NOTES ON THE KODAIRA VANISHING THEOREM

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Let X be a scheme of finite type over a noetherian ring A. Given a line bundle $\mathcal{L}$ on X, recall the notion of *ampleness* (see [Har77]): $\mathcal{L}$ is ample if for any coherent sheaf $\mathcal{F}$ on X, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ is generated by global sections for $k \gg 0$. In particular, $\mathcal{L}^{\otimes k}$ is itself generated by global sections for $k \gg 0$. Line bundles generated by global sections determine maps to projective space, and for $k \gg 0$ we get a map

$$X \to \mathbf{P}(\Gamma(\mathcal{L}^{\otimes k})),$$

where the last object is the projectivization of the A-module $\Gamma(\mathcal{L}^{\otimes k})$. For $k \gg 0$, ampleness guarantees that this map will be an *immersion.* As a result (because we can replace $\Gamma(\mathcal{L}^{\otimes k})$ with a finitely generated submodule thereof), having an ample line bundle is basically equivalent to quasi-projectivity (over $\mathrm{Spec} A$) of the scheme X.

In essence, ampleness is a *positivity* condition. It is easy to show, for instance, that the tensor product $\mathcal{L} \otimes \mathcal{L}'$ of two ample line bundles $\mathcal{L}, \mathcal{L}'$ is ample, and that if a tensor power $\mathcal{L}''^{\otimes k}$ of a line bundle $\mathcal{L}''$ is ample, so is $\mathcal{L}''$. Moreover, on a quasi-projective scheme, any line bundle is the "difference" of two ample line bundles (which follows easily from the fact that if $\mathcal{L}, \mathcal{L}'$ are line bundles with $\mathcal{L}'$ ample, then $\mathcal{L} \otimes \mathcal{L}'^{\otimes k}$ is ample for $k \gg 0$). On the Picard group $\mathrm{Pic}(X)$, we thus get a partial ordering compatible with the group structure.

The Kodaira vanishing and embedding theorems help clarify the equation of "positivity" with ampleness. One consequence of them, for instance, is that a line bundle on a smooth projective variety over $\mathbb{C}$ is ample if and only if it admits a metric whose associated connection satisfies a positivity condition; another is a cohomological characterization of when a compact Kähler manifold can be realized as a projective variety.

The vanishing theorem can be stated as follows:

Theorem 1 (Kodaira vanishing)**.** *If $\mathcal{L}$ is an ample line bundle on a smooth projective variety X over a field of characteristic zero, then*

$$H^p(X, \Omega^q \otimes \mathcal{L}) = 0 \quad \textit{for } p + q > n.$$

We will deduce this result using *analytic* methods, over $\mathbb{C}$; in fact, the result is false in characteristic p. Let M be a compact Kähler manifold. If L is a holomorphic line bundle on M, we will describe a notion of "positivity" of the line bundle in terms of the Chern classes. We will then have:

Theorem 2 (Kodaira-Nakano)**.** *If L is a positive line bundle on a compact Kähler manifold M, then*

$$H^p(M, \Omega^q(L)) = 0 \quad \textit{for } p + q > n.$$

Moreover, if E is any holomorphic vector bundle on M, then $H^p(M, E \otimes L^r) = 0$ for $r \gg 0$ (if $p > 0$).

Here the cohomology of $\Omega^q(L)$ is the sheaf cohomology of the associated sheaf of holomorphic sections. One consequence of this result (the embedding theorem) will be that a positive line bundle on such a manifold is "ample" and leads to a realization of the manifold as a smooth projective variety.

In these notes, we describe a proof of the Kodaira vanishing theorem due to Akizuki and Nakano. In order to fully state and prove this result, we will have to develop some background material. We start with the basics of connections and curvature on vector bundles, using the Cartan formalism of the "moving frame." Next, we discuss Chern classes, and the Chern-Weil theory that shows that they can be computed "analytically" (in terms of a given connection). We then develop the basic theory of Kähler manifolds, including Hodge theory and the "Kähler identities" in detail; after these identities

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are established, the vanishing theorem follows by a positivity argument. We finally discuss various applications (e.g. the Lefschetz hyperplane theorem, Picard groups of hypersurfaces).

1. Connections and curvature

We now want to do some hermitian differential geometry. We will define what it means to have a *connection* in a complex vector bundle E, and construct its curvature as an E-valued global 2-form. In the Riemannian case, there is a *canonical* connection (the Levi-Civita connection) on the tangent bundle when a metric is fixed. There will be an analog for a hermitian metric on a holomorphic vector bundle, the *Chern connection*, which we will find highly useful.

The necessity of this in the proof of the Kodaira vanishing theorem is as follows: in it, one wishes to prove that a certain operator (the $\overline{\partial}$-Laplacian) defined on E-valued (p,q)-forms has trivial kernel for appropriate $p+q$, in view of the Hodge isomorphism of cohomology with harmonic forms. By a miracle of the Kähler structure, the $\overline{\partial}$-Laplacian turns out to be a strictly positive operator plus a variant of the Hodge Laplacian on E. However, the Hodge Laplacian on E-valued forms *cannot* be defined in terms of the exterior derivative d (as it is defined for ordinary differential forms): one has to have a connection to make sense of it. We shall use the Chern connection here.

For a smooth (for now, real) manifold M, we let $\mathcal{A}^1(M)$ denote the space of smooth $\mathbb{C}$-valued 1-forms, and $\mathcal{A}(M)$ denote smooth functions. Similarly, for a complex bundle $E \to M$, we let $E(M)$ denote (smooth, $\mathbb{C}$-valued) sections defined over M. $T_{\mathbb{C}}M$ or simply $T_{\mathbb{C}}$ will denote the complexified tangent bundle, and $T^*_{\mathbb{C}}M$ or simply $T^*_{\mathbb{C}}$ the (complexified) cotangent bundle.

1.1. **Connections.** Recall that a *connection* on a vector bundle E over a smooth manifold M is a $\mathbb{C}$-homorphism

$$\nabla : E(M) \to (T^*_{\mathbb{C}} \otimes E)(M)$$

that maps global sections of M to global sections of $T^*_{\mathbb{C}} \otimes E$ (i.e. E-valued 1-forms), which satisfies the **Leibnitz rule**

$$\nabla(fs) = (df)s + f\nabla s, \quad s \in E(M), f \in \mathcal{A}(M). \tag{1}$$

This is essentially a way of differentiating sections of E, because for any vector field X on M and section $s \in E(M)$, we can define $\nabla_X s \in E(M)$, the *covariant derivative* (with respect to this connection) of s in the direction of X. We can obtain $\nabla_X s$ by contracting ∇s with X.

The covariant derivative satisfies, by (1):

(1) $\nabla_{fX}(s) = f\nabla_X s$.
(2) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

In fact, these two properties *characterize* a connection, as is easily checked, and we might as well have defined in terms of the covariant derivative. Moreover, one checks that the definition of a connection is *local*: that is, given a connection on E, one gets connections on each restriction to an open neighborhood.

We can describe connections locally, in terms of *frames.* Recall that a *frame* of an n-dimensional vector bundle E, over an open subset $U \subset M$, is a family of sections $e_1, \dots, e_n \in E(U)$ that forms a basis at each point; thus $\{e_1, \dots, e_n\}$ defines an isomorphism of vector bundles the trivial n-dimensional bundle over U and $E|_U$. A frame is thus to a vector bundle as local coordinates are to a manifold.

Let $\{e_1, \dots, e_n\}$ be a frame over U for E. Then, a connection ∇ is *determined* over U by the elements $\nabla e_1, \dots, \nabla e_n \in (T^*_{\mathbb{C}} \otimes E)(U)$. For, any section s of $E(U)$ can be written as $s = \sum f_i e_i$ for the f_i uniquely determined smooth functions, and consequently

$$\nabla s = \sum (df_i) e_i + \sum f_i \nabla e_i.$$

In other words, if we use the frame $\{e_i\}$ to identify each section of $E(U)$ with the tuple $\{f_i\}$ such that $s = \sum f_i e_i$, then ∇ acts by applying d and by multiplying by a suitable matrix corresponding to the ∇e_i.

In view of this, we make:

Definition 3. Given a frame $\mathfrak{F} = \{e_1, \dots, e_n\}$ of the vector bundle E over U and a connection ∇, we define the n-by-n matrix $\theta(\mathfrak{F})$ of 1-forms via

$$\nabla \mathfrak{F} = \theta(\mathfrak{F})\mathfrak{F}.$$

In other words, $\nabla e_i = \sum_j \theta(\mathfrak{F})_{ij} e_j$ for each j.

Note that θ itself makes no reference to the bundle: it is simply a matrix of 1-forms.

Given a frame $\mathfrak{F}$, and given $g : U \to \mathrm{GL}_n(\mathbb{C})$, we can define a new frame $g\mathfrak{F}$ by multiplying on the left. We would like to determine how a connection *transforms* with respect to a change of frame, so that we can think of a connection in a different way. Namely, we have:

$$\nabla(g\mathfrak{F}) = (dg)\mathfrak{F} + g\nabla\mathfrak{F} = (dg)\mathfrak{F} + g\theta(\mathfrak{F})\mathfrak{F}.$$

where dg is considered as a matrix of 1-forms. As a result, we get the *transformation law*

$$\theta(g\mathfrak{F}) = (dg)g^{-1} + g\theta(\mathfrak{F})g^{-1}, \quad g : U \to \mathrm{GL}_n(\mathbb{C}). \tag{2}$$

Conversely, if we have for each local frame $\mathfrak{F}$ of a vector bundle $E \to M$, a matrix $\theta(\mathfrak{F})$ of 1-forms as above, which satisfy the transformation law (2), then we get a connection on E. We could, in fact, get carried away and *define* a connection on E as an association of a matrix $\theta(\mathfrak{F})$ of 1-forms to each frame of E, satisfying the transformation law (2); we could then check that all constructions we did with connections (e.g. forming the curvature) transformed appropriately. Needless to say, this is very unpleasant.

Proposition 4. *Any vector bundle $E \to M$ admits a connection.*

The proof is a standard partition-of-unity argument, analogous to the proof that any vector bundle admits a metric.

Proof. It is easy to see that a convex combination of connections is a connection. So, in each coordinate patch U over which E is trivial with a fixed frame, we choose the matrix θ arbitrarily and get some connection ∇'_U on $E|_U$. Let these various U's form an open cover $\mathfrak{A}$. Then, we can find a partition of unity $\phi_U, U \in \mathfrak{A}$ subordinate to $\mathfrak{A}$, and we can define our global connection via

$$\nabla = \sum_U \phi_U \nabla'_U.$$

□

A connection is fundamentally a means of identifying different fibers of a vector bundle, and more generally of a principal G-bundle, by parallel transport. However, we shall not touch on this, although we shall occasionally allude to it for motivation.

1.2. **Curvature.**

We want to now describe the *curvature* of a connection. A connection is a means of differentiating sections; however, the differentiation may not satisfy the standard result for functions that mixed partials are equal (connections that do are called *flat* connections, and are quite special: they correspond to modules over the sheaf of differential operators). The curvature will be the measure of how much flatness fails. It will also, surprisingly, allow us to compute purely the Chern classes (in terms of de Rham cohomology), and this explains the relevance to this paper.

Let M be a smooth manifold, $E \to M$ a smooth complex vector bundle. Given a connection ∇ on E, the curvature is going to be a global section of the vector bundle $\bigwedge^2 T^*_{\mathbb{C}} \otimes \hom(E, E)$: in other words, the bundle of global differential 2-forms with coefficients in the vector bundle $\hom(E, E)$.

Proposition 5. *Let s be a section of E, and X, Y vector fields. The map:*

$$s, X, Y \mapsto R(X, Y, s) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})s$$

is a bundle map $E \to E$, and is $\mathcal{A}(M)$-linear in X, Y.

Proof. Calculation, typically done to define the Riemann curvature tensor in the case of the tangent bundle and the Levi-Civita connection. See for instance []. □

Since the quantity $R(X,Y,s)$ is $\mathcal{A}(M)$-linear in all three quantities (X,Y,s), and clearly alternating in X,Y, we can think of it as a global section of the bundle $\bigwedge^2 T^*_{\mathbb{C}} \otimes \hom(E,E)$.

Definition 6. The above element of $\left(\bigwedge^2 T^*_{\mathbb{C}} \otimes \hom(E,E)\right)(M)$ is called the **curvature** of the connection ∇, and is denoted Θ.

We now wish to think of the curvature in another manner. To do this, we start by extending the connection ∇ to maps $\nabla : (E \otimes \bigwedge^p T^*_{\mathbb{C}})(M) \to \left(E \otimes \bigwedge^{p+1} T^*_{\mathbb{C}}\right)(M)$. The requirement is that the Leibnitz rule hold: that is,

$$\nabla(\omega s) = (d\omega)s + (-1)^p \omega \wedge \nabla s, \tag{3}$$

whenever ω is a p-form and s a section of $E \otimes \bigwedge^q T^*_{\mathbb{C}} M$ for some q. We can define this map locally (by the above formula), and the local definitions necessarily glue.

Thus:

Proposition 7. *One can extend uniquely ∇ to maps $(E \otimes \bigwedge^p T^*_{\mathbb{C}})(M) \to \left(E \otimes \bigwedge^{p+1} T^*_{\mathbb{C}}\right)(M)$ satisfying* (3).

Proof. To be more precise, we should define ∇ locally. Given a frame $\mathfrak{F} = \{e_1, \dots, e_n\}$ of E (over some open subset U), we have seen that there is a matrix $\theta(\mathfrak{F})$ of 1-forms such that ∇ can be described as follows. If s is any section of E over U, say $s = f\mathfrak{F}$ (for f a row vector of functions), then

$$\nabla s = (df)\mathfrak{F} + f\theta(\mathfrak{F})\mathfrak{F}.$$

As a result, we can make the extension easily: if we have a section s' of $E \otimes \bigwedge^p T^*_{\mathbb{C}}$, say $s' = \omega\mathfrak{F}$ (for ω a row vector of p-forms), then we set

$$\nabla s' = (d\omega)\mathfrak{F} + (-1)^p \omega\theta(\mathfrak{F})\mathfrak{F}.$$

So, locally, if a form is represented by a matrix ω, then ∇ amounts to applying d and adding that to $(-1)^p\omega\theta(\mathfrak{F})$. It is clear that this is the only way we could extend ∇, and the uniqueness allows one to extend locally and patch. □

The moral of (3) is that ∇ induces a "derivation-like" operator on the space of differential forms with E-coefficients. In general, we *cannot* define the exterior derivative d with twisted coefficients: the extension of a connection in this manner is the necessary substitute. However, it is not a perfect substitute, since the (extended) covariant derivative, unlike the exterior derivative, does not make the space of E-valued differential forms into a complex. The obstruction to this is, again, curvature.

Given such an extension of ∇ (denoted the same), we can consider the map

$$\nabla^2 : E(M) \to \left(E \otimes \bigwedge^2 T^*_{\mathbb{C}}\right)(M).$$

This is $\mathcal{A}(M)$-*linear*. Indeed, we can check this by computation:

$$\begin{aligned} \nabla^2(fs) &= \nabla(\nabla(fs)) = \nabla(dfs + f\nabla s) \\ &= d^2 fs + (-1)df(\nabla s) + df(\nabla s) + f\nabla^2 s \quad \text{by (3)} \\ &= f\nabla^2 s. \end{aligned}$$

Clearly, as described above, ∇^2 is the obstruction to the covariant differentials forming a complex. We want to now connect this $\mathcal{A}(M)$-linear map (which can be identified with a $\hom(E,E)$-valued 2-form) with the earlier curvature form.

Proposition 8. *The vector-bundle map ∇^2 is equal to the curvature form Θ.*

Proof. We can work in local coordinates, and assume that X,Y are the standard commuting vector fields ∂_a, ∂_b. We want to show that, given a section s, we have

$$\nabla^2(s)(\partial_a, \partial_b) = \left(\nabla_{\partial_b}\nabla_{\partial_a} - \nabla_{\partial_a}\nabla_{\partial_b}\right) s \in E(M).$$

To do this, we should check exactly how the ∇ was defined. Namely, we have, by definition

$$\nabla s = \sum_i dx_i \nabla_{\partial_i} s,$$

and consequently, by (3),

$$\nabla^2 s = -\sum_i dx_i \wedge \nabla\left(\nabla_{\partial_i} s\right) = -\sum_i dx_i \wedge \left(\sum_j dx_j \nabla_{\partial_j} \nabla_{\partial_i} s \right) = -\sum_{i<j} dx_i \wedge dx_j \left(\nabla_{\partial_j} \nabla_{\partial_i} s - \nabla_{\partial_i} \nabla_{\partial_j} s \right).$$

It is easy to see that this, evaluated on (∂_a, ∂_b), gives the desired quantity. It follows that ∇^2 can be identified with Θ. ∎

As a result of the last computation, we may calculate the curvature in a frame. Let E be a vector bundle with a connection ∇, let $\mathfrak{F} = \{e_1, \dots, e_n\}$ be a frame, and let $\theta(\mathfrak{F})$ be the connection matrix. Then we can obtain an n-by-n *curvature matrix* $\Theta(\mathfrak{F})$ of 2-forms such that

$$\nabla^2 \mathfrak{F} = \Theta(\mathfrak{F})\mathfrak{F}.$$

This transforms very simply with respect to a change of frame $\mathfrak{F} \mapsto g\mathfrak{F}$: namely: $\Theta(g\mathfrak{F}) = g\Theta(\mathfrak{F})g^{-1}$, because Θ is a morphism of vector bundles.

The following result enables us to compute $\Theta(\mathfrak{F})$.

Proposition 9 (Cartan equation)**.**

$$\Theta(\mathfrak{F}) = d\theta(\mathfrak{F}) - \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}). \tag{4}$$

Note that $\theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})$ is not zero in general! The reason is that one is working with *matrices* of 1-forms, not just plain 1-forms. The wedge product is thus the matrix product, in a sense. However, it *will* be zero when $\theta(\mathfrak{F})$ is a one-dimensional matrix, i.e., when E is a *line bundle.* This will show that $\Theta(\mathfrak{F})$, for a line bundle, is always a closed 2-form, and consequently represents an element of H^2 of the manifold: this turns out, up to a scalar, to be the first Chern class.

Proof. Indeed, we need to determine how ∇^2 acts on the frame $\{e_i\}$. Namely, with an abuse of notation:

$$\nabla^2(\mathfrak{F}) = \nabla(\nabla\mathfrak{F}) = \nabla(\theta(\mathfrak{F})\mathfrak{F}) = d\theta(\mathfrak{F})\mathfrak{F} - \theta(\mathfrak{F}) \wedge (\theta(\mathfrak{F})\mathfrak{F}).$$

We have used the formula that describes how ∇ acts on a product with a form. As a result, in the frame $\{e_i\}$, the matrix for ∇^2 is what was claimed. ∎

Finally, we shall need an expression for $d\Theta$. We shall state this in terms of a local frame.

Proposition 10 (Bianchi identity)**.** *With respect to a frame $\mathfrak{F}$, $d\Theta(\mathfrak{F}) = [\theta(\mathfrak{F}), \Theta(\mathfrak{F})]$.*

Here the right side consists of matrices of differential forms, so we can talk about the *commutator.* We shall use this identity at a crucial point in showing that the Chern-Weil homomorphism below is even well-defined.

Proof. This is a simple computation. For, by Cartan's equations,

$$d\Theta(\mathfrak{F}) = d\left(d\theta(\mathfrak{F}) - \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})\right) = -d\theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) + \theta(\mathfrak{F}) \wedge d\theta(\mathfrak{F}).$$

Similarly,

$$[\theta(\mathfrak{F}), \Theta(\mathfrak{F})] = [\theta(\mathfrak{F}), d\theta(\mathfrak{F}) + \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})] = [\theta(\mathfrak{F}), d\theta(\mathfrak{F})]$$

because $[\theta(\mathfrak{F}), \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})] = 0$. ∎

1.3. **The Chern connection.** So far, all our general theory of connections has applied to real or complex vector bundles on any manifold, real or complex. We now want to specialize to the case of primary interest to us in the future, that of a hermitian holomorphic bundle on a complex manifold. On a Riemannian manifold, there is a canonical connection in the tangent bundle (the Levi-Civita connection) which is torsion-free and such that the metric has zero covariant derivative. On a complex manifold, a hermitian holomorphic bundle similarly has a canonical connection which is parallel with respect to the metric and satisfies an integrability condition.

Let M be a complex manifold. Given a complex vector bundle $E \to M$ (not necessarily holomorphic), recall that a *hermitian metric* on E is a smoothly varying family of hermitian metrics $(\cdot, \cdot) : E \otimes \overline{E} \to \mathbb{C}$ on the fibers $E_x, x \in M$. A bundle together with a hermitian metric is called a *hermitian* bundle.

We now want to show that, given a *holomorphic* hermitian bundle E over M, there is a unique connection on E which is compatible with the metric and which satisfies a differential condition below.

Proposition 11 (Existence of the Chern connection). *Given a hermitian holomorphic bundle $E \to M$, there is a unique connection ∇ on E such that:*

(1) *∇ is holomorphic: that is, if e is a local holomorphic section, then ∇e takes values in $T_{\mathbb{C}}^{*(1,0)} \otimes E$.*
(2) *∇ is compatible with the hermitian metric. That is, parallel transport preserves the metric, or equivalently, if e, e' are local sections,*

$$d(e, e') = (De, e') + (e, De').$$

In the first assertion, holomorphicity of ∇ can be expressed as follows: the composite

$$E(M) \xrightarrow{\nabla} (T_{\mathbb{C}}^* \otimes E)(M) \to (T_{\mathbb{C}}^{*(0,1)} \otimes E)(M)$$

is given by the derivative $\overline{\partial}$ (here the second map is projection). In the second assertion, note that we have extended the metric $E \otimes \overline{E} \to \mathbb{C}$ to E-valued forms.

This connection will be called the *Chern connection.*

Proof. Fix a holomorphic frame $\mathfrak{F} = \{e_1, \dots, e_n\}$ defined locally, so that the hermitian metric corresponds to functions h_{ij} (which form a hermitian matrix, denoted simply by h). Consider a connection ∇ with connection matrix $\theta = \theta(\mathfrak{F})$. We then have, for any such connection,

$$dh_{ij} = (\nabla e_i, e_j) + (e_i, \nabla e_j) = (\sum_k \theta_{ik} e_k, e_j) + (e_i, \sum_k \theta_{jk} e_k) = \sum_k \theta_{ik} h_{kj} + \sum_k \overline{\theta_{jk}} h_{ik}$$

However, we require that the θ_{ij} consist of $(1,0)$-forms (by holomorphicity). We thus find the two identities

$$\partial h = \theta h, \quad \overline{\partial} h = h \overline{\theta}^t.$$

Naturally, the first equation forces $\theta = \partial h h^{-1}$, while the second forces $\overline{\theta}^t = h^{-1} \overline{\partial} h$; we just need to see that the two equations are compatible. However, this follows because $\overline{h} = h^t$, so $\overline{\partial} h = \overline{\partial \overline{h}} = \overline{\partial h^t}$. Consequently, the second equation yields $\overline{\partial h^t} = \overline{h^t \theta^t}$, precisely the conjugate transpose of the first. ☐

In fact, we have obtained a *formula* for the Chern connection as a matrix of $(1,0)$-forms with respect to a holomorphic frame: we have

$$\theta = \partial h h^{-1}. \tag{5}$$

In such a holomorphic frame, we can also compute the curvature:

Corollary 12. *Hypotheses as above, we have*

$$\Theta(\mathfrak{F}) = \overline{\partial}(\partial h h^{-1}). \tag{6}$$

In particular, the curvature of the Chern connection on a holomorphic hermitian vector bundle consists of a skew-symmetric matrix of $(1,1)$-forms.

Proof. Indeed, we know that the curvature is given locally by $d\theta - \theta \wedge \theta$. This is true in *any* choice of frame. However, if we choose a unitary choice of frame, then θ becomes skew-symmetric (from the definitions). As a result, $d\theta - \theta \wedge \theta$ is skew-symmetric in any frame, because being skew-symmetric is an invariant property for a *tensor*. (It is not true that the connection matrix is always skew-symmetric, because the connection matrix does not transform as a tensor.) It thus follows that the curvature matrix is always skew-symmetric.

We need only see that it consists of $(1,1)$-forms. To do this, however, we note that Θ has no component of the form $(0,2)$, because $\Theta(\mathfrak{F}) = d\theta - \theta \wedge \theta$ and θ is of the form $(1,0)$ if we choose a *holomorphic* frame. As a result, it follows that $\Theta(\mathfrak{F})$ is equal to its $(1,1)$-component, which is easily seen to be $\overline{\partial}(\partial h h^{-1})$ by (5). ☐

2. Chern classes

To start with, we will need to describe what the Chern classes really are, in order to describe the topological meaning of "positivity."

2.1. Generalities.

Let X be a (suitably nice) topological space. The Chern classes are natural maps

$$c : \mathrm{Vect}_{\mathbb{C}}(X) \to H^*(X;\mathbb{Z}),$$

from $\mathrm{Vect}_{\mathbb{C}}(X)$, the isomorphism classes of complex vector bundles on X, to the cohomology ring. In other words, to each vector bundle $E \to X$, we will have an element $c(E) \in H^*(X;\mathbb{Z})$. In order for this to be *natural*, we are going to want that, for any map $f : Y \to X$ of topological spaces,

$$c(f^*E) = f^*c(E) \in H^*(Y;\mathbb{Z}).$$

In other words, we are going to want the map $\mathrm{Vect}_{\mathbb{C}}(X) \to H^*(X;\mathbb{Z})$ to be *functorial* in X, when both are considered as contravariant functors in X.

It turns out that each subfunctor $\mathrm{Vect}_{n,\mathbb{C}}$ (of isomorphism classes of n-dimensional complex vector bundles) and $H^k(X;\mathbb{Z})$ is *representable* on the appropriate homotopy category. Indeed, an n-dimensional complex vector bundle is the same as a principal $\mathrm{GL}_n(\mathbb{C})$ bundle over X, and such are classified by homotopy classes of maps $X \to B\mathrm{GL}_n(\mathbb{C})$. One can explicitly write down what the classifying space $B\mathrm{GL}_n(\mathbb{C})$ is: it is the infinite Grassmannian $\mathrm{Gr}_n(\mathbb{C}^\infty)$ of n-planes in $\mathbb{C}^\infty$. There is a canonical n-dimensional vector bundle V_n on this Grassmannian, consisting of pairs

$$(x, v) \in \mathrm{Gr}_n(\mathbb{C}^\infty) \times \mathbb{C}^\infty,$$

where v belongs to the plane corresponding to x. This bundle is *universal*: every vector bundle on a (reasonable) space X is the pull-back of V_n by a map $X \to \mathrm{Gr}_n(\mathbb{C}^\infty)$, unique up to homotopy. Although it does not matter for our purposes, the functors $H^k(X;\mathbb{Z})$ are also representable, by the Eilenberg-Maclane spaces $K(\mathbb{Z};k)$.

By Yoneda's lemma, to give such a "characteristic class" is to give an element in the cohomology ring of each Grassmannian $\mathrm{Gr}_n(\mathbb{C}^\infty)$. One can explicitly compute the cohomology ring of the Grassmannian and thus construct the Chern classes. However, let us just state the axioms that we want Chern classes to satisfy:

1. c of the trivial bundle is 1.
2. c of an n-dimensional bundle has terms in the cohomology ring only in even degrees $\leq 2n$.
3. $c(E \oplus E') = c(E)c(E')$, for E, E' vector bundles.
4. c of the tautological line bundle on $\mathbb{CP}^1$ is a fixed generator of $H^2(\mathbb{CP}^1;\mathbb{Z})$ (e.g. 1 with the positive orientation).

These conditions are actually going to *determine* the Chern classes.

Theorem 13. *There exists a unique natural transformation* $c : \mathrm{Vect}(X) \to H^*(X;\mathbb{Z})$ *satisfying the above four axioms.*

We shall simply *assume* that they exist, and satisfy these axioms. They are constructed in [MS74], for instance. We will, however, prove uniqueness below by the so-called *splitting principle.*

2.2. **Chern classes of a line bundle.** However, let's step back and try now to compute explicitly the Chern class $c(L)$ for a line bundle L on a connected space X. Note that $c(L) = 1 + c_2(L)$ for c_2 homogeneous of degree two: this is because, firstly, of the axioms, and the fact that the total Chern class $c(E)$ of a vector bundle E is always invertible (because any vector bundle has a complement E' such that $E \oplus E'$ is trivial). The last observation is necessary to argue that the zeroth term of $c(L)$ is 1.

Let $\mathcal{A}_c$ the sheaf of complex-valued continuous functions on X. Then line bundles on X can be described as elements of $H^1(X; \mathcal{A}_c^*)$, because a line bundle can be constructed by "gluing," and this is what Cech 1-cocycles measure. There is an exact sequence

$$0 \to \mathbb{Z} \to \mathcal{A}_c \stackrel{f \mapsto e^{2\pi\sqrt{-1}f}}{\to} \mathcal{A}_c^* \to 0$$

of sheaves, which leads to a map

$$\mathrm{Pic}(X) = H^1(X; \mathcal{A}_c^*) \to H^2(X; \mathbb{Z}).$$

The claim is that we can describe the first Chern class of a complex *line* bundle in this way. Let us call the above class constructed from sheaf cohomology $c_2'(L) \in H^2(X; \mathbb{Z})$; we need to show that $c_2(L) = c_2'(L)$.

Proof. Given a line bundle L over X, the above construction of an element of $H^2(X; \mathbb{Z})$ via the coboundary map is clearly *natural* in L, because pulling back L corresponds to pulling back the 1-cocycle that defines it. As a result, we just need to show that c_2' of the tautological line bundle on $\mathbb{CP}^\infty$ is a generator of the cohomology ring, because of the universality of this line bundle. In fact, because there is a map

$$\mathbb{CP}^1 \to \mathbb{CP}^\infty,$$

which induces an isomorphism on $H^2(\cdot; \mathbb{Z})$, it suffices to do this for the tautological line bundle on $\mathbb{CP}^1$. This, however, can be done explicitly. **However, this needs to be added.** □

2.3. **The splitting principle.** We now discuss a technique that often enables questions about Chern classes of vector bundles to be reduced to the case of line bundles. In particular, it implies that if we have two sets of "Chern classes" that agree on line bundles, they agree totally. As a result, we will easily find that the Chern classes are unique.

Theorem 14. *Let $E \to X$ be a vector bundle. Then there is a space $p : Y \to X$ such that $p^* : H^*(X) \to H^*(Y)$ is injective and p^*E splits as a sum of line bundles.*

In general, there is no reason to expect a vector bundle to split as a sum of line bundles. Incidentally, on $\mathbb{CP}^1$, this is the case (for holomorphic or algebraic line bundles) by a theorem of Grothendieck, which is an application of the sheaf cohomology of line bundles on $\mathbb{CP}^1$.

Proof. We shall find a map $p : Y \to X$ such that p^*E splits as a sum of a line bundle and another bundle, and p^* is injective in cohomology. Repeating this construction, we shall get the claim.

To do this, we take $Y = \mathbf{P}(E)$, the projectivization of E. This is a fiber bundle over E, whose fiber over $x \in X$ is the projective space $\mathbf{P}(E_x)$ of lines through the origin in the $\mathbb{C}$-vector space E_x. There is a *tautological* line bundle on Y: namely, the subset of $Y \times_X E$ consisting of pairs (y, e) (lying over, say, $x \in X$), such that $e \in E_x$ belongs to the line corresponding to y. This is clearly a complex line bundle, and it is a subbundle of p^*E by construction.

So all we need to see is that the map in cohomology is injective. This follows from the Leray-Hirsch theorem. That is, the cohomology $H^*(Y)$ is a free module over $H^*(X)$, generated by (ironically) the first Chern class of the tautological line bundle on Y. Namely, we have already essentially seen that if E is a *trivial* vector bundle over X, then the first Chern class of the tautological line bundle in $\mathbf{P}(E)$ generates $H^*(\mathbf{P}(E))$ as a free module over $H^*(X)$; this is the Künneth formula and the previous computation on $\mathbb{CP}^n$. Locally, the claim is thus true. Since this first Chern class is globally defined, we can appeal to the Leray-Hirsch theorem. □

As a result, we may see quickly that the Chern classes, if they exist, are uniquely determined. That is, any two natural transformations c, c' from vector bundles to the cohomology ring satisfying the required axioms coincide. Namely, because $c = c'$ on the universal line bundle on $\mathbb{CP}^\infty$, it follows that $c = c'$ on any line bundle (by naturality, since any line bundle is a pull-back of the universal one). It follows that $c = c'$ on any vector bundle which is a sum of line bundles. By the splitting principle, it follows now that $c = c'$ for all vector bundles.

In fact, as in [Gro58], we can even *construct* the Chern classes of a general vector bundle in a purely axiomatic form from the first Chern class of a line bundle. Namely, given $E \to B$, to define $c(E)$, we consider the tautological line bundle on $\mathbf{P}(E)$ and its first Chern class $\eta \in H^2(\mathbf{P}(E); \mathbb{Z})$; we have that $H^*(\mathbf{P}(E); \mathbb{Z})$ is a free module of rank $\dim E$ over $H^2(B; \mathbb{Z})$. As a result, we can write

$$\eta^n = \sum_{i=0}^{n-1} q_i \eta^i$$

for uniquely determined cohomology classes $q_i \in H^2(B; \mathbb{Z})$. We then let $c_i(E) = (-1)^i q_i$. It takes a bit of work, of course, to show that doing this actually satisfies the axioms.

2.4. **Chern-Weil for line bundles.** We are now interested in discussing Chern-Weil theory, which gives an analytic description of the Chern classes of a complex vector bundle on a smooth manifold, modulo torsion.

We shall start by warming up with a special case. Let M be a smooth manifold, $L \to M$ a complex line bundle. Let ∇ be a connection on L, and let Θ be the curvature. Thus, Θ is a global section of $\wedge^2 T^*_{\mathbb{C}} \otimes \hom(L, L)$; but since L is a line bundle, this bundle is *canonically* identified with $\wedge^2 T^*_{\mathbb{C}}$. So the curvature is a global 2-form.

Proposition 15 (Chern-Weil for line bundles)**.** *Θ is a closed form, and the image in $H^2(M; \mathbb{C})$ is $2\pi\sqrt{-1}$ times the first Chern class of the line bundle L.*

Proof. Let us suppose that we have an open cover $\{U_\alpha\}$ of M such that each finite intersection of elements of this cover is either empty or contractible; we can do this by choosing a Riemannian metric on M, and then taking geodesically convex neighborhoods. The line bundle L is described by nonvanishing, complex-valued continuous functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, satisfying the usual cocycle condition. To compute the first Chern class, we take the family of functions $\log g_{\alpha\beta}$ (this can be done, since each intersection is contractible!), and take their Cech 2-coboundary. That is, for a triple α, β, γ, we consider the *integer*

$$\frac{1}{2\pi\sqrt{-1}} \left(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma} \right);$$

these integers, for α, β, γ varying, form a Cech 2-cocycle (integer-valued), which is the first Chern class, by *definition* of the connecting homomorphisms in sheaf cohomology.

Now let us try to understand where the curvature Θ lives in de Rham cohomology, and first that it is actually a closed 2-form. Over each U_α, the bundle L is trivial, and we have chosen an isomorphism of it with the trivial bundle (this is what choosing the $g_{\alpha\beta}$ amounted to), so we have a canonical frame $\{e_\alpha\}$ over U_α (so $e_\alpha \in L(U_\alpha)$). The transition from e_α to e_β over $U_\alpha \cap U_\beta$ is given by

$$e_\beta = g_{\alpha\beta} e_\alpha.$$

This is the definition of the local trivializations.

Now, we know that the connection form is a simply a 1-form θ_α for each α (as it's a one-by-one matrix) such that $\nabla e_\alpha = \theta_\alpha e_\alpha$, and the transition rule is, by (2),

$$\theta_\beta = g_{\alpha\beta} \theta_\alpha g_{\alpha\beta}^{-1} + (dg_{\alpha\beta}) g_{\alpha\beta}^{-1} = \theta_\alpha + d(\log g_{\alpha\beta}). \tag{7}$$

The curvature form is given by, locally,

$$\Theta_\alpha = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha, \tag{8}$$

because θ_α is a 1-form (and not a large matrix), so $\theta_\alpha \wedge \theta_\alpha = 0$.

With these preliminaries established, we can figure out what is happening. Note first that (8) implies that Θ_α is a closed 1-form, and consequently Θ (which is obtained by gluing the Θ_α together) is a closed 1-form itself.

We next need to figure out where Θ_α maps to in $H^2(M;\mathbb{C})$. To do this, we need to unwind the de Rham isomorphism, and use (7). Namely, the de Rham isomorphism came from the sheaf-theoretic resolution (where $\mathcal{A}^i$ denotes the sheaf of i-forms)

$$0 \to \mathbb{C} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to \dots.$$

So, if we have a global closed 2-form ω, and we want to figure out where it goes in $H^2(M;\mathbb{C})$ (as a Cech 2-cocycle), we need to start by lifting ω over each U_α: that is, we need to find 1-forms $\tau_\alpha \in \mathcal{A}^1(U_\alpha)$ such that $d\tau_\alpha = \omega|_{U_\alpha}$. Then, we need to form the associated 1-cocycle $(\alpha,\beta) \mapsto \tau_\beta - \tau_\alpha$, which is an element of $H^1(d\mathcal{A}^0)$. Then we have to unapply d, and take the 2-coboundary of this. This is how the de Rham isomorphism works.

Locally we can lift Θ to $d\theta_\alpha$. So we can take the $\tau_\alpha = \theta_\alpha$. The differences $\tau_\beta - \tau_\alpha$ are given by $d(\log g_{\alpha\beta})$, by the transition rules. Now we have to unapply d and take the 2-coboundary of this. But then we get precisely the differences $\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma}$ which were used to define the first Chern class.

□

2.5. **Invariant polynomials.** For a general vector bundle, the curvature Θ (of a connection) will not in itself be a form, but rather a differential form with coefficients in $\hom(E,E)$. In order to get a differential form from this, we shall have to apply an invariant polynomial.

Definition 16. An **invariant k-linear form** on n-by-n matrices is a multilinear map $\phi(\cdot,\dots,\cdot) \to \mathbb{C}$, whose k inputs are elements of the ring of n-by-n complex matrices $M_n(\mathbb{C})$; this is required to satisfy

$$\phi(A_1,\dots,A_k) = \phi(gA_1g^{-1},\dots,gA_kg^{-1}), \quad A_1,\dots,A_k \in M_n(\mathbb{C}), g \in \mathrm{GL}_n(\mathbb{C}).$$

We shall write Φ for the function $M_n(\mathbb{C}) \to \mathbb{C}$ given by $\Phi(A) = \phi(A,A,\dots,A)$.

Given such a form, and given a global section τ of $\hom(E,E) \otimes \bigwedge^2 T^*_{\mathbb{C}}$, we can define $\Phi(\tau) \in \mathcal{A}^{2k}(M)$. Indeed, Φ induces a *canonical* map (not linear) $\hom(E,E) \to \mathbb{C}$. The invariance with respect to conjugation assures this: it does not depend which basis we choose for a fiber of E. More generally, given $\tau_1,\dots,\tau_k \in (\hom(E,E) \otimes \bigwedge^2 T^*_{\mathbb{C}})(M)$, we can define an ordinary differential form $\phi(\tau_1,\dots,\tau_k) \in \mathcal{A}^{2k}(M)$. Note the identity

$$d\phi(\tau_1,\dots,\tau_k) = \sum_i \phi(\tau_1,\dots,d\tau_i,\dots,\tau_k),$$

which follows from the identity for $d(\omega \wedge \eta)$ and the fact that everything has even degree. This will be how we use the curvature form to get differential forms.

Let us note that any invariant homogeneous polynomial of degree k, $\Phi : M_n(\mathbb{C}) \to \mathbb{C}$, can be obtained by restricting a multilinear map $M_n(\mathbb{C})^k \to \mathbb{C}$ to the diagonal. The construction is explicit combinatorics, given in [GH78] in the section on Chern-Weil theory. In fact, Φ is enough to determine ϕ. We omit the details.

In the course of the next proof, we shall need:

Lemma 17. *Let $A_1,\dots,A_k \in M_n(\mathbb{C})$, and let $B \in M_n(\mathbb{C})$. We have:*

$$\sum_{i=1}^{k} \phi(A_1,\dots,A_{i-1},[B,A_i],A_{i+1},\dots,A_k) = 0. \tag{9}$$

Proof. Indeed, we have by invariance:

$$\phi(\exp(tB)A_1\exp(-tB),\dots,\exp(tB)A_k\exp(-tB)) = \phi(A_1,\dots,A_n).$$

Differentiating with respect to t now gives the result, because $\frac{d}{dt}|_{t=0}\exp(tB)A\exp(-tB) = [B,A]$.

□

2.6. **The Chern-Weil homomorphism.** Let M be a smooth manifold, $E \to M$ a smooth vector bundle of dimension n. We saw in the last subsection that, given a global E-valued 2-form, the application of an invariant polynomial Φ allows one to obtain a global $2k$-form (with no twisting). The curvature of any connection is such a global E-valued 2-form.

Theorem 18. *Let ϕ be an invariant k-linear form $M_n(\mathbb{C})^k \to \mathbb{C}$. For any connection ∇ on E with curvature form Θ, the $2k$-form $\Phi(\Theta)$ is a closed form. The cohomology class of $\Phi(\Theta)$ is independent of the choice of connection.*

Proof. The first assertion (closedness) is local, so let us work in a fixed frame $\mathfrak{F}$ where the connection matrix is $\theta(\mathfrak{F})$ and the curvature matrix is $\Theta(\mathfrak{F})$. Then $\Phi(\Theta) = \Phi(\Theta(\mathfrak{F}))$, locally, where $\Theta(\mathfrak{F})$ is considered as a matrix of forms and Φ a function on matrices.

We have $\Phi(\Theta(\mathfrak{F})) = \phi(\Theta(\mathfrak{F}), \dots, \Theta(\mathfrak{F}))$, so, because we are working with even-degree forms,

$$\begin{aligned} d\Phi(\Theta) &= \sum_i \phi(\Theta(\mathfrak{F}), \dots, d\Theta(\mathfrak{F}), \dots, \Theta(\mathfrak{F})) \quad i\text{th position} \\ &= \sum_i \phi(\Theta(\mathfrak{F}), \dots, [\theta(\mathfrak{F}), \Theta(\mathfrak{F})], \dots, \Theta(\mathfrak{F})) \quad \text{by the Bianchi identity} \\ &= 0 \quad . \end{aligned}$$

This proves that $\Phi(\Theta)$ is a closed form. Now we need to show that the cohomology class is independent of the choice of connection. Suppose given two connections ∇_0, ∇_1 on E. The strategy will be to consider the one-parameter family of connections $\nabla_t = (1-t)\nabla_0 + t\nabla_1$ and the respective one-parameter family of curvature forms Θ_t (which will vary smoothly). We will show that the cohomology class of $\Phi(\Theta_t)$ is constant in t.

To do this, consider the vector bundle $E' \to M \times [0,1]$ given by pull-back of E. We define a connection ∇' on E by $\nabla' = (1-t)\nabla_0 + t\nabla_1$ on $M \times \{t\}$. Namely, we pull back ∇_0 and ∇_1 to E' (we can pull back connections along with vector bundles), if t is the projection on the second coordinate, we define ∇' as the convex combination $\nabla' = (1-t)\nabla_0 + t\nabla_1$.

We have two inclusions $i_0, i_1 : M \hookrightarrow M \times [0,1]$ given by inclusion on $M \times \{0\}$ and $M \times \{1\}$. Then clearly $i_0^* E', i_1^* E'$ are *canonically* isomorphic to E. Moreover, i_0^* of ∇' is ∇_0 while i_1^* of ∇' is ∇_1.

Let Θ' be the curvature of ∇', and let Θ_0, Θ_1 be the curvatures of ∇_0, ∇_1. By *naturality* of the curvature, we have

$$i_0^* \Theta' = \Theta_0, \quad i_1^* \Theta' = \Theta_1.$$

But by the "homotopy invariance" of de Rham cohomology, the i_0^* and i_1^* of a closed form on $M \times [0,1]$ are cohomologous. This implies the result. ∎

(This slick argument is essentially from [Dup78].)

2.7. **The Chern classes.** Now, we want to use the general theory of the previous section to describe the Chern classes: that is, we are going to fix polynomials Φ_k, and then associate invariants $c_k(E)$ to a (smooth) complex vector bundle.

Namely, let $\Phi_k : M_n(\mathbb{C})^k \to \mathbb{C}$ be the invariant polynomial described by

$$\Phi_k(A) = \mathrm{Tr}(\bigwedge^k A).$$

The same Φ_k will be used for matrices of any degree. We shall use the following fact: if A, B are matrices, then

$$\Phi_k(A \oplus B) = \sum_{i+j=k} \Phi_i(A)\Phi_i(B). \tag{10}$$

This follows from the canonical decomposition $\bigwedge^k (V \oplus W) = \bigoplus \sum_{i+j=k} \bigwedge^i V \otimes \bigwedge^j W$ for vector spaces V, W.

Theorem 19. *The Chern classes $c_k(E)$ of a complex vector bundle $E \to M$ can be calculated as follows: choose a connection on E with curvature form Θ, and then*

$$c_k(E) = \Phi_k\left(\frac{\sqrt{-1}}{2\pi}\Theta\right) \in H^{2k}(M;\mathbb{C}).$$

Here, of course, we are identifying singular cohomology with de Rham cohomology.

Proof. It suffices to show that the above construction, which we denote by $d_k(E)$, satisfies the usual axioms for Chern classes. Then formal arguments, as given earlier, will imply that they coincide with the topological Chern classes (in particular, that they actually come from $H^\bullet(M;\mathbb{Z})$, which is not obvious here).

(1) The construction $E \mapsto d_k(E) = \Phi_k\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)$ is natural in E. Indeed, this follows because if given a map (say, smooth) $f : N \to M$, then we can pull back a connection ∇ on E to a connection $f^*\nabla$ on f^*E. The curvature also pulls back in the natural way.
(2) If $d(E) = \sum d_k(E)$, then $d(E)$ is multiplicative: $d(E \oplus E') = d(E)d(E')$. Here we use the fact that if ∇, ∇' are connections on E, E', then there is a connection $\nabla + \nabla'$ on $E \oplus E'$. (The parallel transport for this corresponds to the direct sum of the parallel transports on E, E' given by ∇, ∇'.) If Θ, Θ' are the curvatures on E, E', then the curvature on $E \oplus E'$ with respect to this new connection is the $E \oplus E'$-valued 2-form $\Theta \oplus \Theta'$. This follows by easy computation in a local frame: the θ-matrices just add. Now the multiplicativity claim about d follows from the result (10), which expresses the analogous multiplicativity on the functions Φ_k.
(3) $d(E) = c(E)$ if E is a line bundle. This follows in view of the computation already done earlier; see theorem 15.

With these axioms verified, we can now conclude that the characteristic classes constructed by Chern-Weil theory are ordinary Chern classes. □

Note that since $H^\bullet(M;\mathbb{R}) \subset H^\bullet(M;\mathbb{C})$, one consequence is that the above construction provides real differential forms (or rather, cohomology classes represented by real forms).

What we have essentially done is, for a manifold M with a smooth complex vector bundle $E \to M$ of dimension n, to give a homomorphism from the algebra of invariant polynomials on n-by-n matrices to the cohomology ring $H^*(M;\mathbb{C})$. In fact, an analog of this theory exists for principal G-bundles over any Lie group G. (Recall that there is an equivalence of categories between n-dimensional vector bundles and principal $\mathrm{GL}_n(\mathbb{C})$-bundles, or alternatively with $U(n)$-bundles.) That is, if $\mathfrak{g}$ is the Lie algebra of the compact Lie group G, then there is a morphism

$$(\mathbf{Sym}\mathfrak{g}^\vee)^G \to H^\bullet(M;\mathbb{R})$$

defined for every M with a principal G-bundle. So to every G-invariant polynomial function on the Lie algebra $\mathfrak{g}$, we can define a "characteristic class" of the principal G-bundle.

Moreover, with $M = BG$ and with the canonical bundle $EG \to BG$, this map is an isomorphism. (This should be taken with a pinch of salt, because BG is not a manifold!) In other words, characteristic classes of principal G-bundles (that is to say, elements of $H^\bullet(BG;\mathbb{R})$) are the same thing G-invariant polynomials on the Lie algebra $\mathfrak{g}$. This is quite a bit more than we have shown, even in the special case of $G = U(n)$: it implies that the Chern-Weil construction gives *all* possible (real-valued) characteristic classes. The actual construction of characteristic classes is given in a similar way, though; one chooses a connection on a principal G-bundle on a manifold, takes its curvature (a $\mathfrak{g}$-valued 2-form on the total space), applies an invariant polynomial, and projects down. See [Dup78].

3. Kähler manifolds

A Kähler manifold has a complex structure, a symplectic structure, and a hermitian structure, all of which are compatible in some sense. These compatibilities will give us several remarkable relations between the exterior and $\overline{\partial}$ derivatives, which we will use in proving the vanishing theorem.

3.1. **Linear algebra.** Let V be a complex vector space with a hermitian metric $(\cdot,\cdot) : V \times V \to \mathbb{C}$. We are going to show that this data determines an orthogonal structure on the real vector space $V_{\mathbb{R}}$ underlying V, as well as a symplectic structure on $V_{\mathbb{R}}$. Indeed, we can write

$$(\cdot,\cdot) = \frac{1}{2}S(\cdot,\cdot) - \sqrt{-1}A(\cdot,\cdot),$$

where S, A are $\mathbb{R}$-bilinear forms on $V_{\mathbb{R}}$ with values in $\mathbb{R}$. It follows from sesquilinearity that S is symmetric and positive-definite (so defines an inner product on V), and A is alternating. Moreover, A is nondegenerate.

To see this last claim, choose elements $\{e_j\} \subset V$ which form an orthonormal $\mathbb{C}$-basis. Thus $\left\{e_1, \sqrt{-1}e_1, e_2, \sqrt{-1}e_2, \dots\right\}$ forms an $\mathbb{R}$-basis for V. Note that $A(e_j, \sqrt{-1}e_j) \neq 0$ for each j, while e_j is orthogonal (under $(\cdot,\cdot)$, hence under A) to $e_{j'}$ and $\sqrt{-1}e_{j'}$ for $j \neq j'$. This implies that A induces a symplectic structure on $V_{\mathbb{R}}$.

The hermitian metric itself can be expressed as follows: there exist elements $\phi_i \in V^*$ such that

$$(\cdot,\cdot) = \sum \phi_i \otimes \overline{\phi_i} \in V^* \otimes \overline{V}^*.$$

To find these elements, we need only choose an orthonormal basis for the hermitian metric, and let ϕ_i be the dual elements. Now write $\phi_i = \alpha_i + \sqrt{-1}\beta_i$, for the α_i, β_i the real and imaginary parts, which are $\mathbb{R}$-linear functionals. We have then

$$S = 2\sum \alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i.$$

Indeed, we see that $\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i$ corresponds to the $\left\{e_i, \sqrt{-1}e_i\right\}$ piece. That is, α_i is zero on the $\mathbb{R}$-orthogonal complement of e_i, and β_i is zero on the $\mathbb{R}$-orthogonal complement of $\sqrt{-1}e_i$, while $\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i$ defines an inner product on the $\mathbb{R}$-vector space spanned by $\left\{e_i, \sqrt{-1}e_i\right\}$. Next, the associated 2-form A is

$$A = \sqrt{-1}\sum \phi_i \wedge \overline{\phi_i} = \sqrt{-1}\sum(\alpha_i + \sqrt{-1}\beta_i) \wedge (\alpha_i - \sqrt{-1}\beta_i) = 2\sum \alpha_i \wedge \beta_i. \tag{11}$$

We have thus seen how to obtain from a hermitian structure on V both an orthogonal and a symplectic structure on $V_{\mathbb{R}}$. We now want to show that the symplectic form A determines the *volume* form on V (when considered as a real vector space). Since V has an orthogonal structure and is canonically oriented (as a complex vector space), there will be a natural volume form. Indeed, an oriented $\mathbb{R}$-basis for V, orthonormal with respect to S, is given by $\left\{\frac{1}{2}e_1, \frac{\sqrt{-1}}{2}e_1, \dots, \frac{1}{2}e_n, \frac{\sqrt{-1}}{2}e_n\right\}$. The corresponding elements of $V_{\mathbb{R}}^*$ are the $2\alpha_i$ and $2\beta_i$. As a result, the volume form on $\bigwedge^{\dim_{\mathbb{R}} V} V$ is given by

$$2^n \alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \dots \wedge \alpha_n \wedge \beta_n.$$

This, however, is $\frac{1}{n!}A^n$. Consequently, the symplectic form A determines the volume form.

3.2. **Hermitian metrics.** Suppose now M is a complex manifold of complex dimension n, and g is a hermitian metric on M. We should be careful what we mean by this, so let us elaborate:

Recall that if M is a complex manifold, then the complexified tangent bundle $T_{\mathbb{C}}M = (TM) \otimes_{\mathbb{R}} \mathbb{C}$ splits canonically as a sum of the holomorphic and antiholomorphic bundles, $T'_{\mathbb{C}}M$ and $T''_{\mathbb{C}}M$. If z is an element in a holomorphic local coordinate system, then $\frac{\partial}{\partial z} \in T'_{\mathbb{C}}M$ and $\frac{\partial}{\partial \overline{z}} \in T''_{\mathbb{C}}M$. There is a canonical, $\mathbb{C}$-antilinear isomorphism $T'_{\mathbb{C}}M \simeq T''_{\mathbb{C}}M$ given by conjugation.

Definition 20. A **hermitian metric** g on the complex manifold M is a hermitian metric on the holomorphic tangent bundle $T'_{\mathbb{C}}M$.

First, let us show that there is a nice local parametrization of a hermitian metric. By choosing a local unitary frame of $T'_{\mathbb{C}}M$, and an associated dual "co"frame $\{\phi_i\}$ of $(1,0)$-forms (note that the $(1,0)$-forms are the dual to the holomorphic tangent bundle), we can write (by hypothesis) $g = \sum \phi_i \otimes \overline{\phi_i}$. In writing this, we have identified the antiholomorphic cotangent space (that is, the space of $(0,1)$-forms) with the conjugate of the bundle of $(1,0)$-forms. Moreover, we recall that a hermitian vector bundle can be seen as a section of $T'^{\vee}_{\mathbb{C}}M \otimes \overline{T'^{\vee}_{\mathbb{C}}M}$, because it is sesquilinear; here $\overline{T'^{\vee}_{\mathbb{C}}M}$ is the conjugate of the dual bundle $T'^{\vee}_{\mathbb{C}}(M)$ (that conjugate is the bundle of $(0,1)$-forms). More generally, given a complex vector bundle E, a hermitian metric on E is the same thing as a section of $E^\vee \otimes \overline{E^\vee}$ (with the relevant positivity). So in our case, a hermitian metric is a special type of section of the tensor product of the bundles of $(1,0)$ and $(0,1)$-forms on M.

If we have such a metric, we can extend it in a natural way to the *entire* complexified tangent bundle $T_{\mathbb{C}}M$. We can do this in such a way that $g(\overline{v}, \overline{w}) = \overline{g(v,w)}$ for tangent vectors v, w, and such that the holomorphic and antiholomorphic tangent spaces are orthogonal; the extension is unique. Thus, we may regard the hermitian metric as existing on the *entire* complexified tangent bundle $T_{\mathbb{C}}M$. If we consider the inclusion

$$T_{\mathbb{R}}M \hookrightarrow T_{\mathbb{C}}M,$$

then we see that, since g is *real* (i.e. commutes with conjugation), g restricts to a *Riemannian* metric on the real tangent space $T_{\mathbb{R}}M$. A hermitian structure thus determines a Riemannian structure on the underlying real manifold.

Now, we want to construct a 2-form from a hermitian metric. As above, a hermitian metric on M is a section of the bundle $T_{\mathbb{C}}'^{\vee} M \otimes \overline{T_{\mathbb{C}}'^{\vee} M}$. This bundle, however, can be identified with $T_{\mathbb{C}}^{*(0,1)} \otimes T_{\mathbb{C}}^{*(0,1)} \simeq T_{\mathbb{C}}^{*(1,1)}$: it is the bundle of $(1,1)$-forms. Thus, by making this identification of vector bundles, and by multiplying by $\sqrt{-1}$, we can *canonically* associate a $(1,1)$-form to a hermitian metric.

Definition 21. We denote the $(1,1)$-form associated with a hermitian metric by ω.

Let $g = \sum \phi_i \otimes \overline{\phi_i}$ for a coframe $\{\phi_i\}$ of $(1,0)$-forms, which we can make true locally. From the definition, we see that

$$\omega = \sqrt{-1} \sum \phi_i \wedge \overline{\phi_i},$$

showing that this is essentially the same construction as before (i.e. as in (11)), and the associated Riemannian (not hermitian) metric is $\sum \Re(\phi_i) \otimes \Re(\phi_i) + \Im(\phi_i) \otimes \Im(\phi_i)$.

Proposition 22. *ω is a real, nondegenerate 2-form, and, up to scaling, $\frac{1}{n!}\omega^n$ is the volume form on M (with respect to the induced Riemannian metric).*

Proof. We first check realness. Indeed, we need to see that $\overline{\omega} = \omega$, where is the conjugation operator on the complex vector of 2-forms; this will imply that ω comes from a real 2-form. However, $\overline{\omega} = \omega$ follows from the description $\omega = \sqrt{-1} \sum \phi_i \wedge \overline{\phi_i}$.

We can think of ω in the following manner. For $z \in M$, consider the map $T_{z,\mathbb{R}}M \to T_{z,\mathbb{C}}M \to T'_{z,\mathbb{C}}M$, which establishes an isomorphism between the real tangent space and the holomorphic (complex) tangent space. We have a hermitian metric on the latter, which leads to a $\mathbb{R}$-linear 2-form; the pull-back of this to $T_{z,\mathbb{R}}M$ corresponds to ω.

Finally, we need to check the claim about the volume form. But if we choose an orthonormal basis for the real tangent space $T_{\mathbb{R}}M$ at a point, then that will become a $\mathbb{C}$-basis for the complexified tangent space, which is orthonormal (up to a factor) with respect to the *hermitian* metric on $T_{\mathbb{C}}M$. Now the result is clear from the discussion in the previous section. □

Since it is a little annoying to worry about normalization of the volume, we make:

Definition 23. Given a hermitian metric with associated $(1,1)$-form ω, we define the **hermitian volume** to be $\frac{1}{n!}\omega^n$.

So the hermitian volume is a real $2n$-form which is proportional (by a factor depending only on the dimension) to the Riemannian volume, for the induced Riemannian metric. Admittedly this definition is something of a hack to prevent us from worrying about factors of 2^n in the future.

3.3. **Kähler manifolds.** Thus, a hermitian metric determines a Riemannian metric on M when considered as a real manifold. It almost determines a symplectic structure: the problem is that ω, while nondegenerate, is not necessarily closed.

Definition 24. A hermitian complex manifold is **Kähler** if the associated 2-form ω is closed.

Note that a Kähler manifold is naturally a symplectic manifold. Indeed, ω is a symplectic form: to see this, we recall that ω is a real form.

Proposition 25. *Let M be an n-dimensional, compact Kähler manifold (where n is the complex dimension). For each $i \leq n$, we have $H^{2i}(M;\mathbb{R}) \neq 0$. That is, the even Betti numbers of M are positive.*

The argument below works for any compact symplectic manifold.

Proof. Indeed, we show that ω^i (which has degree $2i$, and is a real form) is closed but not exact for each $i \leq n$. It is closed because ω is closed. However, it is not exact, because then ω^n would be exact, and then

$$\frac{1}{n!}\int_M \omega^n = 0$$

by Stokes's theorem, a contradiction since this is the hermitian volume of M. (In general, on any compact oriented manifold, a nowhere vanishing top form has nonzero integral, so is not exact.) $\square$

In [GH78], ω is defined as half of what it is here. We have followed [SS85], because it makes the Kähler identities simpler. But as a result, the volume form with respect to the induced Riemannian metric becomes off by a factor of 2^n. The actual do not seem to be particularly important in the end, and we have taken the shortcut of defining a "hermitian volume."

Example 26 (The Fubini-Study metric)**.** Complex projective space $\mathbb{CP}^n$ or $\mathbb{P}^n_{\mathbb{C}}$ is the canonical example of a Kähler manifold. Indeed, we shall define a metric on this space via the fundamental 2-form. Let $U \subset \mathbb{CP}^n$ be an open set and let $s : U \to \mathbb{C}^{n+1}$ be a holomorphic section of the natural projection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$. Then we let:

$$\omega = \sqrt{-1}\partial\overline{\partial}\left|s\right|^2.$$

It is easy to check that this does not depend on the choice of lifting, and that it is a $(1,1)$-form. It turns out that this $(1,1)$-form actually represents a hermitian metric.

This metric is called the *Fubini-Study metric* on projective space. Since (as is easily checked), the restriction of a Kähler metric to a complex submanifold is Kähler, it follows that any complex submanifold of projective space is naturally a Kähler manifold.

Example 27. The euclidean metric $\sum dz_i \otimes d\overline{z_i}$ on $\mathbb{C}^n$ with canonical coordinates $(z_1, \dots, z_n)$ is Kähler: indeed, the fundamental $(1,1)$-form $\sqrt{-1}\sum dz_i \wedge d\overline{z_i}$ is clearly closed. As a result, any complex torus with the induced metric (note that translations are isometries on $\mathbb{C}^n$ with this metric) becomes a Kähler manifold. However, not every complex torus can be imbedded in projective space. Consequently, there are compact Kähler manifolds that are non-projective.

3.4. **The Hodge star.** Let M be a hermitian complex manifold, with hermitian metric g and associated $(1,1)$-form ω. There is a metric on the vector bundle $T'_{\mathbb{C}}M$; we extend this to a hermitian metric on the bundles $T^{*(p,q)}M$ of (p,q)-forms. Namely, if $\phi_1, \dots, \phi_n \in T'^*_{\mathbb{C}}M$ is a unitary coframe, so that $\sum \phi_i \otimes \overline{\phi_i}$ is the hermitian metric, then we take the $\phi_{i_1} \wedge \cdots \wedge \phi_{i_p} \wedge \overline{\phi_{j_1}} \wedge \cdots \wedge \overline{\phi_{j_q}}$ as an orthonormal frame for $T^{*(p,q)}$. We will still denote this hermitian metric by g.

We define an inner product on the space of global sections $\mathcal{A}^{(p,q)}(M)$ of $T^{*(p,q)}$ by

$$(\alpha, \beta) = \int_M g(\alpha, \beta) d\mu, \tag{12}$$

for μ the (hermitian) volume form $\frac{1}{n!}\omega^n = (\sqrt{-1})^n \phi_1 \wedge \cdots \wedge \phi_n \wedge \overline{\phi_1} \wedge \cdots \wedge \overline{\phi_n}$. We thus get a metric on the space of *all* differential forms on M, taking the various spaces $\mathcal{A}^{(p,q)}(M)$ orthogonal.

Definition 28. The *Hodge star* of a (p,q)-form β is the $(n-q, n-p)$-form $*\beta$ such that

$$(\alpha, \beta) = \int_M \alpha \wedge \overline{*\beta}.$$

This is uniquely defined if it exists. In fact, we can check that it exists locally: for instance, if we write $\phi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$ if $I = \{i_1, \dots, i_p\}$ and similarly for $\overline{\phi_J}$, then we find that $\phi_I \wedge \overline{\phi_J}$ will be mapped to $\overline{\phi_{I^c}} \wedge \phi_{J^c}$, up to a root of unity. We then have the stronger relation

$$g(\alpha, \beta)\mu = \alpha \wedge \overline{*\beta}. \tag{13}$$

We could use this to define the Hodge star.

The Hodge star is a $\mathbb{C}$-linear map. In fact, it is even linear over the space of smooth $\mathbb{C}$-valued functions, so it is a map of bundles. The Hodge star can, naturally, be defined locally at each point: one checks that the Hodge star of a form at a point depends only on the tangent space at that point, and the metric there. We could have constructed it with simple linear algebra. It is also a *real* operator, i.e. it commutes with conjugation. This is because the hermitian metric is real.

From the explicit expression in a local frame we see, furthermore, that the Hodge star is an *isometry* on each tangent space, hence globally. That is,

$$(\alpha, \beta) = (*\alpha, *\beta).$$

Proposition 29. $* * \alpha = (-1)^k \alpha$ *for a global* k*-form* α.

Proof. Indeed, this will follow from a few elementary manipulations and the fact that $*$ is an isometry. (It could also be checked locally, though the sign is a pain.) Namely, by definition of $*$ and its realness, we have (for β a form of the same type as α)

$$\begin{aligned} *\beta \wedge * * \alpha &= g(*\beta, \overline{*\alpha})\mu \quad \text{definition} \\ &= g(*\alpha, \overline{*\beta})\mu \quad \text{hermitianness} \\ &= g(\alpha, \overline{\beta})\mu \quad \text{isometry} \\ &= \alpha \wedge *\beta \quad \text{definition} \\ &= (-1)^k * \beta \wedge \alpha. \end{aligned}$$

Thus the wedge products of $* * \alpha$ and $(-1)^k \alpha$ with any form of the form $*\beta$ are the same. Since, however, $*$ is an isometry, we are done. ∎

With the Hodge star and a suitable metric on the space of (p, q)-forms constructed, we want to describe adjoints.

Recall that $\mathcal{A}^{(p,q)}(M)$ denotes the space of global, smooth (p, q)-forms on M.

Definition 30. We define $^*\overline{\partial}$ to be the adjoint to the operator $\overline{\partial} : \mathcal{A}^{(p,q)}(M) \to \mathcal{A}^{(p,q+1)}(M)$ with respect to the natural inner products (see (12)) on these spaces (given in terms of a hermitian structure on M). Similarly, we define $^*d$ to be the adjoint to the exterior derivative, and $^*\partial$ the adjoint to ∂.

In particular, these adjoint operators decrease the degree. It is not obvious at this point that the adjoints exist: we are not working with Hilbert spaces or even bounded operators. However, we can give an explicit description:

Proposition 31. *We can take* $^*\overline{\partial} = - * \circ \partial \circ *$. *Similarly, we can take* $^*d = - * \circ d \circ *$.

The adjoints are uniquely determined. Similarly, there is an adjoint $^*\partial$ to ∂: $^*\partial = - * \circ \overline{\partial} \circ *$.

Proof. This is a direct computation. We do it for d; the result for $\overline{\partial}$ follows by taking the appropriate component (the adjoint to $\overline{\partial}$ should lower the bidegree by $(0, 1)$). Namely, given a p-forms α and a $p+1$-form β (note that we consider the space of forms as simply a graded, and not a bigraded, algebra here), we have

$$\int d(\alpha \wedge *\beta) = 0$$

by Stokes's theorem. As a result, we get

$$\int d\alpha \wedge *\beta + (-1)^p \int \alpha \wedge d * \beta = \int d\alpha \wedge *\beta + (-1)^p (-1)^{2n-p} \int \alpha \wedge * * d * \beta = 0,$$

or, with the usual inner products on global differential forms,

$$(d\alpha, \overline{\beta}) + (\alpha, *d * \overline{\beta}) = 0.$$

Replacing $\overline{\beta}$ by β, we find that this is precisely what we wanted to show. ∎

3.5. **Hodge theory.** Hodge theory allows for a representation of sheaf cohomology groups in terms of harmonic forms, and will be an essential tool in the proof of the vanishing theorem. To motivate this, recall the *Dolbeaut resolution* of the sheaf $\Omega^p = \Omega^{(p,0)}$ of holomorphic p-forms on a complex manifold M:

$$0 \to \Omega^p \to \mathcal{A}^{(p,0)} \overset{\overline{\partial}}{\to} \mathcal{A}^{(p,1)} \overset{\overline{\partial}}{\to} \dots .$$

Since the sheaves $\mathcal{A}^{(p,m)}$ of *smooth* (p,m)-forms are sheaves of modules over the sheaf of continuous functions, they are *soft* and have trivial cohomology. This is thus an acyclic resolution, and we can use it to compute sheaf cohomology. In particular, $H^m(M, \Omega^p)$ can be represented naturally as

$$H^m(M, \Omega^p) = \frac{\overline{\partial}\text{–closed } (p,m)\text{–forms}}{\overline{\partial}\text{–exact } (p,m)\text{–forms}}.$$

Suppose given a hermitian metric on M. Now, the claim is going to be that every cohomology class in $H^m(M, \Omega^p)$ has a *canonical* representative in the space of $\overline{\partial}$-closed (p,m)-forms. Before stating it, let us loosen the hypotheses: consider a holomorphic vector bundle $E \to M$. We can define sheaves $\Omega^p(E)$ and $\mathcal{A}^{(p,q)}(E)$ of holomorphic E-valued forms and E-valued (p,q)-forms. Since E is linear over the ring of holomorphic functions, we can define operators

$$\overline{\partial} : \mathcal{A}^{(p,q)}(E) \to \mathcal{A}^{(p,q+1)}(E).$$

We can do this precisely because $\overline{\partial}$ annihilates holomorphic functions; we cannot, in general, define an analog of the exterior derivative. (This is the whole reason we need connections.)

We are going to define the canonical representative of each cohomology class as a *harmonic* form, i.e. one which is annihilated both by $\overline{\partial}$ and its "adjoint." To define the adjoint, however, one has to define an inner product on the space of (p,q) forms with coefficients in E: this is done by taking the induced metric on $T_{\mathbb{C}}^{*(p,q)} \otimes E$ (from the metrics on E and the metric on the tangent bundle), and then making a definition analogous to (12).

Let us first do this in the case of E the trivial bundle.

Definition 32. The **Hodge $\overline{\partial}$-Laplacian** is the map $\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + {}^*\overline{\partial}\overline{\partial}$, which is a differential operator on the space $\mathcal{A}^{(p,q)}(M)$ of (p,q)-forms. Forms in the kernel of this operator are called **harmonic.**

Using elliptic operator theory, one may prove:

Theorem 33 (Hodge theorem)**.** *There is a direct sum decomposition of $\mathcal{A}^{(p,q)}(M)$ into*

$$\mathcal{A}^{(p,q)}(M) = \overline{\partial}\mathcal{A}^{(p,q+1)}(M) \oplus {}^*\overline{\partial}\mathcal{A}^{(p,q-1)}(M) \oplus \mathcal{H},$$

where $\mathcal{H}$ is the subspace of harmonic *forms. In particular, the cohomology $H^q(M, \Omega^p)$ is canonically isomorphic to the space of harmonic (p,q)-forms.*

The essential idea of this argument is that $1 + \Delta_{\overline{\partial}}$ turns out to be *invertible*, as $\Delta_{\overline{\partial}}$ is a postive elliptic operator. Once this is known, the rest is fairly straightforward.

In fact, this much of the story is true on *any* compact complex manifold with a hermitian metric. The interest in the Kähler condition comes from the compatibility of this decomposition with *another* flavor of Hodge theory, for compact oriented Riemannian manifolds: there one defines the Hodge Laplacian $\Delta_d = {}^*dd + d^*d$ on p-forms. Then one shows that every cohomology class in ordinary de Rham cohomology is represented by a unique Δ_d-harmonic form. In particular, elements of $H^p(M; \mathbb{R})$ are canonically with associated to harmonic p-forms on M.

On a compact Kähler manifold, Δ_d is *proportional* to Δ_∂, as we shall show below and consequently a Δ_d-harmonic m-form has its (p,q)-components all harmonic as well. In particular, one obtains a connection between the hermitian Hodge theory (which enabled one to compute the cohomology of sheaves of $\mathcal{O}_M$-modules) and Riemannian Hodge theory (which enabled one to compute the cohomology of the constant sheaf). One finds a decomposition

$$H^n(M; \mathbb{C}) = \bigoplus_{p+q=n} H^p(M; \Omega^q).$$

This will not be relevant to the vanishing theorem, however.

Let us briefly explain how the ideas generalize to forms with coefficients in a holomorphic vector bundle. Recall that the $\overline{\partial}$-operator makes sense on E-valued forms (though the exterior derivative does not). Suppose now given a hermitian metric on M, and a hermitian metric on the holomorphic vector bundle E. As we saw, we can define an inner product on $\mathcal{A}^{(p,q)}(E)$ for each (p,q). Indeed, if $\{\phi_i\}$ is a unitary coframe of $(1,0)$-forms for the Kähler metric, and $\{e_j\}$ a unitary frame of continuous sections of E, then we set $\{\phi_i \otimes e_j\}$ as a local frame: this gives a metric on the vector bundle, which gives as before a metric on sections.

Given this, we can define an adjoint ${}^*\overline{\partial} : \mathcal{A}^{(p,q)}(E) \to \mathcal{A}^{(p,q-1)}(E)$ to the $\overline{\partial}$ operator, as before. As a result, one similarly defines a $\overline{\partial}$ operator. One similarly has:

Theorem 34 (E-valued Hodge theorem)**.** *There is a direct sum decomposition of $\mathcal{A}^{(p,q)}(E)$ into*

$$\mathcal{A}^{(p,q)}(E) = \overline{\partial}\mathcal{A}^{(p,q+1)}(E) \oplus {}^*\overline{\partial}\mathcal{A}^{(p,q-1)}(E) \oplus \mathcal{H},$$

where $\mathcal{H}$ is the subspace of harmonic *E-valued forms. In particular, the cohomology $H^q(M, \Omega^p(E))$ is canonically isomorphic to the space of harmonic E-valued (p,q)-forms.*

The last statement follows by using E-valued Dolbeaut cohomology, as before. One consequence is the following:

Theorem 35 (Serre duality)**.** *There is an isomorphism $H^p(M, \Omega^q(E)) \simeq H^{n-p}(M, \Omega^{n-q}(E^\vee))$.*

The point is that a slight variant of the Hodge star induces an isomorphism with harmonic (p,q)-forms with coefficients in E and harmonic (p,q)-forms with coefficients in $E^\vee$.

3.6. **The Kähler identities.** Let M be a compact Kähler manifold, with fundamental 2-form ω. Consider the operator

$$L(\eta) = \omega \wedge \eta$$

defined on the space of differential forms $\bigwedge^\bullet(M) \to \bigwedge^\bullet(M)$. Since there is an inner product on this space (cf. the discussion of the Hodge star), we can define an *adjoint* operator Λ.

We will need this in more generality, so we define:

Definition 36. Given a form Ω, we define the operator E_Ω on $\bigwedge^\bullet(M)$ as $\eta \mapsto \Omega \wedge \eta$. The adjoint to the operator $E_{\overline{\Omega}}$ will be denoted I_Ω. The two operators will be called **exterior** and **interior** multiplication, respectively.

Actually, we can construct the interior operator:

Proposition 37. *We have*

$$I_\Omega = \pm * E_\Omega *, \tag{14}$$

where the sign depends on the degree: for Ω of degree k, and for an input form of degree m, it is $(-1)^{(k+1)m}$. In particular, the adjoint exists.

Proof. In fact, we have, by theorem 29,

$$\begin{aligned}(\alpha, *E_\Omega * \beta) &= \int \alpha \wedge * * \overline{E_\Omega * \beta}) \\ &= (-1)^m \int \alpha \wedge \overline{\Omega \wedge *\beta} \\ &= (-1)^m \int (\alpha \wedge \overline{\Omega}) \wedge \overline{*\beta} \\ &= (-1)^{(k+1)m} (\overline{\Omega} \wedge \alpha, \beta) = (-1)^{(k+1)m} (E_{\overline{\Omega}}\alpha, \beta).\end{aligned}$$

□

As a result of this proposition, we can speak of Λ without offense. Note that if Ω is a *one-form*, then we simply have $I_\Omega = *E_{\overline{\Omega}}*$.

We want to discuss the interior multiplication operator in more detail. Suppose η is a *cotangent* vector. We want to show that the interior multiplication is in fact a *derivation.* Namely, since all this is local (i.e. forming adjoints of linear operators commutes with restriction to open submanifolds), we can assume that we have a unitary coframe $\phi_1, \dots, \phi_n$ in $T_{\mathbb{C}}'^* M$ with respect to the Kähler metric. Given a form of the form

$$\phi_I \wedge \overline{\phi_J}, \quad I, J \subset \{1, 2, \dots, n\},$$

we see that applying the exterior operator E_η for η one of the $\{\phi_i\}$ or its conjugates simply augments either I or J, as the case may be (up to a sign).

Now consider $\eta = \phi_i$. Then, $I_\eta(\phi_I \wedge \overline{\phi_J}) = 0$ if $i \notin I$, because the interior product of a form containing a ϕ_i with $\phi_I \wedge \overline{\phi_J}$ is then zero. If $i \in I$, then (up to a sign) the interior product lops off the i from ϕ_I. From this, one can check that the interior product with respect to a *1-form* is in fact a derivation (with the sign): i.e. $I_\eta(\alpha \wedge \beta) = I_\eta \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge I_\eta \beta$: in fact, it is the familiar interior product with respect to the dual of η (with respect to the metric). With this in mind, we can easily compute the commutator of I_η, L:

Proposition 38. *For a holomorphic 1-form η.*

$$[I_\eta, L] = -\sqrt{-1} E_\eta. \tag{15}$$

Proof. Fix a form α. Then $[I_\eta, L]\alpha = I_\eta(\omega \wedge \alpha) - \omega \wedge I_\eta \alpha$, by definition. Since, however, I_η is a derivation, the first term becomes $I_\eta \omega \wedge \alpha + \omega \wedge I_\eta \alpha$, and we need only compute the difference. So we need to compute $I_\eta \omega$. If we can prove that this is $-i\eta$, then we win.

We can do this, for instance, by using local coordinates $\omega = \sqrt{-1} \sum \phi_i \wedge \overline{\phi_i}$ and computing I_η in them for η one of the ϕ_i. Here is a more conceptual argument. Let α be a $(1,0)$-form. We write

$$(I_\eta \omega, \alpha) = -(\omega, \alpha \wedge \overline{\eta})$$

because I_η and $E_{\overline{\eta}}$ are adjoints. However, the function $g(\omega, \overline{\eta} \wedge \alpha)$ is $\sqrt{-1}$ times the hermitian metric applied to $g(\eta, \alpha)$, by construction of ω and the inner product. We get that $(I_\eta \omega, \alpha) = -i(\eta, \alpha)$, which is what we wanted. (Note that $I_\eta \omega$ is a $(1,0)$-form.) ☐

Anyway, we now want to obtain relations between the $\overline{\partial}$ and usual Laplacian. Note first that L commutes with $\overline{\partial}, \partial$: this is because ω is a closed form (in all possible ways!). However, it is not clear how the adjoint Λ behaves with respect to differentiation. We shall need:

Theorem 39 (Kähler identities)**.**

$$[\Lambda, \partial] = \sqrt{-1}^* \overline{\partial}, \quad [\Lambda, \overline{\partial}] = -\sqrt{-1}^* \partial. \tag{16}$$

In [GH78], the Kähler identities are proved by the following strategy: they are verified (quite directly) for $\mathbb{C}^n$ with the euclidean metric.[1] Then, since (as we do not show), a Kähler metric is one approximable to order 2 by a euclidean metric, the "first-order" Kähler identities follow on any Kähler manifold. I tried without success to understand the fancier argument in [Wel08], but could not: at any rate, there seems to be a certain amount of *real* work necessary to prove these identities. Nevertheless, being uncomfortable with computation, I tried (with, perhaps, slightly more success) to understand the more conceptual approach in [SS85], and this is what I follow.

In order to do this, we shall need a factorization of the exterior derivative $d : \bigwedge^\bullet(M) \to \bigwedge^\bullet(M)$ into

$$\bigwedge^\bullet(M) \to \mathcal{A}^1(M) \otimes_{C^\infty(M)} \bigwedge^\bullet(M) \to \bigwedge^\bullet(M),$$

where the last map is the wedge product. The first map will behave well with respect to the Hodge star, while for the second will be the exterior product: as a result, the expression $d = - * d *$ will be made more accessible.

Namely, let us consider the Chern connection ∇ on the holomorphic complex cotangent bundle $T_{\mathbb{C}}'^* M$ and its wedge powers, extend by conjugation to the antiholomorphic cotangent bundle and *its* wedge powers, and then take the tensor product.

[1]Naturally this space is not compact, so one has to work with compactly supported forms.

Let $\bigwedge^\bullet$ denote the vector bundle of *all* differential forms on M: in short, the sum of the exterior powers of the cotangent bundle. As a result, there is a *canonical* connection on $\bigwedge^\bullet$. Thus there is a map

$$\nabla : \bigwedge^\bullet(M) \to \left(T^*_{\mathbb{C}} \otimes \bigwedge^\bullet\right)(M).$$

In other words, this sends a form to a form with some 1-form attached to it. It is *homogeneous* by construction. Moreover, it is a derivation with respect to the wedge product, without a sign.

To state the next proposition, we define the map of bundles

$$E : T^*_{\mathbb{C}} \otimes \bigwedge^\bullet \to \bigwedge^\bullet$$

by wedging. This is the global form of the exterior multiplication maps earlier.

Proposition 40. *On a Kähler manifold, we have*

$$d = E\nabla. \tag{17}$$

This is where the Kähler condition enters. In fact, it implies that the Chern connection coincides with the Levi-Civita connection associated with the metric, and this statement is true for the Levi-Civita connection (because the Levi-Civita connection is torsion-free).

Proof. Indeed, one checks directly that $E\nabla$ satisfies the following conditions: $E\nabla f = df$ for a function f, and $E\nabla$ behaves well with respect to wedge products, i.e., $E\nabla(\omega_1 \wedge \omega_2) = (E\nabla\omega_1) \wedge \omega_2 + (-1)^{\deg \omega_1}\omega_1 \wedge (E\nabla\omega_2)$. So $E\nabla$ is very close to being an exterior derivative, except we have to check that its square is zero. This is now an explicit computation given that the matrices representing ∇ in a local frame can be explicitly computed, as is the case for ω. See [SS85]. □

Next, we want to show that ∇ as defined commutes with the Hodge star operator $*$. Namely, we define $*$ as an endomorphism of the bundle $T^*_{\mathbb{C}} \otimes \bigwedge^\bullet$ by doing nothing on the first factor and doing the Hodge on the second factor.

Proposition 41. $\nabla * = *\nabla$.

Note that the two stars refer to slightly different things: one refers to the Hodge star, one to the tensor product of that with the identity on the cotangent bundle. This shows that ∇ can be useful when d is not (e.g. d does not commute with $*$). We shall *not* use the Kähler condition for this. In effect, it is saying that the Hodge star is parallel: this, however, is clear because the orthogonal structure is parallel (with respect to the Chern connection), as is the orientation, and this is what the Hodge star depends on.

Proof. Let us recall what ∇ satisfies: namely, $\nabla(\eta_1 \wedge \eta_2) = \nabla\eta_1 \wedge \eta_2 + \eta_1 \wedge \nabla\eta_2$, because of how we defined the connection on $\bigwedge^\bullet$ (in terms of the tensor product). Note that this is *not* the extension of the Chern connection on the tangent bundle to forms; this is completely separate, since the *ground bundle* itself consists of all the forms. The aforementioned identity is how one extends a connection to an exterior (or tensor) power.

Next, we note that the volume form $\mu = \frac{1}{n!}\omega^n$ is parallel with respect to ∇: that is, $\nabla\mu = 0$, since $\nabla\omega = 0$. (This in turn follows because the metric is parallel, and ω is constructed directly from the metric.) As a result, we find for forms α, β of the same type:

$$\nabla(\alpha \wedge *\beta) = \nabla\alpha \wedge *\beta + \alpha \wedge \nabla * \beta.$$

However, by (13), we find that the left side is also

$$\nabla(g(\alpha, \overline{\beta})\mu) = dg(\alpha, \overline{\beta})\mu = (g(\nabla\alpha, \overline{\beta}) + g(\alpha, \nabla\overline{\beta}))\mu.$$

Applying the same definitions, we get

$$g(\nabla\alpha, \overline{\beta})\mu = \nabla\alpha \wedge *\beta, \quad g(\alpha, \nabla\beta)\mu = \alpha \wedge *\nabla\beta.$$

If we equate these two expressions, we get the claim. □

Let us see what this implies. We get in fact a new expression for ${}^*d$. Define the operator $I : T^*_{\mathbb{C}} \otimes \bigwedge^\bullet \to \bigwedge^\bullet$ by interior multiplication by the 1-form part. As a result, we find by (17),

$$I = *E*.$$

We now get the promised simple expression for ${}^*d$.

Proposition 42. ${}^*d = -I\nabla$.

Proof. Indeed, we have

$$\begin{aligned} {}^*d &= -*d* \\ &= -*E\nabla* \\ &= -*E*\nabla \\ &= -I\nabla. \end{aligned}$$

□

Let us take homogeneous components in the previous result. Namely, we saw that ${}^*d = -I\nabla$. We recall that $\nabla = \nabla' + \nabla''$ where ∇' is the holomorphic part (which lies as a section of $T'^* \otimes \bigwedge^\bullet$), while ∇'' is the anti-holomorphic part. Note that $E\nabla'$ will, when applied to a (p,q)-form, be a $(p+1,q)$-form, whiel $E\nabla''$ will be a $(p,q+1)$-form: it follows that

$$E\nabla' = \partial, \quad E\nabla'' = \overline{\partial}. \tag{18}$$

With this in mind, we can obtain expressions for the adjoints. But note the antisymmetry.

Proposition 43. *The adjoint* ${}^*\partial$ *of* ∂ *is* $-I\nabla''$*; the adjoint* ${}^*\overline{\partial}$ *of* $\overline{\partial}$ *is* $-I\nabla'$*.*

Proof. This follows by taking homogeneous components of the proved identity ${}^*d = -I\nabla$ and looking at homogeneous components (since $I = *E*$). Namely, on a (p,q)-form, ∇'' will go to a product of an antiholomorphic cotangent vector and a (p,q)-form; $*$ will send that to the product of an antiholomorphic cotangent vector and an $(n-q, n-p)$-form; E will send that to a $(n-q, n-p+1)$-form; finally, $*$ again will send that to a $(p-1,q)$-form. It follows that $-I\nabla''$ shifts the bidegree by $(-1,0)$. Similarly, the other piece $-I\nabla'$ shifts the bidegree by $(0,-1)$. □

Finally, we shall be able to complete the proof of the Kähler identities.

Proof of (16). Namely, we are required to compute $[\Lambda, \overline{\partial}]$, or equivalently, by adjointness, $[L, {}^*\overline{\partial}]$. However, because L commutes with ∇ (as ω is parallel), and consequently $[L, \nabla'] = 0$, we have

$$\begin{aligned} [L, {}^*\overline{\partial}] &= [L, -I\nabla'] \\ &= [I, L]\nabla' \quad \text{by the Jacobi identity for commutators} \\ &= -\sqrt{-1}E\nabla' \quad \text{by (15), as the tangent vectors are holomorphic} \\ &= -\sqrt{-1}\partial \quad \text{by the expression (18).} \end{aligned}$$

Consequently, taking adjoints, we find

$$[\Lambda, \overline{\partial}] = -\sqrt{-1}({}^*\partial).$$

If we take the conjugate of this, we get the other Kähler identity. □

3.7. **Relations between the Laplacian.** Let E be a holomorphic hermitian vector bundle on the Kähler manifold M. Then E has a canonical connection ∇, the Chern connection. As a result, given an E-valued differential form, we can apply ∇ and produce a new E-valued differential form of one higher degree. We also have another differential structure: there is the $\overline{\partial}$-operator $\mathcal{A}^{(p,q)}(E) \to \mathcal{A}^{(p,q+1)}(E)$, as E is holomorphic.

As a result, there are several Laplacians we can construct. Namely, there is the $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}}$ as before, defined as

$$\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + {}^*\overline{\partial}\overline{\partial},$$

where ${}^*\overline{\partial}$ is the adjoint. (Using similar formulas as before, one can show that all these adjoints exist.) Next, we can think ∇ as a variant of exterior differentiation—our best substitute, as E is a vector bundle—and consider

$$\Delta = {}^*\nabla\nabla + \nabla^*\nabla,$$

for ${}^*\nabla$ the adjoint to ∇. Here we are not even using the holomorphic structure of E. Similarly, using the decomposition $\nabla = \nabla' + \nabla''$ (holomorphic and antiholomorphic parts), we can define a partial Laplacian

$$\Delta' = {}^*\nabla'\nabla' + \nabla'^*\nabla'.$$

(Of course, since $\nabla'' = \overline{\partial}$, doing the same for ∇'' gives us $\Delta_{\overline{\partial}}$ as before.) So in total, we have three Laplacians:

(1) The $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}}$.
(2) The total Laplacian Δ.
(3) The "holomorphic" Laplacian Δ'.

If E is the trivial line bundle with the trivial metric, then the Chern connection is the trivial one, and the associated ∇ operator is just the exterior derivative. So we should think of Δ as the generalization of the Laplacian Δ for a Riemannian manifold.

Our goal is to show that these are related.

Theorem 44 (Kodaira-Nakano identity)**.**

$$\Delta_{\overline{\partial}} = \Delta' + \sqrt{-1}[\nabla^2, \Lambda]. \tag{19}$$

Here Λ is the operator on E-valued differential forms which is the adjoint to wedging with the $(1,1)$-form ω from the Kähler structure.

Note that ∇^2 is just the curvature tensor. This will be the key point in the proof of Kodaira vanishing theorem, to estimate $\Delta_{\overline{\partial}}$ from below by the nonnegative operator Δ' (since we will explicitly compute $[\nabla^2, \Lambda]$).

Proof. Let us start by proving this when E is the trivial bundle, so ∇ is just exterior differentiation, and ∇' is just ∂. In this case, $\nabla^2 = 0$ in fact and the identity becomes very simple: $\Delta_{\overline{\partial}} = \Delta' = {}^*\partial\partial + \partial^*\partial$. We thus get an identification (up to a constant) of the Hodge Laplacian and the $\overline{\partial}$-Laplacian.

The strategy is to use the various expressions we have for ${}^*\overline{\partial}$ and $\overline{\partial}\Lambda$. Namely, we have by the Kähler identities:

$${}^*\overline{\partial} = -\sqrt{-1}[\Lambda, \partial],$$

so

$$\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + {}^*\overline{\partial}\overline{\partial} = -\sqrt{-1}(\overline{\partial}[\Lambda, \partial] + [\Lambda, \partial]\overline{\partial}) = -\sqrt{-1}\left(\overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}\right).$$

We now need to move some terms around, and thus need commutators. So, $\overline{\partial}\Lambda = [\overline{\partial}, \Lambda] + \Lambda\overline{\partial}$. By the Kähler identities, this is $\sqrt{-1}^*\partial + \Lambda\overline{\partial}$. Similarly, we find that $\Lambda\overline{\partial} = \overline{\partial}\Lambda - \sqrt{-1}^*\partial$ (by subtracting from the previous identity!). Putting this together, we find

$$\begin{aligned}\Delta_{\partial} &= -\sqrt{-1}\left((\sqrt{-1}^*\partial + \Lambda\overline{\partial})\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial}\right) \\ &= {}^*\partial\partial - \sqrt{-1}(-\overline{\partial}\partial\Lambda - \partial\Lambda\overline{\partial}) \\ &= {}^*\partial\partial - \sqrt{-1}(-\overline{\partial}\partial\Lambda - (\partial\overline{\partial}\Lambda - \sqrt{-1}\partial^*\partial)) \\ &= \Delta'.\end{aligned}$$

To prove this result for general vector bundles E, one has to establish the analogs of the Kähler identities (16) for E-valued forms. After this, one can follow the above argument (though the statements $\partial^2 = 0$ fail when ∂ is replaced by the connection analog). In fact, the analogs of (16) can be deduced directly the case we have handled. This is done in [GH78], for instance. □

4. The vanishing theorem

We are now mostly ready to approach the vanishing theorem. The key step will be the Kodaira-Nakano identity (19). Given a positive line bundle, we will choose a Kähler metric on our manifold such that the curvature tensor (with respect to the Chern connection) is simply the $(1,1)$-form ω of the metric. As a result, in (19), the objects Λ—the adjoint to wedging with ω—and ∇^2 will be closely related. In particular, we will be able to bound below this operator, and thus show that the $\overline{\partial}$-Laplacian is *strictly positive* in appropriate degrees.

4.1. **Positive line bundles.**

First, however, we need to discuss the positivity line bundles. Let M be a complex manifold. We do not yet assume M Kähler.

Definition 45. A holomorphic line bundle $L \to M$ is **positive** if there is a hermitian metric on L whose curvature form Θ (which is a skew-symmetric $(1,1)$-form with values in $\hom(L,L) = \mathcal{O}$) is such that $i\Theta$ is positive. A line bundle is **negative** if its dual is positive.

Recall the meaning of positivity of a $(1,1)$-form: a $(1,1)$-form is *positive* if it comes (as the fundamental form) from a hermitian metric on M.

The following is straightforward: if $M' \to M$ is an immersion of complex manifolds, and $L \to M$ a positive line bundle, then the pull-back $L \times_M M' \to M'$ is a positive line bundle too.

Example 46. The key example of a positive line bundle is $\mathcal{O}(1)$ on $\mathbb{P}^n$. Indeed, consider its dual $\mathcal{O}(-1)$: this can be identified with the tautological line bundle of points (x,v) in $\mathbb{P}^n \times \mathbb{C}^{n+1}$ such that v belongs to the line represented by x. We can put a metric on this dual by taking the usual euclidean metric on $\mathbb{C}^{n+1}$: i.e., the inner product of (x,v) and (x,w) is $v \cdot \overline{w}$.

Let $s : U \to \mathbb{C}^{n+1}$ be a local section of the line bundle, for U an open subset of $\mathbb{P}^n$. This means that s maps into $\mathbb{C}^{n+1} - \{0\}$, and $s(x)$ always lies in the line represented by x. We can compute the curvature of $\mathcal{O}(-1)$ with respect to this metric (with respect to the Chern connection) by (6): namely, if $(s(x), s(x)) = h$, then

$$\Theta = \overline{\partial}\partial \log h.$$

Since h was just $|s|^2$ for a local holomorphic lifting s of $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n_{\mathbb{C}}$, we see that, up to a constant, $-i\Theta$ is the fundamental form of the Fubini-Study metric on $\mathbb{P}^n_{\mathbb{C}}$. This implies that $\mathcal{O}(-1)$ is negative, so $\mathcal{O}(1)$ is positive.

Consequently, there is always a positive line bundle on a submanifold of projective space, by restricting $\mathcal{O}(1)$. However, there are obstructions to a complex manifold's admitting a positive line bundle. For instance, such a manifold *must* be Kähler. Indeed, the curvature form is necessarily closed—as we saw in the discussion of Chern-Weil theory—and it provides a hermitian metric on the holomorphic bundle (whose fundamental form is just this curvature form again).

In fact, the strategy to prove the vanishing theorem will be to *choose* the Kähler metric to match the curvature form.

4.2. **A commutator identity.**

Let M be a Kähler manifold. Let $E \to M$ be a holomorphic vector bundle with a hermitian metric. On M, we have a fundamental 2-form ω, and consequently we can define an operator L on the vector bundle $\bigwedge^\bullet \otimes E$ by wedging with ω, as we did before. Let us review some of the formalism developed earlier in this slightly more general context.

The bundle $\bigwedge^\bullet \otimes E$ inherits a hermitian metric (from the metrics on E and $\bigwedge^\bullet$, the latter from the Kähler metric). As a result, one can define an adjoint operator Λ.

Namely, let $\phi_1, \dots, \phi_n$ be a unitary coframe, so that $\omega = \sqrt{-1}\sum \phi_i \wedge \overline{\phi_i}$. If we define E_η, for a 1-form η, as before on $\bigwedge^\bullet \otimes E$, then we see that

$$L = \sqrt{-1}\sum E_{\phi_i} E_{\overline{\phi_i}}.$$

One similarly defines adjoint operators I_η (which is the adjoint to $E_{\overline{\eta}}$) on $\bigwedge^\bullet \otimes E$, just as in the case of E trivial. In particular, we can take

$$I_\eta = * E_{\overline{\eta}} *,$$

for a 1-form η, by the analog of (14). (We can easily reduce to (14) by working locally and assuming E admits an orthonormal frame; note that the holomorphic structure was irrelevant here.) As a result, we have

$$\Lambda = -\sqrt{-1}\sum I_{\overline{\phi_i}} I_{\phi_i}.$$

Here I_η is not quite a derivation, but it satisfies the identity

$$I_\eta(\alpha \wedge \beta) = I_\eta \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge I_\eta \beta,$$

valid for a 1-form α and an E-valued 1-form β.

Proposition 47. *$[L, \Lambda] = p + q - n$ on (p, q)-forms.*

Proof. We shall use the identity

$$[I_\eta, L] = -\sqrt{-1}E_\eta,$$

valid for a holomorphic 1-form, proved as before in (15); it is valid for E-valued forms, too. We have

$$[L, \Lambda] = -\sqrt{-1}\sum [L, I_{\overline{\phi_i}} I_{\phi_i}] = -\sqrt{-1}\sum \left([L, I_{\overline{\phi_i}}]I_{\phi_i} + I_{\overline{\phi_i}}[L, I_{\phi_i}]\right).$$

By the quoted identity, and its conjugate $[I_{\overline{\eta}}, L] = \sqrt{-1}E_{\overline{\eta}}$, we get

$$[L, \Lambda] = -\sqrt{-1}\sum \left(-\sqrt{-1}E_{\overline{\phi_i}} I_{\phi_i} + \sqrt{-1}I_{\overline{\phi_i}} E_{\phi_i}\right)$$

Using the explicit expression of I_ϕ as an interior product (i.e., lopping off a factor) and E_ϕ as an exterior product (i.e., adding a product), the identity follows. □

4.3. **Proof of the vanishing theorem.** We shall prove:

Theorem 48. *Let E be a negative line bundle on the compact complex manifold M. Then $H^p(M, \Omega^q(E)) = 0$ for $p + q < n$.*

By Serre duality, this will imply the Kodaira vanishing theorem as usually stated.

Proof. Choose a hermitian metric on E whose $(1, 1)$-curvature form Θ (for the Chern connection) is $\sqrt{-1}$ times a positive form ω. This positive form is closed, and is associated with a Kähler metric on M whose associated $(1, 1)$-form ω is precisely $\frac{1}{\sqrt{-1}}\Theta$.

By Hodge theory, it suffices to show that for $p + q < n$, there are no $\Delta_{\overline{\partial}}$-harmonic E-valued (p, q)-forms. We will do this by showing that $\Delta_{\overline{\partial}}$ is a strictly positive operator on (p, q)-forms with values in E. Indeed, by (19), we have

$$\Delta_{\overline{\partial}} = \Delta' + \sqrt{-1}[\nabla^2, \Lambda].$$

Here Δ' is the "holomorphic" Laplacian obtained from the holomorphic part of the Chern connection on E; it is a nonnegative operator, in that $(\Delta'\alpha, \alpha) \geq 0$ for all α (by construction). Now ∇^2 is just given by wedging with $\Theta = \sqrt{-1}\omega$, so it is also the operator iL described in terms of the Kähler metric. However, we have seen that $[L, \Lambda] = p + q - n$. As a result,

$$\sqrt{-1}[\nabla^2, \Lambda] = \sqrt{-1}[\sqrt{-1}L, \Lambda] = n - (p + q).$$

But if A is an endomorphism of an inner product space such that A is nonnegative (i.e., $(Av, v) \geq 0$ for all v), then $A + rI$ has trivial kernel for each $r > 0$. It follows that $\Delta_{\overline{\partial}}$ has trivial kernel if $p + q < n$. This is what we wanted to prove. □

We next prove an "asymptotic" version of the same result.

Theorem 49. *Let $L \to M$ be a positive line bundle on the compact complex manifold M. For any holomorphic vector bundle $E \to M$, we have*

$$H^p(M, E \otimes L^r) = 0, \quad r \gg 0$$

for all p.

This is analogous to a theorem of Serre in the algebraic setting: if $X \to \operatorname{Spec} A$ is a projective scheme, and $\mathcal{L}$ an ample line bundle on X, then for any coherent sheaf $\mathcal{F}$ on X, we have $H^p(X, \mathcal{F} \otimes \mathcal{L}^r) = 0$ for $r \gg 0$. The result stated here is only for vector bundles, because we need the tools of Hodge theory, but in fact it holds for coherent *analytic* sheaves E as well, as follows from the Kodaira *embedding* theorem along with GAGA. (I wish I knew a direct method to prove the generalization of this method to coherent analytic sheaves: that would also provide a proof of the Kodaira imbedding theorem without blow-ups.)

Proof. Of course, there are only finitely many admissible p (for which the cohomology can *ever* be nonzero), so it is sufficient to handle the case of a given p. In fact, only cohomology can occur in degrees $\leq n$, by Hodge theory.[2] By Serre duality, we may prove the following variant:

> Let $L \to M$ be a negative line bundle, and $E \to M$ any holomorphic vector bundle.
> Then if $p < n$, we have $H^p(M, E \otimes L^{-r}) = 0$ for $r \gg 0$.

This is what we shall prove. So, assume L is *negative.*

As before, choose a hermitian metric on L such that the associated curvature form Θ_L is $\sqrt{-1}\omega$ for ω a positive $(1,1)$-form. Then, choose a Kähler metric on M such that ω is the associated $(1,1)$-form for that Kähler metric. Choose any hermitian metric on E.

There are Chern connections ∇_L, ∇_E on L, E, respectively. If we take the induced metric on each $E \otimes L^r$, then we get a Chern connection on $E \otimes L^r$ whose associated curvature is $\Theta_E + 1 \otimes r\Theta_L$. The strategy is going to be to prove that for $r \gg 0$, the curvature—which is dominated by the term $1 \otimes r\Theta_L$, which makes the $\overline{\partial}$-Laplacian a positive operator. Consider the Kodaira-Nakano identity (19):

$$\Delta_{\overline{\partial}} = \Delta' + \sqrt{-1}[\nabla^2, \Lambda].$$

We apply it to the bundle $E \otimes L^r$. The goal is to show that $i[\nabla^2, \Lambda]$ is a strictly positive operator for $r \gg 0$ (for any degree). Since Δ' is nonnegative, this will imply the result. Here, however, Λ is defined only in terms of ω (and adjointness). But ∇^2 is the curvature of $E \otimes L^r$, which changes as $r \to \infty$; we have

$$\nabla^2 = \Theta_E + 1 \otimes r\Theta_L.$$

So the term $\sqrt{-1}[\nabla^2, \Lambda]$ becomes, on the space of $(p,0)$-forms:

$$\sqrt{-1}[\Theta_E, \Lambda] + r\sqrt{-1}[1 \otimes \Theta_L, \Lambda] = \sqrt{-1}[\Theta_E, \Lambda] + r\sqrt{-1}[\sqrt{-1}L, \Lambda] = \sqrt{-1}[\Theta_E, \Lambda] + r(n-p),$$

because $\Theta_L = \sqrt{-1}\omega$. Recall that $p < n$. Then, the first term on the right is a uniformly bounded family of operators on $E \otimes L^r$ (with respect to the natural metric), while the second is positive and grows arbitrarily large. As $r \to \infty$, the sum becomes a positive operator. Consequently for $r \gg 0$, $\Delta_{\overline{\partial}}$ is strictly positive on p-forms and we get that $H^p(M, E \otimes L^r) = 0$ for $p < n$. □

5. Applications

Using long exact sequences, we shall be able to extract further information from the Kodaira vanishing theorem.

5.1. Algebraic version.

We now want to prove the algebraic version of the Kodaira vanishing theorem.

Theorem 50. *If $\mathcal{L}$ is an ample line bundle on a smooth projective variety X over a field k of characteristic zero, then*[3]

$$H^p(X, \Omega^q \otimes \mathcal{L}) = 0 \quad \textit{for } p + q > n.$$

Proof. We shall start by proving the result for $k = \mathbb{C}$. Then, we know that there is an associated *analytic space* X^{an}, which is a complex manifold as X is smooth; there is an *analytification functor* from coherent sheaves on X to coherent analytic sheaves on X^{an}. Serre's GAGA theorem (see [Ser56]) states that this functor is an equivalence of categories, and the sheaf cohomologies are the same. Note

[2] For sheaves in general, the cohomological dimension is $2n$; for sheaves of $\mathcal{O}$-modules, the bound is n.

[3] Here Ω^q is the qth wedge power of the sheaf of differentials $\Omega_{X/k}$.

moreover that this functor sends the sheaf $\Omega^q = \bigwedge^q \Omega_{X/k}$ to the sheaf of holomorphic q-differentials on X^{an}, which we also denote by Ω^q.

Now, the claim is that the analytification $\mathcal{L}^{an}$ is a *positive* line bundle. In fact, it suffices to show that some power of $\mathcal{L}^{an}$ is postive, so we can assume that $\mathcal{L}$ is very ample. But then $\mathcal{L}$ is the pull-back of $\mathcal{O}(1)$ on projective space by an immersion; since $\mathcal{O}(1)$ on projective space is a positive line bundle (see Example 46), we see that $\mathcal{L}$ is positive. Thus, the analytic version of the vanishing theorem completes the proof.

Now let k be an arbitrary field of characteristic zero. There is a smooth projective variety X_0 over a field $k' \subset k$ with an ample line bundle $\mathcal{L}_0$ such that $X \simeq X_0 \times_{k'} k$ and $\mathcal{L} \simeq \mathcal{L}_0 \otimes_{k'} k$, where $k'/\mathbb{Q}$ is finitely generated; this follows by "noetherian descent" arguments as in [Gro66], sec. 8. Since cohomology commutes with base change by a flat morphism (see [Har77], III-9), we find that $H^\bullet(X, \Omega^p \otimes \mathcal{L}) \simeq H^\bullet(X_0, \Omega^p_{X_0} \otimes \mathcal{L}_0) \otimes_{k'} k$, and we are thus reduced to proving the result for the pair $(X_0, \mathcal{L}_0)$. Now k' imbeds in $\mathbb{C}$, and the same argument reduces us to proving the result for $(X_0 \times_{k'} \mathbb{C}, \mathcal{L}_0 \otimes_{k'} \mathbb{C})$, which follows by the previous discussion. (We have essentially used the "Lefschetz principle.") □

The result is false in characteristic p.

5.2. **Divisors.** Let M be a complex manifold. A (Cartier) **divisor** is a system of holomorphic local equations $f_\alpha = 0$ for meromorphic functions f_α on open sets $\{U_\alpha\}$ covering M; we require that the quotients f_α/f_β be units in the corresponding intersections. In other words, a divisor is a section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$, where $\mathcal{M}^*$ is the sheaf of nonzero meromorphic functions (under multiplication).

Any divisor determines a line bundle in a natural way, Namely, we consider the subsheaves $\mathcal{O}f_\alpha^{-1} \subset \mathcal{M}$; these glue in a natural way because of the condition on the quotients. These clearly form a line bundle on M. As the story is similar to the algebraic case, we shall not belabor this point. (See [Har77].)

Given an analytic subvariety $D \subset M$ which is cut out locally by one equation (e.g., a submanifold of codimension one), we can associate to it a divisor in a natural way. Namely, the local equations over an open cover define the divisor. As a result, there is a line bundle $\mathcal{O}(D)$ associated with the subvariety. In [GH78], it is shown that the Chern class of $\mathcal{O}(D)$ is the same as the associated *cycle* under the Gysin map. If $\mathcal{O}(-D)$ is the dual of $\mathcal{O}(D)$, note that there is an exact sequence of sheaves

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0, \tag{20}$$

where $\mathcal{O}_D$ is the sheaf of holomorphic functions on the analytic subvariety D (identified with a sheaf on M by the lower star). (We use the language of analytic subvarieties for more generality, but in fact we could restrict to submanifolds for many interesting cases.) One reason, however, to allow subvarieties is the following: if M sits inside projective space $\mathbb{P}^n_{\mathbb{C}}$, then an intersection of M with a hyperplane (that does not contain M) need not be a submanifold, but it is always an analytic subvariety and is locally cut out by a single (nonzero) equation.

Recall that $\mathcal{O}(1)$ is the line bundle on $\mathbb{P}^n_{\mathbb{C}}$ corresponding to any hyperplane under the above correspondence. It is a *positive* line bundle. As a result, if M is a (compact) submanifold[4] of $\mathbb{P}^n_{\mathbb{C}}$ and D the intersection of M with a hyperplane, then the corresponding line bundle $\mathcal{O}(D)$ on $\mathbb{P}^n_{\mathbb{C}}$ is the pull-back of $\mathcal{O}(1)$ to M. As a result, we are going to apply the Kodaira vanishing theorem to (20). For instance, let us note that if $\dim M \geq 2$, then $H^1(M, \mathcal{O}(-D)) = 0$ by the vanishing theorem; indeed, $\mathcal{O}(-D)$ is the pull-back of $\mathcal{O}(-1)$, and the pull-back of a negative line bundle by an immersion is negative. Thus, as a simple example, we find:

Proposition 51. *The map $H^0(M, \mathcal{O}) \to H^0(D, \mathcal{O}_D)$ is an epimorphism if $\dim M \geq 2$. In other words, a global regular function can be extended from a divisor.*

Finally, for use in the future, we prove a useful formula. Let $N \subset M$ be an inclusion of complex manifolds, such that N is of (complex) codimension one. We have defined a holomorphic line bundle

[4]We have been slightly loose with identifying submanifolds and smooth subvarieties of $\mathbb{P}^n_{\mathbb{C}}$; we can do this by Chow's theorem, which states that any analytic subset of projective space is algebraic.

$\mathcal{O}(N)$ on M; since N is of codimension one, we can define another as the cokernel of

$$T'N \hookrightarrow T'M|_N;$$

this quotient of (holomorphic) tangent bundles is the *conormal sheaf* and is denoted $\mathcal{N}_{M/N}$. It is a line bundle on N.

Proposition 52 (Adjunction formula)**.** $\mathcal{N}_{M/N} = \mathcal{O}(-N)|_N$.

Proof. Indeed, if $\{f_\alpha = 0\}$ is a system of holomorphic local equations for N over a cover $\{U_\alpha\}$, then for each α, df_α does not vanish along N. If the $g_{\alpha\beta}$ are the transition functions with $f_\beta = g_{\alpha\beta} f_\alpha$, then on N,

$$df_\beta = g_{\alpha\beta} df_\alpha,$$

because $f_\alpha = 0$ on N. As a result, the $\{df_\alpha\}$ when restricted to N satisfy the same transition equations as $\mathcal{O}(-N)|_N$, and since they do not vanish (by smoothness), they generate a nowhere zero section of $\mathcal{O}(-N)_N \otimes \mathcal{N}^*_{M/N}$. (They vanish on tangent vectors to N, so each df_α is a section of $\mathcal{N}^*_{M/N}$.) This line bundle is thus trivial bundle as a result, and we get the formula. □

There is another useful variant of this for the *canonical bundles.* Namely, we have

$$\omega_N = \omega_X|_N \otimes \mathcal{O}(-N).$$

To get this, we use the exact sequence $0 \to T'N \to T'M|_N \to \mathcal{O}(-N)|_N$, and take the top exterior power.

5.3. **Connectedness.** Let X be a smooth, irreducible projective variety, say inside $\mathbb{P}^n_{\mathbb{C}}$. Recall:

Theorem 53 (see [Har77], II.8)**.** *For a generic hyperplane $H \subset X$, $H \cap X$ is a smooth variety.*

This result is true over any algebraically closed field, not necessarily $\mathbb{C}$; we are, however, going to show connectedness. Bertini's theorem is useful in induction arguments, by replacing a smooth variety by one of smaller dimension.

We want to show that the intersection is always *connected.*

Theorem 54. *If X is a connected, smooth variety in $\mathbb{P}^n_{\mathbb{C}}$ of dimension ≥ 2, then $X \cap H$ is connected for every hyperplane $H \subset \mathbb{P}^n_{\mathbb{C}}$.*

Note that by dimension theory, the intersection is nonempty (see [Har77], I.7). This is essentially the argument in [Har77], III.7, though the result there is slightly more general.

Proof. Recall that on any locally ringed space, the connected components are in bijection with the idempotents in the ring of global sections. Let $D = X \cap H$. Now $H^0(M, \mathcal{O}) \to H^0(D, \mathcal{O}_D)$ is surjective, by Proposition 51. By the proper mapping theorem (which we do not prove!), $H^0(M, \mathcal{O})$ is a finite-dimensional $\mathbb{C}$-algebra. Since it admits no zero-divisors, it must be $\mathbb{C}$, and consequently $H^0(D, \mathcal{O}_D) = \mathbb{C}$ (since D is nonempty by the previous remarks). □

Ultimately, in the above proof we used the rather fancy proper mapping theorem. We needed it: in general a ring with no nontrivial idempotents can surject on a ring with many, e.g. consider $\mathbb{Q}[x] \twoheadrightarrow \mathbb{Q}[x]/(x-2)(x-3)$. There is, however, a simpler argument if we admit Hodge theory, and assume $X \cap H$ is smooth. It suffices thus to show that the map

$$H_0(H \cap X; \mathbb{C}) \to H_0(X; \mathbb{C})$$

is an isomorphism. Since $H \cap X$ is always a divisor (though it need not be smooth), we are reduced to comparing the (co)homology of divisors with the (co)homology of ambient varieties. We will do this in the Lefschetz theorem below.

5.4. **The Lefschetz hyperplane theorem.** Let $M \subset \mathbb{P}^n_{\mathbb{C}}$ be a smooth variety, and let $N = H \cap V$ be a hyperplane section which is also smooth; many such exist by Bertini's theorem. We are interested in comparing the cohomology $H^*(M;\mathbb{C})$ with the cohomology $H^*(N;\mathbb{C})$.

Theorem 55 (Lefschetz)**.** *The inclusion $N \hookrightarrow M$ induces an isomorphism in cohomology for degrees $\leq n-2$; the map $H^*(M;\mathbb{C}) \to H^*(N;\mathbb{C})$ is an injection.*

Proof. We will use the Hodge decompositions

$$H^m(M;\mathbb{C}) = \bigoplus_{p+q=m} H^p(M,\Omega^q)$$

$$H^m(N;\mathbb{C}) = \bigoplus_{p+q=m} H^p(N,\Omega^q)$$

The claim is that for $p+q < n-1$, the map

$$H^p(M,\Omega^q) \to H^p(N,\Omega^q)$$

is an isomorphism (with corresponding injectivity when $p+q \leq n-1$). This will establish the claim. For clarity, we denote below $\Omega^\bullet_M, \Omega^\bullet_N$ for the sheaves of holomorphic forms to distinguish. We have an exact sequence of sheaves

$$0 \to \Omega^q_M(-N) \to \Omega^q_M \to \Omega^q_M|_N \to 0.$$

For $p+q < n-1$ (resp. $p+q \leq n-1$), we see that $H^p(M,\Omega^q_M(-N) = H^{p+1}(M,\Omega^q_M(-N)) = 0$ (resp. just the first claim). The exact sequence shows that

$$H^p(M,\Omega^q_M) \to H^p(M,\Omega^q_M|_N)$$

is an isomorphism (resp. an injection).

Next, we need to compare $H^p(M,\Omega^q_M|_N)$ with $H^p(N,\Omega^q_N)$. By pull-back, we have a surjection $\Omega^q_M|_N \to \Omega^q_N \to 0$. We want to fit this into a short exact sequence so as to use cohomology. For this, we note that the dual is

$$0 \to \wedge^q T'N \to \wedge^q T'M|_N,$$

which is obtained by applying $\wedge^q$ to the inclusion $T'N \hookrightarrow T'M|_N$. Since the cokernel of this last map is the *line bundle* $\mathcal{N}_{M/N}$, we find that there is a natural exact sequence

$$0 \to \wedge^q T'N \to \wedge^q T'M|_N \to \wedge^{q-1} T'N \otimes \mathcal{N}_{M/N} \to 0,$$

which upon dualizing and applying the adjunction formula gives

$$0 \to \Omega^{q-1}_N \otimes \mathcal{O}(-N) \to \Omega^q_M|_N \to \Omega^q_N \to 0.$$

We see, by using the long exact sequence in cohomology and the vanishing theorem, as before, that

$$H^p(M,\Omega^q_M|_N) \to H^p(N,\Omega^q_N)$$

is either an isomorphism or an injection, in the respective cases. Composing these two arguments, we get the claim. ☐

Note in the proof we have actually shown more: the restriction map $H^p(M,\Omega^q_M) \to H^p(M,\Omega^q_M|_N)$ is an isomorphism for $p+q < n-1$ (resp. an injection for $p+q \leq n-1$). We shall use this in the following application on hypersurfaces in projective space.

Let us first compute the Picard groups of hypersurfaces. Recall that the Picard group (here of holomorphic line bundles, or equivalently algebraic ones by GAGA) on a smooth projective manifold (or variety) $X \subset \mathbb{P}^n_{\mathbb{C}}$ can be computed as $H^1(X,\mathcal{O}^*)$. Although $\mathcal{O}^*$ is not even an $\mathcal{O}$-module, we do have the useful exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \stackrel{f \mapsto e^{2\pi\sqrt{-1}f}}{\to} \mathcal{O}^* \to 0, \tag{21}$$

which we now apply in the following situation. Let X be a smooth variety in $\mathbb{P}^n_{\mathbb{C}}$ of dimension at least 4, and consider a smooth hyperplane section $Y = X \cap H$. Then $H^i(Y,\mathcal{O}) = H^i(X,\mathcal{O})$ for $i = 1,2$ and $H^i(Y;\mathbb{Z}) = H^i(X;\mathbb{Z})$ for $i = 1,2$, by the Lefschetz hyperplane theorem (and the additional refinement thereof). Here we have used the slightly stronger refinement, which states that the maps are isomorphisms in $\mathbb{Z}$-cohomology. By the long exact sequence in cohomology associated with (21) and the five-lemma, we find that $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism.

We find:

Proposition 56. *Let X be a smooth projective variety of dimension ≥ 4, and H a hypersurface. such that $X \cap H$ is smooth. Then $\mathrm{Pic}(X) \to \mathrm{Pic}(X \cap H)$ is an isomorphism.*

Proof. Everything follows from the above discussion if H is a hyperplane. To reduce to this case, imbed everything in a larger projective space using the d-uple embedding (if H is of degree d). Then intersecting with H corresponds to intersecting with a hyperplane. □

In particular, if $n \geq 4$, then a smooth hypersurface in $\mathbb{P}^n_{\mathbb{C}}$ has Picard group $\mathbb{Z}$, because $\mathrm{Pic}(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}$. This result can be proved using algebraic means, and is true if $\mathbb{C}$ is replaced with any algebraically closed field. In fact, the following *stronger* result is true: a complete intersection (not necessarily smooth!) in $\mathbb{P}^n_k$ of dimension ≥ 3, for k an algebraically closed field, has $\mathbb{Z}$ as its Picard group. See [Gro05], sec. 3 of expose 12.

Proposition 57. *Let X_1, X_2 be smooth hypersurfaces of degree $d \neq n+1$ in $\mathbb{P}^n$. If $X_1 \simeq X_2$ as abstract varieties, then there exists an automorphism of $\mathbb{P}^n$ that induces an isomorphism $X_1 \simeq X_2$.*

Since every automorphism of $\mathbb{P}^n$ is linear (i.e. the automorphism group is $\mathrm{PGL}(n+1,k)$), it follows that X_1, X_2 are related by a linear map (when considered as imbedded in projective space).

Proof. Indeed, let us consider the imbeddings $i_1 : X_1 \hookrightarrow \mathbb{P}^n, i_2 : X_2 \hookrightarrow \mathbb{P}^n$ and the corresponding line bundles $\mathcal{L}_1$ on X_1, and $\mathcal{L}_2$ on X_2. There is an isomorphism $g : X_1 \simeq X_2$, but a priori the corresponding very ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ do not correspond. The crux of the proof is that they do in this case.

Indeed, we shall use the adjunction formula. Note that the isomorphism $g : X_1 \to X_2$ preserves the (intrinsically defined) canonical bundles $\omega_{X_1}, \omega_{X_2}$. Since $\omega_{\mathbb{P}^n_{\mathbb{C}}} = \mathcal{O}(-n-1)$, we find by the adjunction formula

$$\omega_{X_1} = i_1^* \mathcal{O}(-n-1) \otimes i_1^* \mathcal{O}(d) = i_1^* \mathcal{O}(d-n-1),$$

and similarly

$$\omega_{X_2} = i_2^* \mathcal{O}(-n-1) \otimes i_2^* \mathcal{O}(d) = i_2^* \mathcal{O}(d-n-1).$$

However, we know that i_1^*, i_2^* are *isomorphisms* $\mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X_1), \mathrm{Pic}(\mathbb{P}^n) \to \mathrm{Pic}(X_2)$: in particular, both are $\mathbb{Z}$. It follows that, since ω_{X_2} is pulled back to ω_{X_2} by g, we have

$$g^*(i_2^* \mathcal{O}(1)) = i_1^* \mathcal{O}(1),$$

which is to say that the very ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X_1, X_2 correspond. From this the rest is easy. □

References

[Dup78] Johan L. Dupont. *Curvature and Characteristic Classes.* Springer, 1978. Lecture Notes in Mathematics.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[Gro58] Alexander Grothendieck. La théorie des classes de Chern. *Bull. Soc. Math. France*, 86:137–154, 1958.

[Gro66] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[Gro05] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2).* Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 4. Société Mathématique de France, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes.* Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[Ser56] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

[SS85] Bernard Shiffman and Andrew John Sommese. *Vanishing theorems on complex manifolds*, volume 56 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1985.

[Wel08] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia-Prada.