+N/p^2+\cdots+N/p^i)}A_0.$$
    Since the geometric series converges, it follows that the subring
$$\bigcup_{i\geq 0}\phi^{-i}(A_0) \subset A$$
    (which contains $A_0$) is actually bounded, and hence a ring of definition. Thus, $A$ has a ring of definition which is perfect (since this ring is also just $(A_0)_{\mathrm{perf}}$).

## 7. The tilting equivalence

Our next goal is to explain the proof of the following result.

**Theorem 122.** *Let $K$ be a perfectoid field. Then any finite extension $L/K$ is perfectoid, and under the construction $L \mapsto L^\flat$ the categories of finite extensions of $K$ and $K^\flat$ are identified (via an equivalence which preserves degrees). In particular, the Galois groups of $K, K^\flat$ are identified.*

**Proposition 123.** *Let $K$ be a perfectoid field. Suppose $K^\flat$ is algebraically closed. Then $K$ is algebraically closed.*

*Proof.* Suppose $K^\flat$ is algebraically closed; we show that $K$ is also algebraically closed. Let $\omega \in K^\circ$ be a perfectoid pseudo-uniformizer with compatible system of $p$-power roots, which corresponds to an element $\omega^\flat \in K^{\circ\flat}$, so that we have the usual formula $K^\circ/\omega = K^{\circ\flat}/\omega^\flat$.

From this, we conclude that the value group of $K$ is the same as the value group of $K^\circ$. This is because the value group of $K$ (resp. $K^\circ$) is generated by the absolute values of elements whose absolute values are between $|\omega| = |\omega^\flat|$ and 1. In particular, the absolute value group of $K$ is a $\mathbb{Q}$-vector space. Finally, any monic polynomial in $K^\circ$ admits a root modulo $\omega$. Applying the next result, we conclude. $\square$

**Proposition 124.** *Let $K$ be a complete NA field. Suppose that:*

(1) *The image of $|\cdot| : K^\times \to \mathbb{R}_{>0}$ is a $\mathbb{Q}$-vector space.*
(2) *There exists a pseudouniformizer $\omega \in K$ such that any monic polynomial in $K^\circ$ has a root in $K^\circ/\omega$.*

*Then $K$ is algebraically closed.*

*Proof.* Let $f(x) \in K^\circ[x]$ be a monic irreducible polynomial of some degree $d$. It suffices to show that $f$ has a root in $K$.

Fix $\epsilon > 0$ and suppose that we have $\alpha \in K^\circ$ with $|f(\alpha)| = \epsilon$. Then we claim that there exists $\alpha' \in K^\circ$ such that:

(1) $|\alpha - \alpha'| \leq \epsilon^{1/d}$.
(2) $|f(\alpha')| \leq \epsilon|\omega|$.

First, by replacing $f$ by $f(x-\alpha)$, we can assume that $\alpha = 0$. Since $f$ is irreducible, all the roots of $f$ have absolute value $\epsilon^{1/d}$. If we have $f(x) = \sum_{i=0}^{d} a_i x^{n-i}$, we will have $|a_i| \leq \epsilon^{i/d}$ by this.

Choose $v \in K^\circ$ such that $|v|^d = \epsilon$. Then we can consider the new polynomial $\widetilde{f}$ defined via $\widetilde{f}(x) = v^{-d} f(vx)$. Our assumptions show that $\widetilde{f}(x)$ has coefficients in $K^\circ$, and its zeroth coefficient is a unit. By assumption, we have $\beta \in K^\circ$ such that $|\widetilde{f}(\beta)| \leq |\omega|$. This means equivalently that $|f(v\beta)| \leq \epsilon|\omega|$. Thus we can take $\alpha' = v\beta$.

Now we can use the above to obtain a successive approximation argument which converges to a root of $f$, QED. $\square$

Next, we describe the tilting correspondence for fields (following Kedlaya-Liu).

**Lemma 125** (Artin)**.** *Let $L$ be a field, and let $G$ be a finite group of automorphisms acting faithfully on $L$. Then $K = L^G$ has the property that $L/K$ is a Galois extension with group $G$ (in particular, $[L : K] = |G|$).*

**Lemma 126** (Krasner)**.** *Let $K$ be a complete NA field. Let $\alpha \in \overline{K}$ be an element and let $\epsilon$ be the minimal distance between $\alpha$ and its conjugates. Let $\beta \in \overline{K}$ be such that $|\beta - \alpha| < \epsilon$. Then $K(\alpha) \subset K(\beta)$.*

**Construction 127** (Tilting correspondence for certain Galois extensions)**.** Let $K$ be a perfectoid field of characteristic zero with tilt $K^\flat$. Let $L/K^\flat$ be a finite extension; note that $L$ is finite separable, and also perfect and nonarchimedean hence perfectoid. Then we can form the *untilt*

$$L^\sharp = (W(L^\circ) \otimes_{W(K^{\circ\flat})} K^\circ)[1/p]$$

We have seen that $L^\sharp$ is a perfectoid field too (by working with valuation rings).

Suppose now that $L/K^\flat$ is $G$-Galois, so $L^G = K^\flat$. Then $G$ acts on $L^\sharp$ and the fixed points are $K$ (since taking $G$-fixed points are an exact an operation for $\mathbb{Q}$-vector spaces). Also, $G$ acts faithfully on $L^\sharp$ since after tilting we get $L$ again. By Artin's lemma, it follows that $L^\sharp$ is a $G$-Galois extension of $K$. Moreover, the finite extensions between $K/L^\sharp$ are in one-to-one correspondence with the finite extensions between $K^\flat/L$.

**Proposition 128.** *Let $K$ be a perfectoid field. Let $E/K$ be a finite extension. Then $E$ is perfectoid. The construction $E \mapsto E^\flat$ induces an equivalence between finite extensions of $K$ and $K^\flat$.*

*Proof.* We have already seen that this construction is an equivalence between perfectoid extensions of $K$ and $K^\flat$. Consider the union of all $L^\sharp$, as $L/K^\flat$ ranges over finite extensions. This is a nonarchimedean field whose completion is necessarily perfectoid, and whose tilt is the completed union of $L/K^\flat$ and hence algebraically closed. It follows that $\bigcup_{L/K^\flat} L^\sharp$ is algebraically closed (by Krasner's lemma). Using the correspondence of Galois groups, everything should now follow. □

Fix a perfectoid field $K$, and let $\omega \in K$ be a perfectoid pseudo-uniformizer with $p$-power roots $\omega^{1/p^n}$ (we always assume that $|p| \leq |\omega| < 1$). This also determines a perfectoid pseudo-uniformizer $\omega^\flat$ in $K^{\circ\flat}$.

**Definition 129** (Tilting perfectoid algebras)**.** Let $R$ be an integral perfectoid $K^\circ$-algebra. We consider the *tilt* $R^\flat$ of $R$, as a $K^{\circ\flat}$-algebra.

By construction, this is the inverse limit perfection of $R/p$; it's also easy to see that this is the inverse limit perfection of $R/\omega$. As a multiplicative monoid we have $R^\flat = \varprojlim_{x\mapsto x^p} R$. The following results are analogous to what happens for valuation rings which we have already discussed.

**Proposition 130.** *The tilt $R^\flat$ is an integral perfectoid $K^{\circ\flat}$-algebra, with*

$$R^\flat/\omega^\flat = R/\omega.$$

**Proposition 131.** *Let $v \in W(K^{\circ\flat})$ be a generator of the kernel of the map $\theta : W(K^{\circ\flat}) \to K^\circ$, so $v$ is primitive of degree one. Then for any integral perfectoid $K^\circ$-algebra $R$, the map*

$$\theta : W(R^\flat) \to R$$

*is surjective, with kernel generated by $v$.*

**Theorem 132** (Tilting equivalence for integral perfectoid algebras)**.** *The functor $R \mapsto R^\flat$ establishes an equivalence of categories between integral perfectoid $K^\circ$ and $K^{\circ\flat}$-algebras. Moreover, if $R$ is saturated, then so is $R^\flat$. The inverse of the tilting construction sends an integral perfectoid $K^{\circ\flat}$-algebra $S$ to $W(S) \otimes_{W(K^\circ)} K^\circ$.*

*Proof.* The strategy is the same as for valuation rings. One also checks that if $R$ and $R'$ are two integral perfectoid algebras which are *almost equal* (e.g., $R \subset R'$ and they have the same saturation), then the tilts also have the same saturation, and vice versa. □

**Corollary 133** (Tilting equivalence for perfectoid $K$-algebras)**.** *Let $K$ be a perfectoid field. There is an equivalence of categories between perfectoid Banach $K$-algebras and perfectoid Banach $K^\flat$-algebras. The equivalence sends $A$ to $A^\flat$, which as a multiplicative monoid is $\varprojlim_{x \mapsto x^p} A$.*

## 8. ALMOST RING THEORY

**Definition 134.** We will work in the following setup for almost ring theory: $R$ is a commutative ring, and $I \subset R$ is an ideal such that:

(1) $I$ is flat.
(2) $I^2 = I$ (equivalently, $I \otimes_R I \simeq I$).

Note that this can only happen in the non-noetherian setup (excluding the trivial cases $I = R, I = 0$).

**Example 135.** Let $V$ be a rank $1$ valuation ring whose maximal ideal is nondiscrete (e.g., the ring of integers in a perfectoid field). Then the maximal ideal $\mathfrak{m}$ has these properties. Any torsion-free module over a valuation ring is flat, and non-discreteness implies that $\mathfrak{m}^2 = \mathfrak{m}$. Thus we can take $(R, I) = (V, \mathfrak{m})$.

**Example 136.** Let $R$ be a perfect $\mathbb{F}_p$-algebra and let $t \in R$. Then we can consider almost mathematics with respect to the ideal $I = (t^{1/p^\infty})$. Note that this ideal is flat: this is easy when $t$ is a nonnzerodivisor, but we showed it more generally earlier.

**Remark 137.** Part of this condition is that $R/I \otimes_R^L R/I \simeq R/I$. In fact, using the short exact sequence $0 \to I \to R \to R/I \to 0$, it suffices to show that $I \otimes_R^L R/I = 0$. But this is also just the underived tensor product $I \otimes_R R/I$ (since $I$ is flat), and we know that this is $I/I^2 = 0$ by our assumptions. We saw something like this (vanishing of higher Tor's) in the case of perfect $\mathbb{F}_p$-algebras.

**Definition 138** (Almost zero modules)**.** Fix a pair $(R, I)$ as above (for almost ring theory). We say that an $R$-module $M$ is *almost zero* if $IM = 0$, i.e., if $M$ is an $R/I$-module.

**Remark 139.** This equivalently happens if $I \otimes_R M = I \otimes_R^L M = 0$. In fact, we have a surjection $I \otimes_R M \twoheadrightarrow IM$, so one direction is clear. Conversely, we also know that $I \otimes_R R/I = 0$, so the same is true with $R/I$ replaced by any module over it.

In such a situation, we can form an interesting category, obtained from the category of all $R$-modules while neglecting the almost zero ones. This uses the following idea.

**Definition 140** (Serre subcategories)**.** Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* is a full subcategory $\mathcal{B} \subset \mathcal{A}$ such that:

(1) $\mathcal{B}$ is an abelian subcategory, i.e., $\mathcal{B}$ is closed under direct sums, kernels, and cokernels.
(2) $\mathcal{B}$ is closed under extensions.

**Construction 141** (The Serre quotient)**.** Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory. Then one can form a *quotient category* $\mathcal{A}/\mathcal{B}$ with the following properties:

(1) $\mathcal{A}/\mathcal{B}$ is an abelian category, receiving an exact functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ which sends every object of $\mathcal{B}$ to zero.
(2) Given an abelian category $\mathcal{C}$ and an exact functor $F : \mathcal{A} \to \mathcal{C}$ which annihilates all objects in $\mathcal{B}$, then $F$ factors canonically over $\mathcal{A}/\mathcal{B}$ (this is the universal property of the quotient).

Explicitly, the category $\mathcal{A}/\mathcal{B}$ can be constructed as follows. The objects are the same as that of $\mathcal{A}$. Given objects $X, Y$, we have that $\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ is the filtered colimit of $\mathrm{Hom}_{\mathcal{A}}(X', Y)$ where $X' \to X$ ranges over all maps with kernel and cokernel in $\mathcal{B}$.

**Example 142.** Let $\mathcal{A}$ be the category of abelian groups, and let $\mathcal{B} \subset \mathcal{A}$ be the subcategory of torsion abelian groups. Then $\mathcal{A}/\mathcal{B}$ is the category of $\mathbb{Q}$-vector spaces.

**Proposition 143.** *Let $(R, I)$ be as above. Then the category of almost zero $R$-modules is a Serre subcategory.*

*Proof.* Here one uses the fact that $I = I^2$ to see closure under extensions. □

**Remark 144.** The idea is that the category of almost zero $R$-modules is sort of like the category of torsion abelian groups. However, being almost zero is a *much* stronger condition. For instance, it's not true that an infinite product of torsion abelian groups is torsion, while it is true that any product of almost zero modules is almost zero.

**Definition 145** (The almost category)**.** The *almost category* $\mathrm{Mod}^a(R)$ is the quotient category of $R$-modules by the category almost zero modules. We have an exact functor $\mathrm{Mod}(R) \to \mathrm{Mod}^a(R)$, called *almostification.* We will also call objects of the almost category $R^a$-modules and write the functor as $(\cdot)^a$.

By definition, given $R$-modules $M, N$, we have that

$$\mathrm{Hom}_{\mathrm{Mod}^a(R)}(M^a, N^a) = \varinjlim_{M' \to M} \mathrm{Hom}_{\mathrm{Mod}(R)}(M', N),$$

where $M' \to M$ ranges over all almost isomorphisms. In general, one expects this colimit to be somewhat inexplicit. But in this case one has an initial object of the category of all maps $M' \to M$: namely, the map $I \otimes_R M \to M$.

**Remark 146** ($\mathrm{Mod}^a(R)$ is a tensor category)**.** The tensor product on $R$-modules has the property that Therefore, it descends to a tensor product on $\mathrm{Mod}^a(R)$.

**Construction 147** (The functors relating $\mathrm{Mod}^a(R), \mathrm{Mod}(R)$)**.** The analogy is that if $X$ is a topological space, and $U \subset X$ an open subset, then one has a functor

$$j^* : \mathrm{Sh}(X) \to \mathrm{Sh}(U),$$

from sheaves on $X$ to sheaves on $U$. This functor is exact, and has two adjoints $j_!, j_*$ (extension by zero and pushforward). The functor $j_!$ is exact, whereas $j_*$ is only left exact.

Similarly, the almostification functor

$$\mathrm{Mod}(R) \to \mathrm{Mod}^a(R)$$

admits adjoints in both directions, which we write as

$$()_!, ()_* : \mathrm{Mod}^a(R) \to \mathrm{Mod}(R),$$

and $()_!$ is exact. To describe these functors, it suffices to describe them on (honest) $R$-modules and check that they respect almost isomorphisms:

(1) $M_! = I \otimes_R M$.
(2) $M_* = \mathrm{Hom}_R(I, M)$.

In fact, we've already seen that if $M, N$ are $R$-modules, then $\mathrm{Hom}_{\mathrm{Mod}^a(R)}(M, N) = \mathrm{Hom}_{\mathrm{Mod}(R)}(I \otimes_R M, N) = \mathrm{Hom}_{\mathrm{Mod}(R)}(M, \mathrm{Hom}_{\mathrm{Mod}(R)}(I, N))$. This easily implies the desired adjunctions. Moreover, since $I$ is flat, $()_!$ is exact.

**Remark 148** (The almost category as a subcategory of modules)**.** An equivalent description of the almost category is the collection of all $R$-modules $M$ such that $I \otimes_R M \to M$ is an isomorphism. Thus, the almost category can be embedded fully faithfully (via $()_!$) as a subcategory of $\mathrm{Mod}(R)$. If we interpret it this way, then the tensor structure is just the $R$-linear tensor product, but now we replace the unit with $I$.

**Example 149.** Suppose $I = (t^{1/p^\infty})$ for a nonzerodivisor $t \in R$ admitting all $p$-power roots. Consider the object $(R/t)$. An example of an almost element of this module is $\sum_{n>0} t^{1-1/p^n}$.

**Remark 150.** The category $\mathrm{Mod}^a(R)$ has all limits and colimits, and almostification preserves them.

## 9. Finiteness and flatness conditions

There are analogs of standard notions in module theory for the almost category. Let $(R, I)$ be a setup for almost ring theory. Let $A$ be an $R$-algebra (or even an $R^a$-algebra). We consider the almost category $\mathrm{Mod}^a(A)$, i.e., $A$-modules in $\mathrm{Mod}^a(R)$.

**Definition 151** (Almost finitely generated and almost finitely presented objects)**.** An object $M^a$ of $\mathrm{Mod}^a(A)$ (for $M \in \mathrm{Mod}^a(A)$) is *almost finitely generated* if for every $\epsilon \in I$, there exists a finitely generated $A$-module $M'$ and a map $M' \to M$ whose cokernel is annihilated by $\epsilon$. (This doesn't depend on the choice of $M$.) Similarly, one has the notion of *almost finitely presented.*

**Example 152.** Consider the ring $R = \mathbb{F}_p[t^{1/p^\infty}]$ and the module $M = \bigoplus_{n>0} R/t^{1/p^n}$. This is almost finitely generated.

**Example 153.** Let $p \neq 2$. Consider the ring $R = \mathbb{F}_p[t^{1/p^\infty}]$ and the algebra $S = \mathbb{F}_p[t^{1/2p^\infty}] = \bigcup_{n>0} R[\sqrt{t^{1/p^n}}]$. This is not a finitely generated module, but it is almost finitely generated. In fact, the cokernel of $R[\sqrt{t^{1/p^n}}] \to S$ is annihilated by smaller and smaller powers of $t$ as $n \to \infty$. Note that in this case, the number of generators needed is uniform in the power $\epsilon$.

Let me mention the (partial) classification of almost finitely generated modules in the case of a perfectoid valuation ring $V = K^\circ$, for $K$ a perfectoid field, considered as a setup for almost mathematics in the usual way.

**Example 154.** Let $I \subset V$ be any ideal, so $I$ is determined (with one possible ambiguity) by a real number $\gamma \in [0,1]$ as $I = I_\gamma := \{x \in V : |x| \leq \gamma\}$ or $I = I'_\gamma := \{x \in V : |x| < \gamma\}$. I claim that $I_\gamma, I'_\gamma$ is almost finitely generated.

When $\gamma$ belongs to the value group, then $I_\gamma$ is principal and hence isomorphic to $V$ itself. Otherwise, we can write $\gamma$ as the limit $\gamma = \lim \gamma_i$ of an increasing sequence $\gamma_i \to \gamma$ with $\gamma_i$ in the value group, so $I_\gamma = \varinjlim_i I_{\gamma_i}$; it's easy to see that this expression shows that $I_\gamma$ is almost finitely generated. Similarly for $I'_\gamma$ (which agrees with $I_\gamma$ when $\gamma$ is not in the value group).

**Example 155.** Let $\gamma_1, \gamma_2, \dots$ be an increasing sequence of nonnegative real numbers which tends to 1 as $i \to \infty$. Then $V/I_{\gamma_1} \oplus V/I_{\gamma_2} \oplus \dots$ is almost finitely generated.

Any almost finitely generated $V$-module can be approximated by one of the previous form, in the following sense.

**Definition 156.** Given $V$-modules $M, N$, we say that $M \approx N$ if for every $\epsilon \in \mathfrak{m}$, there exist maps $f_\epsilon : M \to N$ and $g_\epsilon : N \to M$ such that both composites are multiplication by $\epsilon$. This is an equivalence relation on $V$-modules (or even on $V^a$-modules) which in general is weaker than almost isomorphism.

**Theorem 157** (Scholze)**.** *Any almost finitely generated $V$-module $M$ is almost finitely presented, and one has $M \approx V/I_{\gamma_1} \oplus V/I_{\gamma_2} \oplus \dots$ for a unique nondecreasing sequence $\gamma_1, \gamma_2, \dots \in \mathbb{R}_{\geq 0}$ which tends to 1 as $i \to \infty$. In particular, any almost finitely generated projective $V$-module $M$ is $M \approx V^n$ for some unique $n \geq 0$.*

**Definition 158** (Almost flat and almost projective objects)**.**

(1) We say that an object $M^a \in \mathrm{Mod}^a(R)$ is almost flat if tensoring with $M^a$ is exact in $\mathrm{Mod}^a(R)$. In other words, for $N \in \mathrm{Mod}(R)$, $\mathrm{Tor}_i^R(M, N)$ is almost zero for $i > 0$.

(2) We say that $M$ is *almost projective* if $\mathrm{Ext}^i(M, N) =^a 0$ for all $N \in \mathrm{Mod}^a(R)$.

Note that the almost category (like the category of sheaves) generally has no projective objects. For instance, $R$ itself is not an almost projective object.

**Example 159.** Consider the same ring $R = \mathbb{F}_p[t^{1/p^\infty}]$ as above. Then $R^a$ is not a projective object in $\mathrm{Mod}^a(R)$. Unwinding the definitions, we have that

$$\mathrm{Hom}_{\mathrm{Mod}^a(R)}(R, M) = M_*,$$

the module of almost elements of $M$. Now the sequence

$$0 \to R \xrightarrow{t} R \to R/tR \to 0$$

leads to an exact sequence

$$0 \to R \to R \to (R/tR)_*$$

which is not exact on the right because of elements such as $\sum_{n>0} t^{1-1/p^n}$.

**Remark 160.** The above is saying that the construction $()_*$ has higher derived functors when applied to $R$. If $V$ is the valuation ring of a field which is *spherically complete* (every descending sequence of discs has a nonempty intersection), then this problem goes away: there are no higher derived functors of $()_*$ when applied to $R$.

**Construction 161** (Internal hom in almost modules)**.** The category $\mathrm{Mod}^a(R)$ has internal mapping objects $\mathrm{alHom}(\cdot,\cdot)$. These are internal mapping objects for the tensor product on almost modules, and they are obtained from the usual internal hom on modules by passing to almost modules. So if $M, N \in \mathrm{Mod}^a(R)$, then $\mathrm{alHom}(M, N) \in \mathrm{Mod}^a(R)$.

The notion of almost homomorphisms may make the above notions seem more palatable. Recall the following facts:

**Proposition 162.** (1) *Given an $R$-module $M$, $M$ is finitely generated if and only if for every directed system of $R$-modules $N_\alpha, \alpha \in A$, the map $\varinjlim \mathrm{Hom}(M, N_\alpha) \to \mathrm{Hom}(M, \varinjlim N_\alpha)$ is injective.*
(2) *$M$ is finitely presented if and only if this map is always an isomorphism.*

There is an almost analog of this proposition (all this appears in Gabber-Romero). Moreover, we describe some more almost analogs of basic results about modules.

**Proposition 163.** *If $M \in \mathrm{Mod}^a(R)$, $M$ is almost finitely generated if and only if for every directed system of almost $R$-modules $N_\alpha, \alpha \in A$, the map $\varinjlim \mathrm{alHom}(M, N_\alpha) \to \mathrm{alHom}(M, \varinjlim N_\alpha)$ is injective (in the almost category). Similarly, $M$ is almost finitely presented if this map is always an almost isomorphism.*

**Proposition 164.** *If $M \in \mathrm{Mod}^a(R)$, $M$ is almost projective if and only if $\mathrm{alHom}(M, \cdot)$ is an exact functor.*

**Proposition 165.** *Let $M \in \mathrm{Mod}^a(R)$ be almost finitely generated and (almost) projective. Then for each $\epsilon \in I$, there is a finitely generated free $R$-module $F$ and maps $M \to F \to M$ such that the composite is multiplication by $\epsilon$; moreover, $F$ is almost finitely presented.*

*Proof.* In fact, there exists a finitely generated free module $F$ and a map $F \to M$ whose image $M' \subset M$ contains $\epsilon^{1/2}M$. The map $f : M \to M'$ given by multiplication by $\epsilon^{1/2}$ necessarily has the property that $\epsilon^{1/2}f$ lifts to a map $M \to F$, and the composite $M \to F \to M' \to M$ is then multiplication by $\epsilon$ as desired. $\square$

## 10. Almost purity

There is a general principle that assertions involving perfectoid rings on the generic fiber naturally extend to almost statements integrally. The almost purity theorem is an instance of this when one works with finite étale extensions.

Let's start with something similar along these lines, which will be useful. Let $R$ be a perfect $\mathbb{F}_p$-algebra. Suppose $t \in R$ is an element; we consider almost mathematics with respect to the ideal $(t^{1/p^\infty})$.

**Proposition 166.** *Let $S \to S'$ be a map of perfect $R$-algebras and suppose each is integral over $R$. Suppose $S[1/t] \to S'[1/t]$ is an isomorphism. Then $S \to S'$ is an almost isomorphism.*

*Proof.* Let $s \in S'$. Then the $R$-module $M \subset S'$ generated by $s$ is finitely generated. Now any finitely generated $R$-module $M' \subset S'$ has the property that there exists $N$ with $t^N M' \subset S$, since $S, S'$ become the same after inverting $t$. It follows that $t^N s^M \in S$ for all $M > 0$. Taking $M = p^R$, and then extracting $R$th roots, we get that $t^{N/p^R} s \in S$ for all $R > 0$. This means that $s$ almost belongs to $S$, as desired. $\square$

**Corollary 167.** *There is an equivalence of categories between:*

(1) *Perfect $R[1/t]$-algebras.*
(2) *Perfect $R$-algebras which are integral over $R$, up to almost isomorphism.*

Now let's explain almost purity. Recall the following definition (which is also equivalent to many other well-known definitions).

**Definition 168** (Finite étale algebras)**.** Let $A$ be a commutative ring. An $A$-algebra $B$ is called *finite étale* if:

(1) $B$ is a finitely generated projective $A$-module.
(2) The multiplication map $m : B \otimes_A B \to B$ admits a section in $B \otimes_A B$-modules; that is, there is an idempotent $e \in B \otimes_A B$ such that $e$ generates the kernel of $m$. Equivalently, $B$ is projective as a $B \otimes_A B$-module.

**Remark 169.** Geometrically, condition (2) is saying that the diagonal map $Y \to Y \times_X Y$ is a clopen immersion.

**Remark 170.** Another way of phrasing condition (2) is that if $I$ is the kernel of $B \otimes_A B \to B$, then $I = I^2$.

**Remark 171.** The notion of an *étale* $A$-algebra $B$ means that (1) is replaced by $B$ being finitely presented over $A$ as an algebra. There is also the notion of a *weakly étale* map $A \to B$, which works under fewer finiteness conditions. This means that $B$ is a flat module over $B \otimes_A B$. For instance, any ind-étale map has this property. Weakly étale maps are not too far from being ind-étale.

**Remark 172.** After any base change $A \to k$, for $k$ an algebraically closed field, we find that $B \otimes_A k$ is a product of copies of $k$.

This idempotent $e$ that shows up in the formulation of a finite étale algebra plays an important role. We can also construct it as follows.

**Construction 173** (The trace pairing)**.** Let $B/A$ be finite étale. Then we have a *trace map*

$$\mathrm{tr} : B \to A$$

given by the $A$-trace of multiplication by an element $b \in B$. The trace pairing

$$B \times B \to A, \quad (b_1, b_2) \mapsto \mathrm{tr}(b_1 b_2)$$

is a nondegenerate symmetric bilinear form. As a result, we have an associated *Casimir element*: choose a basis $b_i$ of $B/A$ (Zariski locally on $A$) and dual basis $b'_i$, and take $\sum_{i=1}^n b_i \otimes b'_i \in B \otimes_A B$. In particular, if we can write $B$ as a retract of $A^n$ via the maps $B \to A^n, b \mapsto \{\mathrm{Tr}(b_i b)\}$, $A^n \to B$ via $\{x_i\} \mapsto \sum x_i b'_i$.

**Remark 174.** The nondegeneracy of the pairing is equivalent to finite étaleness. That is, if $B/A$ is an algebra such that $B$ is a finitely generated projective module, then one has such a trace pairing; if it is nondegenerate, then $B$ is finite étale.

**Example 175.** Let $A$ be a $\mathbb{Z}[1/2]$-algebra. Given a unit $x \in A$, then $B = A[\sqrt{x}]$ is a finite étale algebra of rank 2. The associated idempotent can be taken to be

$$\frac{1}{2}(1 + \sqrt{x} \otimes \frac{1}{\sqrt{x}}) \in B \otimes_A B.$$

**Example 176.** Suppose $B = A[x]/(f(x))$ where $f(x)$ is a monic polynomial such that $f(x), f'(x)$ generate the unit ideal. One can write down an explicit formula for the associated dual basis to $1, x, \ldots, x^{n-1}$.

**Theorem 177.** *Let $A$ be a ring which is complete with respect to an ideal $I$. Then the category of finite étale $A$-algebras is equivalent to the category of finite étale $A/I$-algebras.*

This is the "topological invariance" of the étale site. Somehow full faithfulness is easy, but essential surjectivity is harder.

**Remark 178.** This is true for *weakly étale* algebras too, and in the almost category. (This is explained by Gabber-Romero, and relies on the theory of the cotangent complex.)

We can transport this to the almost setting.

**Definition 179.** Let $(R, I)$ be a setup for almost ring theory as above. Let $S$ be an $R^a$-algebra. We say that $S$ is *almost finite étale* over $R$ if:

(1) $S$ is an almost finitely generated projective $R$-module.
(2) The multiplication map $m : S \otimes_R S \to S$ (in $R^a$-algebras) is given by the image of an idempotent. That is, there is an idempotent $e$ in $(S \otimes_R S)_*$ such that $m(e) = 1$ and $e(\ker(m)) = 0$.

**Theorem 180** (Almost purity in characteristic $p$)**.** *Let $R$ be a perfect $\mathbb{F}_p$-algebra, and consider almost mathematics with respect to the ideal $(t^{1/p^\infty})$ as above. Then:*

(1) *If $S$ is a perfect $R$-algebra which is integral over $R$ and such that $S[1/t]$ is finite étale over $R[1/t]$, then $S$ is almost finite étale over $R$.*
(2) *Inverting $t$ induces an equivalence of categories between finite étale $R^a$-algebras and finite étale $R[1/t]$-algebras. (The inverse is given by taking the integral closure and almostification.)*

*Proof.* Without loss of generality $R, S$ are $t$-torsion-free: quotient by the ideal of $t$-power torsion, which is almost zero.

Suppose $S[1/t]$ is finite étale over $R[1/t]$. Let $e \in S[1/t] \otimes_{R[1/t]} S[1/t]$ be the idempotent which generates the kernel of the multiplication map $S[1/t] \otimes_{R[1/t]} S[1/t] \to S[1/t]$. By assumption, $t^N e \in S \otimes_R S$ for some $N \gg 0$. Since $S \otimes_R S$ is perfect, and since $e$ is idempotent, we have that $t^{N/p^R} e \in S \otimes_R S$ for all $R > 0$. This defines an almost idempotent in $S \otimes_R S$. (Maybe worth noting: $S \otimes_R S$ is almost equal to its image in $(S \otimes_R S)[1/t]$, because any $t$-power torsion is almost zero, thanks to perfectness.) This idempotent maps to $1$ in $S$ is the desired one.

Another way of phrasing this: we want to show that, in the almost category, $S \otimes_R S \simeq S \times T$ for another $R^a$-algebra $T$. Since everything is integral over $R$, this is equivalent to the assertion $S[1/t] \otimes_{R[1/t]} S[1/t] \simeq S[1/t] \times T[1/t]$, which is part of finite étaleness, by the equivalence between perfect $R[1/t]$-algebras and perfect integral $R$-algebras (up to almost isomorphism).

One needs to now show that $S/R$ is an almost finitely generated projective module. This uses the idempotent $e$ and its interpretation via the Casimir. Let $e$ be the idempotent in $(S \otimes_R S)[1/t]$ as above. Then $e$ is also the Casimir element for the trace pairing. If we can write $e = \sum_{i=1}^n a_i \otimes b_i$, then we have maps

$$S[1/t] \to R[1/t]^n \to S[1/t]$$

where the first map sends $s \mapsto \{\mathrm{Tr}(a_i s)\}$ and the second map is $\{x_i\} \mapsto \sum x_i b_i$, which exhibit $S[1/t]$ as a retract of $R[1/t]^n$.

In fact, for each $\epsilon \in \mathbb{Z}[1/p]_{>0}$, we have that $t^\epsilon e \in S \otimes_R S$, and we can define maps $S \to R^n, R^n \to S$ by expanding out $t^\epsilon e$ as above. The composite is given by $t^\epsilon$. This shows that $S$ is almost finitely generated projective, as desired. $\square$

**Lemma 181.** *Let $V$ be a perfectoid valuation ring and let $M$ be a $\pi$-adically complete, $\pi$-torsion-free module. Suppose $M/\pi M$ is almost finitely generated. Then so is $M$.*

*Proof.* Fix $\epsilon \in (0, 1)$ and in $\mathbb{Z}[1/p]$. Choose elements $x_1, \dots, x_n \in M/\pi M$ generating a submodule $\overline{M}_\epsilon$ such that $\pi^\epsilon (M/\pi M) \subset \overline{M}_\epsilon$. Let $M_\epsilon \subset M$ be a finitely generated submodule which projects to $\overline{M}_n$. Then for $x \in M$, we can write $\pi^\epsilon x = \pi z + m_0$ for $z \in M, m_0 \in M_\epsilon$. Then we can write $\pi z = \pi^{1-\epsilon}(\pi^\epsilon z) = \pi^{1-\epsilon}(\pi z' + m_1)$ with $m_1 \in M_\epsilon$. We get

$$\pi^\epsilon x = m_0 + \pi^{1-\epsilon}(m_1 + \pi z'),$$

and continuing in this fashion, we get that $\pi^\epsilon x$ is contained in $M_\epsilon$ (which is complete). The infinite sum is in terms of powers of $\pi^{1-\epsilon}$. $\square$

**Proposition 182.** *Let $V$ be a perfectoid valuation ring with perfectoid pseudo-uniformizer $\pi$. Let $W$ be a $V$-algebra which is $\pi$-complete and $\pi$-torsion-free. TFAE:*

1. *$W$ is almost finite étale over $V$.*
2. *$W/\pi$ is almost finite étale over $V/\pi$.*

*Proof.* First, we need to see that $W$ is almost finitely presented over $V$. This follows from the previous lemma. Then, we need the existence of idempotents. This follows from the nilpotent lifting property for idempotents. $\square$

**Theorem 183** (Almost purity for perfectoid fields)**.** *Let $K$ be a perfectoid field. Let $L/K$ be a finite extension. Then $L^\circ/K^\circ$ is almost finite étale.*

*Proof.* Let $\pi$ be a perfectoid pseudo-uniformizer. Then $L^\circ$ is a $\pi$-complete, $\pi$-torsion-free $K^\circ$-module. We want to see that it is almost finite étale. To do this, it suffices to see that $L^\circ/\pi$ is almost finite étale over $K^\circ/\pi$. □

**Example 184.** Consider $\mathbb{Q}_p(p^{1/p^\infty})$ and add a square root of $p$.

**Corollary 185** (Tate)**.** *Let $K \subset \mathbb{C}_p$ be a complete nonarchimedean field which contains all $p$-power roots of unity. Then for every finite extension $L/K$, the trace map is a surjection $\mathfrak{m}_L \to \mathfrak{m}_K$.*

**Definition 186** (Perfectoid almost rings)**.** Let $V = K^\circ$ be a perfectoid valuation ring, with perfectoid pseudo-uniformizer $\pi$. Let $R$ be a $V^a$-algebra. We say that $R$ is *perfectoid* if $R$ is $\pi$-adically complete, $\pi$-torsion-free, and the map $R/\pi^{1/p} \to R/\pi$ given by the Frobenius is an isomorphism (of $V^a$-algebras).

Then one has the following theorem:

**Theorem 187.** *The categories of perfectoid Banach $K$-algebras and perfectoid $V^a$-algebras are equivalent.*

**Theorem 188** (General form of almost purity)**.** *Let $A$ be a perfectoid Banach $K$-algebra. Let $B$ be a finite étale $A$-algebra. Then $B$ is also a perfectoid $K$-algebra and $A^\circ \to B^\circ$ is almost finite étale. In fact, one has an equivalence of categories between finite étale $A$-algebras and almost finite étale $A^\circ$-algebras.*

The general case is proved by reduction to the case of a perfectoid field.