THE DIRAC OPERATOR

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1. First properties

1.1. **Definition.** Let X be a Riemannian manifold. Then the tangent bundle TX is a bundle of real inner product spaces, and we can form the corresponding *Clifford bundle* $\mathrm{Cliff}(TX)$. By definition, the bundle $\mathrm{Cliff}(TX)$ is a vector bundle of $\mathbb{Z}/2$-graded $\mathbb{R}$-algebras such that the fiber $\mathrm{Cliff}(TX)_x$ is just the Clifford algebra $\mathrm{Cliff}(T_xX)$ (for the inner product structure on T_xX).

Definition 1. A **Clifford module** on X is a vector bundle V over X together with an action

$$\mathrm{Cliff}(TX) \otimes V \to V$$

which makes each fiber V_x into a module over the Clifford algebra $\mathrm{Cliff}(T_xX)$.

Suppose now that V is given a connection ∇.

Definition 2. The **Dirac operator** D on V is defined in local coordinates by sending a section s of V to

$$Ds = \sum e_i . \nabla_{e_i} s,$$

for $\{e_i\}$ a local orthonormal frame for TX and the multiplication being Clifford multiplication.

It is easy to check that this definition is independent of the coordinate system. A more invariant way of defining it is to consider the composite of differential operators

$$V \stackrel{\nabla}{\to} T^*X \otimes V \simeq TX \otimes V \hookrightarrow \mathrm{Cliff}(TX) \otimes V \to V, \tag{1}$$

where the first map is the connection, the second map is the isomorphism $T^*X \simeq TX$ determined by the Riemannian structure, the third map is the natural inclusion, and the fourth map is Clifford multiplication.

The Dirac operator is clearly a first-order differential operator. We can work out its symbol fairly easily. The symbol[1] of the connection ∇ is

$$\mathrm{Sym}(\nabla)(v, t) = iv \otimes t, \quad v \in V_x, t \in T_x^*X.$$

All the other differential operators in (1) are just morphisms of vector bundles. As a result, we have:

Proposition 1. The symbol of the Dirac operator is

$$\mathrm{Sym}(D)(v, t) = it.v, \quad v \in V_x, t \in T_xX$$

where t is identified with a cotangent vector via the metric.

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[1]Recall that the *symbol* of a differential operator between vector bundles $E \to F$ is a map between the pull-backs to the cotangent bundle.

We observe that if a nonzero tangent vector $t \in T_x X$ is fixed the morphism $V_x \to V_x$ given by Clifford multiplication by t is an *isomorphism* of vector spaces (in fact, the inverse is a scalar multiple of Clifford multiplication by t again). Consequently:

Corollary 2. The Dirac operator is elliptic.

We also note that the symbol of D^2 is therefore given on (v, t) by $\|t\|^2 v$, and this is precisely the symbol of a Laplacian-type operator. The Dirac operator can be thought of (and was originally introduced) as a square root for the Laplacian.

There are a few further natural conditions to impose here. For instance, we could require that V have a parallel metric such that Clifford multiplication by vectors is skew-adjoint. We could also require Clifford multiplication $TM \otimes V \to V$ to be parallel (with respect to the tensor product connection on $TM \otimes V$ and the given connection on V).

Proposition 3. If X is an oriented Riemannian manifold (so that it inherits a volume form), and V satisfies the hypotheses of the previous paragraph, then D is formally self-adjoint.

Proof. Let s_1, s_2 be global sections of V with compact support. We would like to show that

$$\int_X (Ds_1, s_2) = \int_X (s_1, Ds_2)$$

where the integration is with respect to the global volume form. In fact, the claim is that

$$(Ds_1, s_2)(x) = \mathrm{div} V(x) + (s_1, Ds_2)(x) \tag{2}$$

where V is the vector field defined locally as $V = \sum_i (e_i.s_1(x), s_2) e_i$ for e_i an orthonormal frame for the tangent bundle. Since any divergence has zero integral, the claim follows.

To prove this, note that everything is invariantly defined. Consequently, if we wish to prove (2) at x, we may choose an orthonormal frame $\{e_i\}$ specifically adapted to x, say such that $\nabla_{e_i} e_j(x) = 0$. The vector field V can be locally expressed as $\sum_i (e_i s_1, s_2) e_i$ so $\mathrm{div} V = \sum_i e_i(e_i.s_1, s_2)$. Then, we have

$$\begin{aligned}
(Ds_1, s_2)(x) &= \sum_i (e_i \nabla_{e_i} s_1, s_2)(x) \\
&= \sum_i (\nabla_{e_i} e_i s_1, s_2)(x) \quad \text{because } \nabla_{e_i} e_i(x) = 0 \\
&= \sum_i e_i(e_i s_1, s_2)(x) - \sum_i (e_i s_1, \nabla_{e_i} s_2)(x) \\
&= \sum_i e_i(e_i s_1, s_2)(x) + \sum_i (s_1, e_i \nabla_{e_i} s_2)(x) \\
&= \mathrm{div} V(x) + (s_1, Ds_2).
\end{aligned}$$

∎

A frequent situation that will arise in the following is that the Clifford module V will have a $\mathbb{Z}/2$-grading, $V = V_0 \oplus V_1$, such that V_0, V_1 are orthogonal and such that Clifford multiplication by a tangent vector takes V_0 into V_1 and vice versa. In this case, we can form operators

$$D^+ : V_0 \to V_1, \quad D^- : V_1 \to V_0,$$

given by restricting the operator D above. In fact, the index of D is zero because D is self-adjoint, but we can compute the indices of D^+, D^-. Observe that

$$\mathrm{index} D^+ = \dim\ker D^+ - \dim \mathrm{coker} D^+ = \dim\ker D^+ - \dim\ker D^-, \tag{3}$$

We will see, starting in the next subsection, that the indices of these operators contain significant geometric information, and that—amazingly—they can be computed in terms of the characteristic classes of M and V.

1.2. **The Dirac operator on the exterior algebra.** While most of our attention will come from the irreducible Clifford modules, we can already get a fair bit of information in the simplest case. In this subsection, we show that the Euler characteristic and signature can be described as the indices of Dirac operators. The index theorem can be used to compute these indices and yields, in the latter case, the Hirzebruch signature formula. (In the former case, it shows that the Euler characteristic comes from the Euler class, which is less interesting.)

Example. A simple example of a Clifford module is $\mathrm{Cliff}(TX)$ itself. The Levi-Civita connection on TX lifts to give a connection on $\mathrm{Cliff}(TX)$ (namely, if one thinks of a connection as a collection of isomorphisms from parallel transport, this is obvious). We can determine what the Dirac operator looks like in this case.

It is well-known that there is an isomorphism of Clifford modules

$$\mathrm{Cliff}(TX) \simeq \bigwedge^{\bullet} TX \simeq \bigwedge^{\bullet} T^*X.$$

This arises from the canonical action of the Clifford algebra $\mathrm{Cliff}(V)$ on the exterior algebra $\bigwedge^{\bullet} V$; a vector v acts on a multi-vector $w \in \bigwedge^{\bullet} V$ via $-\iota_v(w) + v \wedge w$ (for ι_v interior multiplication). If we make these identifications, then the connection on $\mathrm{Cliff}(T^*X)$ is just obtained by taking the exterior power of the dual of the Levi-Civita connection on TX.

In particular, we find that the Dirac operator on $\mathrm{Cliff}(T^*X) \simeq \bigwedge^{\bullet} T^*X$ can be expressed by choosing a local orthonormal frame e_i of TX and let η^i be the dual coframe of 1-forms. Then

$$D : \Gamma(\bigwedge^{\bullet} T^*X) \to \Gamma(\bigwedge^{\bullet} T^*X)$$

is given by

$$Ds = -\sum \iota_{e_i} \nabla_{e_i} s + \sum \eta^i \wedge \nabla_{e_i} s.$$

We shall now require the following fact.

Proposition 4. Let X be a Riemannian manifold, ∇ the Levi-Civita connection (extended to differential forms on X). Then we can express exterior differentiation on a form s as

$$ds = \sum \eta^i \wedge \nabla_{e_i} s \tag{4}$$

for e_i a local orthonormal frame of vector fields with η^i the dual coframe. Similarly, $d^*s = -\sum e_i \wedge \nabla_{e_i} s$.

Proof. The strategy is to note that $ds, \eta_i \wedge \nabla_{e_i} s$ are tensorial, so we may prove the equality (4) desired by working in a local coordinate system that is particularly convenient. Fix a point r. To prove (4), we may assume that we have chosen local coordinates $x^1, \dots, x^n$ around r such that $\partial_i = \frac{\partial}{\partial x_i}$ are orthonormal at r and such that

$$g(\partial_i, \partial_j) = \delta_{ij} + O(|x|^2).$$

That is, we are choosing *normal coordinates* such that the metric is to order one the same as the euclidean metric. Equivalently, we are assuming that the Christoffel symbols Γ_{ij}^k vanish at p.

At the point r, we can thus prove (4) by taking $e_i = \partial_i, \eta_i = dx^i$. If s is, say, a form $\sum f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, then we have

$$ds = \sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx^{i_1} \cdots \wedge dx^{i_p}$$

while

$$\nabla_{\partial_i} s = \frac{\partial f}{\partial x_i} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

So, $\eta_i \wedge \nabla_{e_i} \omega = \sum \frac{\partial f}{\partial x_i} dx_i \wedge dx^{i_1} \dots dx^{i_p}$, and the two are equal as desired. The other statement in (4) is similar. □

In particular, we find that the Dirac operator on $\mathrm{Cliff}(TX) \simeq \bigwedge^{\bullet} T^*X$ is just given by $D = d + d^*$. Note that D is a square root of the Hodge Laplacian $\Delta = dd^* + d^*d$.

There are two natural ways to put a $\mathbb{Z}/2$-grading on the Clifford module $\mathrm{Cliff}(TX) \simeq \bigwedge^{\bullet} T^*X$.

Example (The Euler characteristic)**.** One way is to take the $\mathbb{Z}/2$-grading that comes from the $\mathbb{Z}$-grading of the exterior algebra (which gives a $\mathbb{Z}/2$-grading by taking even and odd parts, respectively). Alternatively, we could take the involution of the Clifford algebra which acts by -1 on vectors, and take the ± 1 eigenspaces.

Then the Dirac operator $D = d + d^*$ splits into

$$D^+ : \bigwedge^{\text{even}} T^*X \to \bigwedge^{\text{odd}} T^*X, \quad D^- : \bigwedge^{\text{even}} T^*X \to \bigwedge^{\text{odd}} T^*X.$$

These two operators just describe the de Rham complex of X rolled up. Note that $\ker D^+ = \ker \Delta \cap \bigwedge^{\text{even}} T^*X$ just consists of the even-dimensional harmonic forms, while $\ker D^-$ consists of the odd-dimensional harmonic forms. Consequently, by (3),

$$\mathrm{index} D^+ = \chi(X).$$

As Atiyah observes in [4], this observation together with the yoga of index theory can be used to give a proof of the classical fact that the existence of an everywhere nonzero vector field implies that $\chi(X) = 0$. Suppose X is an everywhere nonzero vector field on X, and consider then the operator R_X of *right* Clifford multiplication by X. Then R_X interchanges the $\mathbb{Z}/2$-grading: it takes $\bigwedge^{\text{even}} T^*X$ into $\bigwedge^{\text{odd}} T^*X$ and vice versa. As a result, we can compare the operators

$$D^+, R_X^{-1} D^- R_X.$$

A key observation is that since the symbol of D^- and D^+ is *left* Clifford multiplication, and the symbol of R_X is *right* Clifford multiplication, the symbols of D^- (or D^+) and that of R_X commute. This means that $D^+, R_X^{-1} D^- R_X$ have the same symbol and thus the same index. In particular,

$$\mathrm{index} D^+ = \mathrm{index} R_X^{-1} D^- R_X = \mathrm{index} D^- = -\mathrm{index} D^+,$$

since D^+, D^- are adjoint. This proves the desired claim. See below for more examples from the paper (which do not have direct proofs).

Example (The signature operator)**.** A more interesting $\mathbb{Z}/2$-grading on $\mathrm{Cliff}(TX) \simeq \bigwedge^{\bullet} T^*X$ comes from the Clifford structure. Suppose $n = \dim X$ is even. We can define the *chirality operator* $\Gamma = (-1)$. ...

The interaction between the two $\mathbb{Z}/2$-gradings of $\mathrm{Cliff}(TX)$ was used in [4] to prove various congruence relations. We give a sketch of their methods.

We can construct more interesting examples of Clifford modules when the tangent bundle TX has a spin structure. In fact, the data we have worked with so far is an orthogonal structure on TX (i.e., a principal $O(n)$-bundle associated to TX), and we have constructed new vector bundles on X (such as $\mathrm{Cliff}(TX)$) by using representations of the orthogonal group $O(n)$, for instance the representation of $O(n)$ on $\mathrm{Cliff}(\mathbb{R}^n)$. Nonetheless, there are Clifford modules that do not admit an $O(n)$-action, or even an $SO(n)$-action.

2. The Spin group and its representations

Suppose now that TX has a spin structure, associated to a principal $\mathrm{Spin}(n)$-bundle P, and let $n = \dim X$. Then, for any $\mathrm{Spin}(n)$-representation V, we can form a new bundle $P \times_{\mathrm{Spin}(n)} V$ on X. The representation theory of $\mathrm{Spin}(n)$ is strictly richer than that of $SO(n)$, so this will actually let us construct new things.

2.1. **The spin representation Δ_{2k}.** We will apply this construction in the following situation. Suppose n is even, $n = 2k$. Then we have an isomorphism

$$\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) = \mathrm{Cliff}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{End}(\mathbb{C}^{2^k}).$$

Consequently, $\mathbb{C}^{2^k}$ becomes a representation of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^n)$, and thus a representation of $\mathrm{Spin}(n)$ (which is a subgroup of the units in $\mathrm{Cliff}(\mathbb{R}^n)$). This is called the **spin representation Δ_{2k}**.

This isomorphism can be written down very explicitly, and the spin representation can be made more concrete. We will do it in stages.

Example. Consider the case when $k = 1, n = 2$. In this case, the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}$ is the associative $\mathbb{C}$-algebra on generators e_1, e_2 satisfying the conditions

$$e_1^2 = e_2^2 = -1 \tag{5}$$

$$e_1 e_2 = -e_2 e_1. \tag{6}$$

There is a vector space basis for the algebra given by $1, e_1, e_2, e_1e_2$. The isomorphism $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^2) \simeq \mathrm{End}(\mathbb{C}^2)$ can be given by the *Pauli spin matrices*

$$\sigma_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We map the Clifford algebra into $\mathrm{End}(\mathbb{C}^2)$ as follows: $1 \mapsto \mathrm{id}, e_1 \mapsto \sigma_1, e_2 \mapsto \sigma_2, e_1e_2 \mapsto \sigma_3$. This is easily checked to be a morphism of algebras, and it is consequently an isomorphism.

Proposition 5. We have an isomorphism

$$\phi : \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{n+2}) \simeq \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \otimes_{\mathbb{C}} \mathrm{End}(\mathbb{C}^2).$$

Proof. We shall describe where the generators $e_1, \dots, e_{n+2}$ of $\mathrm{Cliff}(\mathbb{R}^{n+2})$ are mapped. If σ_3 is the matrix as above, we write $T = i\sigma_3 \in \mathrm{End}(\mathbb{C}^2)$ and observe then that $T^2 = \mathrm{id}$. We set

$$\phi(e_1) = 1 \otimes \sigma_1, \ \phi(e_2) = 1 \otimes \sigma_2, \quad \phi(e_j) = e_{j-2} \otimes T \text{ for } j \geq 2.$$

Since $T^2 = 1$, these elements $\phi(e_j)$ satisfy the relations of the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{n+2})$. The induced map $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{n+2}) \simeq \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \otimes_{\mathbb{C}} \mathrm{End}(\mathbb{C}^2)$ is an *isomorphism* because it takes basis elements to basis elements injectively. $\square$

In particular, the Morita equivalence class of the complexified Clifford algebra is periodic mod 2. For real Clifford algebras, one gets a mod 8 periodicity. This is related to the Bott periodicity theorem and the analogous periodicity in real and complex K-theory.

By induction, we get:

Corollary 6. We have an isomorphism $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2k}) \simeq \mathrm{End}(\mathbb{C}^{2^k})$.

We can write down this isomorphism very explicitly in view of the above proposition. Namely, we find recursively:

$$e_1 \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_1 \tag{7}$$
$$e_2 \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_2 \tag{8}$$
$$e_3 \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_1 \otimes T \tag{9}$$
$$e_4 \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_2 \otimes T \tag{10}$$
$$e_5 \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_1 \otimes T \otimes T \tag{11}$$
(12)

Here there are k tensor factors at every stage. The pattern continues.

Note that since $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$ is a matrix algebra, the representation is its unique irreducible representation.

2.2. **The half-spin representations $\Delta_{2k}^+, \Delta_{2k}^-$.** As before, we write Δ_{2k} for the **spin representation**: this is $\mathbb{C}^{2^k}$ as a Clifford module as above, and it becomes a representation of $\mathrm{Spin}(n)$ (where as before $n = 2k$), which sits inside the Clifford algebra.

Let us now analyze Δ_{2k}. It is obviously a faithful representation, since the action of the Clifford algebra is. The action of the Clifford algebra is also irreducible, but the action of the spin group is not. We will split it into two pieces.

Definition 3. Fix $n = 2k$ as above. The **volume element** Γ is defined via

$$\Gamma = i^k e_1 \dots e_n \in \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n).$$

Observe that $\Gamma^2 = (-1)^k e_1 \dots e_n e_1 \dots e_n = 1$. The element Γ is independent of the choice of *oriented* orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. To see this, observe that the image in the exterior algebra is certainly independent of the oriented orthonormal basis.

It is easy to check that Γ anticommutes with every element $x \in \mathbb{R}^n$, regarded as an element of $\mathrm{Cliff}(\mathbb{R}^n)$. In fact, by making a change of basis, we may assume that x is a multiple of e_1 (cf. the previous remarks about the independence of Γ), in which case one directly checks

$$e_1 \Gamma = -\Gamma e_1. \tag{13}$$

The *spin group* $\mathrm{Spin}(n)$ may be defined as the subgroup of the units in $\mathrm{Cliff}(\mathbb{R}^n)$ generated by elements of the form vv' for $v, v' \in \mathbb{R}^n$. Clearly (13) shows that Γ *commutes* with every element of $\mathrm{Spin}(n)$.

We may summarize the above discussion in the following.

Proposition 7. $\Gamma \in \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$ commutes with every element of $\mathrm{Spin}(n)$ and anticommutes with every vector in $\mathbb{R}^n$.

Consider a complex vector space V with the structure of a $\mathrm{Cliff}(\mathbb{R}^n)$ (or, equivalently, $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$)-module. The element Γ acts on V by an involution, $\Gamma^2 = 1$, of vector spaces, but it is *not* a homomorphism of Clifford modules. Nonetheless, the above discussion that Γ is a homomorphism of $\mathrm{Spin}(n)$-representations. This implies that we can decompose

$$V = V^+ \oplus V^- \in \mathrm{Rep}(\mathrm{Spin}(n))$$

for V^+, V^- the eigenspaces for eigenvalues $1, -1$ of Γ. By (13), Clifford multiplication by an element $x \in \mathbb{R}^n$ sends V^+ into V^- and vice versa. In particular, $\dim V^+ = \dim V^-$.

There is a morphism

$$\mathbb{R}^n \otimes V \to V$$

given by Clifford multiplication. This is in fact a morphism of $\mathrm{Spin}(n)$-representations, because the action of an element $g \in \mathrm{Spin}(n)$ on $\mathbb{R}^n$ is conjugation in the Clifford algebra: g sends x to gxg^{-1}. As a result, we get morphisms of $\mathrm{Spin}(n)$-representations

$$\mathbb{R}^n \otimes V^+ \to V^-, \quad \mathbb{R}^n \otimes V^- \to V^+.$$

Definition 4. The $\mathrm{Spin}(n)$-representations $\Delta_{2k}^+, \Delta_{2k}^-$ (the half-spin representations) are the eigenspaces of Γ on the Clifford module Δ_{2k}. These each have dimension 2^{k-1}.

Proposition 8. The half-spin representations are irreducible.

Proof. Suppose there is a proper subrepresentation $W \subsetneq \Delta_{2k}^+$, not zero. Then W is preserved under action of vv' for any vectors $v, v' \in \mathbb{R}^n$. If v is any nonzero vector, then $W \oplus vW$ is invariant under multiplication by any vector v'; this follows because $vW = v'W$ for any two nonzero vectors v, v' (as v, v' differ by a scalar times something in the spin group). Consequently, $W \oplus vW$ is a proper Clifford module of Δ_{2k}, which is a contradiction. □

In the rest of these notes, we will use the half-spin representations to construct a Dirac operator on any spin manifold. Using the index theorem, we will compute its index in terms of the Pontryagin classes of the manifold, and deduce several geometric corollaries.

3. The Dirac operator on a spin-manifold

3.1. **Construction.** Let M be a spin-manifold of dimension n. By assumption, we are given a principal $\mathrm{Spin}(n)$-bundle $P \to M$ such that $P \times_{\mathrm{Spin}(n)} \mathbb{R}^n \simeq TM$. If $n = 2k$ is even, we can form the spin-representation Δ_{2k} of the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$.

Definition 5. The bundle $P \times_{\mathrm{Spin}(n)} \Delta_{2k}$ is called the **spinor bundle** S. The action $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \otimes \Delta_{2k} \to \Delta_{2k}$ (a morphism of $\mathrm{Spin}(n)$-representations if $\mathrm{Spin}(n)$ acts on the Clifford algebra by conjugation) shows that S is naturally a Clifford module over $\mathrm{Cliff}(M)$.

Observe that, by the discussion of the previous section, we have a splitting of vector bundles

$$S = S^+ \oplus S^-$$

and Clifford multiplication interchanges the two.

In order to construct the Dirac operator, we'll need a connection on S. This is fairly straightforward: since M has a Riemannian structure, we get a canonical connection on TM. Since the Lie algebras of $\mathrm{Spin}(n)$ and $SO(n)$ are isomorphic, we get (by lifting) a connection on the principal bundle P. This yields a connection on $S = P \times_{\mathrm{Spin}(n)} \Delta_{2k}$, which preserves the two half-spinor bundles, which we denote by ∇. When we make these lifts, Clifford multiplication is parallel, and we have

$$\nabla(x.s) = (\nabla x).s + x.\nabla s,$$

for sections x, s of $\mathrm{Cliff}(TX)$ and S.

The Dirac operator as before now makes perfect sense, and yields the operator D on S constructed by Atiyah and Singer in [3]. From the definition of D as $\sum e_i.\nabla_{e_i}$, we see that D maps S^+ into S^- and vice versa. We can thus represent D as a matrix

$$\begin{bmatrix} 0 & D|_{S^+} \\ D|_{S^-} & 0 \end{bmatrix}.$$

We can choose a metric on Δ_{2k} which is invariant under the action of the compact group $\mathrm{Spin}(n)$ and such that $\Delta_{2k}^+, \Delta_{2k}^-$ are orthogonal. This means that the representations of $\mathrm{Spin}(n)$ can be given by morphisms $\mathrm{Spin}(n) \to SO(2^{k-1})$, and in particular we get a metric on S such that S^+, S^- are orthogonal. We saw earlier that with these choices, D is (formally) self-adjoint. (Consequently, the index of D is zero.) This means that $D|_{S^+}$ is the formal adjoint of $D|_{S^-}$. In particular, as in (3),

$$\mathrm{index} D|_{S^+} = \ker D|_{S^+} - \mathrm{coker} D|_{S^+} = \ker D|_{S^+} - \ker D|_{S^-}.$$

Elements in the kernel of D are sometimes called *harmonic* spinors, by analogy with the Laplacian.

3.2. **The index theorem.** Our next goal is to compute the index of $D|_{S^+}$. This is an elliptic operator, and (if M is compact) we can use the Atiyah-Singer index formula. Let us recall the cohomological statement of the index theorem. Cohomology will always mean cohomology with *rational* coefficients.

Theorem 9 (Atiyah-Singer)**.** Let M be an oriented manifold of dimension n, and T an elliptic pseudodifferential operator on M. Suppose that T defines the symbol $s \in K(TM)$. Then

$$\mathrm{index} T = (-1)^{n(n+1)/2} \int_M \mathrm{Th}^{-1}(\mathrm{ch} s)\tau(TM \otimes_{\mathbb{R}} \mathbb{C}). \tag{14}$$

Here τ is the Todd class, and $\mathrm{ch} s \in H^*_c(TM)$ is identified with its image in $H^*(M)$ (under the image of the Thom isomorphism $\mathrm{Th} : \widetilde{H}^*(M) \simeq H^{*+n}(TM, TM \setminus M)$).

Our main goal is to compute the index of the Dirac operator $D|_{S^+}$. In doing so, we know what the symbol is: if $\pi : T^*M \to M$ is the projection, then the symbol is (up to a constant) the map of Clifford multiplication

$$\pi^* S^+ \to \pi^* S^-, \quad (v, s) \mapsto v.s \quad v \in T_x M, \quad s \in S_x^+.$$

The symbol defines an element of $K(TM) = K(BM, SM)$ (for BM, SM the ball and sphere bundles in TM), and we need to compute the Chern character of this, and then pull back to $H^*(M)$ via the inverse of the Thom isomorphism. This was done by Atiyah and Singer.

We could do this computation rather easily by restricting to BM. Then $K(BM) \simeq K(M)$ and the "difference bundle" in the relative K-theory becomes simply the class $[S^+] - [S^-]$ in $K(M)$. Restriction commutes with taking the Chern character. Consequently, if $\iota : (M, \emptyset) \to (BM, SM)$ is the inclusion, we have

$$\iota^* s = [S^+] - [S^-]$$

and consequently

$$\iota^* \mathrm{ch} s = \mathrm{ch} S^+ - \mathrm{ch} S^-.$$

Recall that the Euler class $e(TM)$ of TM has the property that for any cohomology class $t \in H^*(BM, SM)$, we have

$$\iota^* t = e(TM)\mathrm{Th}^{-1} t;$$

that is, the inverse Thom isomorphism is "dividing" by the Euler class $e(TM)$. In particular, we have $e(TM)\mathrm{Th}^{-1}(\mathrm{ch}s) = \mathrm{ch}S^+ - \mathrm{ch}S^-$, and we would like to write:

$$\mathrm{Th}^{-1}(\mathrm{ch}s) = \frac{\mathrm{ch}S^+ - \mathrm{ch}S^-}{e(TM)}.$$

We cannot do this because $e(TM)$ is very far from a nonzerodivisor in $H^*(M)$.

In the next section, we will :

Proposition 10. Let $\pi : V \to X$ be a real vector bundle over the space[2] X with a $\mathrm{Spin}(2k)$-structure P. Consider the element $s \in K(BV, SV)$ defined as follows. Consider the spinor spaces $S^+(V), S^-(V)$ constructed as $P \times_{\mathrm{Spin}(n)} \Delta^+_{2k}, P \times_{\mathrm{Spin}(n)} \Delta^-_{2k}$, and define a map $\phi : \pi^*S^+(V) \to \pi^*S^-(V)$ by Clifford multiplication as above. The map ϕ is an isomorphism restricted to the sphere bundle and defines the class $s \in K(BV, SV)$.

Then,

$$\mathrm{Th}^{-1}\mathrm{ch}s = \hat{A}(-V) \in H^*(M),$$

where $\hat{A}$ is the stable characteristic class associated to the power series $\frac{x/2}{\sinh(x/2)}$.

(ADD THIS.)

Nonetheless, the key observation is that the element thus defined is given by a "universal" formula: namely, a formula that will make sense on the classifying space $B\mathrm{Spin}(n)$ (or rather, the ball and sphere bundles there). Namely, we will express

Very concretely, there is a map

$$M \to B\mathrm{Spin}(n)$$

realizing the spin bundle P. On $B\mathrm{Spin}(n)$, there is a

3.3. **The characteristic classes of S^+, S^-.** Finally, we are reduced to the problem of computing the Chern characters of the bundles S^+, S^- associated to a principal $\mathrm{Spin}(n)$-bundle $P \to M$ in terms of the characteristic classes of $P \times_{\mathrm{Spin}(n)} \mathbb{C}^n$. In our case, $P \times_{\mathrm{Spin}(n)} \mathbb{C}^n$ is the complexified tangent bundle, while S^+, S^- are the spinor bundles.

There is a general formalism described in [3], but we will only need a special case. Essentially, we have two maps

$$\mathrm{Spin}(n) \to SO(n) \to U(n), \quad \mathrm{Spin}(n) \to U(2^{k-1})$$

where the first map comes from the standard representation of $\mathrm{Spin}(n)$ on $\mathbb{C}^n$ (i.e., the one factoring through $SO(n)$); the second map comes from the half-spinor representation Δ^+_{2k}. This induces maps

$$\alpha : B\mathrm{Spin}(n) \to BU(n), \quad \beta : B\mathrm{Spin}(n) \to BU(2^{k-1}),$$

and we are interested in the induced maps on cohomology

$$\alpha^* : H^*(BU(n)) \to H^*(B\mathrm{Spin}(n)), \quad \beta^* : H^*(BU(2^{k-1})) \to H^*(B\mathrm{Spin}(n)).$$

Namely, the Chern character lives as a cohomology class in $H^{**}(BU(2^{k-1}))$; we are interested in expressing $\beta^*\mathrm{ch}$ in terms of the images of $\alpha^* c_i$.

Here is the plan for determining $\beta^*\mathrm{ch}$. We will construct a maximal torus $\mathbb{T} = (S^1)^k \subset \mathrm{Spin}(n)$. By the generalized version of the splitting formula, there is an injection

$$H^*(B\mathrm{Spin}(n)) \to H^*(B\mathbb{T})$$

[2]Not necessarily a manifold!

and consequently we may as well as identify the image of ch under

$$H^{**}(BU(2^k) \to H^*(B\mathrm{Spin}(n)) \to H^*(B\mathbb{T}).$$

This essentially amounts to the following problem. There are k line bundles $\mathcal{L}_1, \dots, \mathcal{L}_k$ on $B\mathbb{T}$ defining a $\mathbb{T}$-bundle; from these we can construct a 2^k-dimensional vector bundle $V \to B\mathbb{T}$ via the representation $\mathbb{T} \to U(2^k)$. The image of ch in $H^{**}(B\mathbb{T})$ is none other than ch$V$. However, V will turn out to be a fairly simple expression in the $\mathcal{L}_i$, and determining ch$V$ in terms of the $c_1(L_i)$ (which are generators for $H^{**}(B\mathbb{T})$) will be straightforward.

Consider the following maximal torus of Spin(n). Define $A_i = e_i e_{i+1}$; then the A_i commute with one another and satisfy $A_i^2 = -1$. This implies in particular that $\exp(tA_i) = \cos t + A_i \sin t$. We take all terms of the form

$$\exp(t_1 A_1 + \cdots + t_k A_k).$$

This generates a maximal torus of Spin(n). In fact, we observe that A_i corresponds to the element $2e_i \wedge e_{i+1} \in \mathfrak{so}_n$—this is the matrix with a 2 in $(i, i+1)$, a -2 in $(i+1, i)$, and zero everywhere else. In particular, $\exp(t_i A_i)$ is

$$f(\exp(t_i A_i)) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & \cos 2t_i & -\sin 2t_i & & \\ & & \sin 2t_i & \cos 2t_i & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}.$$

It follows that the image of the torus thus constructed in Spin(n) in $SO(n)$ is just the standard maximal torus.

Let us work out how this maximal torus acts on the representation Δ_{2k}. We see, for instance, that

$$A_1 = e_1 e_2 \mapsto 1 \otimes \cdots \otimes \cdots \otimes 1 \otimes \sigma_3 \tag{15}$$

$$A_2 = e_3 e_4 \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_3 \otimes 1 \dots A_i \mapsto 1 \otimes \cdots \otimes 1 \otimes \sigma_3 \otimes 1 \otimes \cdots \otimes 1. \tag{16}$$

Here σ_3 is the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and consequently we find that $\Delta_{2k}|_{\mathbb{T}}$ is just the tensor product of the representations

$$\mathbb{C}^2 \boxtimes \mathbb{C}^2 \boxtimes \cdots \boxtimes \mathbb{C}^2$$

where $\mathbb{C}^2$ is regarded as a representation of S^1 via

$$e^{it} \mapsto \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Finally, we have a diagram

$$\begin{array}{ccccc} U(n) & \longleftarrow & B\mathbb{T} & \longrightarrow & BU(2^k) \\ & & \downarrow & & \downarrow = \\ U(n) & \longleftarrow & B\mathrm{Spin}(n) & \longrightarrow & BU(2^k) \end{array}.$$

The takeaway from all this can be phrased as:

Proposition 11. If s is the class in K-theory defined by the symbol of the Dirac operator, then

$$\mathrm{ch} s\tau(TM \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^{n(n+1)/2}\hat{A}(M).$$

In particular, we have computed the index of the Dirac operator.

Theorem 12 (Atiyah-Singer)**.** The index of the Dirac operator D^+ on the spin-manifold M is given by $\hat{A}(TM)[M]$ (that is, the cohomology class $\hat{A}(TM)$ evaluated on the fundamental class).

Definition 6. The **A-roof genus** $\hat{\mathbf{A}}(M)$ of an oriented manifold M is defined as $\hat{\mathbf{A}}(M) = \hat{A}(TM)[M]$. This is a *genus*: that is, $\hat{\mathbf{A}}$ satisfies the properties:

(1) $\hat{\mathbf{A}}(M_1 \sqcup M_2) = \hat{\mathbf{A}}(M_1) + \hat{\mathbf{A}}(M_2)$.
(2) $\hat{\mathbf{A}}(M_1 \times M_2) = \hat{\mathbf{A}}(M_1)\hat{\mathbf{A}}(M_2)$.
(3) $\hat{\mathbf{A}}(M) = 0$ if M is an oriented boundary.

The $\hat{\mathbf{A}}$-genus is a polynomial with rational coefficients in the Pontryagin classes integrated over M, and so it takes values in $\mathbb{Q}$. The $\hat{\mathbf{A}}$-genus is clearly zero if $\dim M$ is not divisible by four. We have seen that for a *spin*-manifold, the $\hat{\mathbf{A}}$-genus is an integer, but this is not true in general.

Example. For instance, the $\hat{\mathbf{A}}$-genus of a four-manifold is given by $-p_1/24$. In the case of $\mathbb{CP}^2$,

$$p_1(T\mathbb{CP}^2) = -c_2(T\mathbb{CP}^2 \oplus \overline{T\mathbb{CP}^2}) = c_1(T\mathbb{CP}^2)^2 - 2c_2(T\mathbb{CP}^2).$$

.... This doesn't come out to be an integer.

4. Applications

4.1. **Integrality theorems.** Historically, the integrality of the $\hat{\mathbf{A}}$-genus suggested to mathematicians in the 1950s that it might be representing a cohomological formula for the index of some operator. The index theorem shows that this is in fact possible, as we have seen, and we will describe some integrality theorems that follow from it.

However, the full strength of the index theorem is not really relevant to the integrality results. Essentially, the corollary of the index theorem used so far has been that for any manifold M and any $s \in K(TM)$, the term $\left((\mathrm{Th}^{-1}\mathrm{ch} s)\tau(TM \otimes_{\mathbb{R}} \mathbb{C}\right)[M]$ is integral—as it is the index of any elliptic operator with symbol s. In fact, this result is completely elementary. Section 2 of [3] is effectively devoted to showing that the term $\left((\mathrm{Th}^{-1}\mathrm{ch} s)\tau(TM \otimes_{\mathbb{R}} \mathbb{C}\right)$ is, up to a sign, equal to the image of s under the homomorphism $K(TM) \to K(*) = \mathbb{Z}$ (the K-theoretical "topological index"). Since this is an integer by construction, we do not need to say anything about the analytical index to see that $\left((\mathrm{Th}^{-1}\mathrm{ch} s)\tau(TM \otimes_{\mathbb{R}} \mathbb{C}\right)[M]$ is an integer. In fact, the integrality of the $\hat{\mathbf{A}}$-genus is a consequence of the "differentiable" Riemann-Roch theorem; see [2].

Nonetheless, in this section, we include two more important examples of "integrality theorems"

Theorem 13 (Rohlin)**.** Let M be a four-dimensional spin-manifold. Then the signature $\sigma(M)$ is divisible by 16. Equivalently, p_1 is divisible by 48.

The equivalence of the two statements comes from Hirzebruch's formula

$$\sigma(M) = \frac{p_1}{3},$$

valid for any oriented four-manifold (and in fact, a consequence of the index theorem).

Proof. We can get divisibility by eight "formally." In fact, we know that the $\hat{\mathbf{A}}$-genus $\hat{\mathbf{A}}(M)$ is an integer, and for a four-manifold,

$$\hat{\mathbf{A}}[M] = -\frac{p_1}{24}. \tag{17}$$

This means that $24 \mid p_1$, or $8 \mid \sigma(M)$.

Remark. This result can also be deduced using the theory of integer quadratic forms. Since M is spin, we have $w_2(M) = 0$. The Wu formula implies that $w_2(M)$ is Poincaré dual to the form $H^2(M; \mathbb{Z}_2) \to \mathbb{Z}/2$ given by Sq^2 followed by pairing with the fundamental class, or equivalently of squaring and pairing with the fundamental class. It follows that the quadratic form $q : H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \to \mathbb{Z}$ given by the cup product is *even* (i.e. $q(v, v)$ is always even), and Poincaré duality implies that it is unimodular. Now it is a theorem of Arf that any even, unimodular quadratic form over $\mathbb{Z}$ has signature divisible by eight.

To get the further divisibility by 2, we will need to work harder. By (17), to say that $16 \mid \sigma(M)$ is equivalent to the condition that the index of the Dirac operator $D^+ : S^+ \to S^-$ is divisible by two. We will do this by showing that there are *quaternionic* structures on S^+ and S^- which anticommute with D^+.

Let us recall the structure of the half-spin representations Δ_4^+, Δ_4^- of $\mathrm{Spin}(4)$. Alternatively, we may recall the structure of Δ_4 as a Clifford module. The spinor space Δ_4 was $\mathbb{C}^2 \otimes \mathbb{C}^2$, and the generators of the Clifford algebra e_1, e_2, e_3, e_4 operated by

$$e_1 \mapsto 1 \otimes \sigma_1, \; e_2 \mapsto 1 \otimes \sigma_2, \; e_3 \mapsto \sigma_1 \otimes T, \quad e_4 \mapsto \sigma_2 \otimes T.$$

Here $\sigma_1, \sigma_2 \in \mathrm{End}(\mathbb{C}^2)$ were the spin matrices constructed earlier. Our goal is to construct a quaternionic structure on Δ_4 which anti-commutes with Clifford multiplication (and which thus commutes with the action of $\mathrm{Spin}(4)$).

Lemma 1. There is a real structure α on $\mathbb{C}^2$ and a quaternionic structure β on $\mathbb{C}^2$ such that α commutes with σ_1, σ_2 and β anticommutes with them. (Consequently, both α, β commute with σ_3.)

Granting this lemma, we can construct a quaternionic structure on Δ_4 via $\alpha \otimes \beta$. This squares to -1 and it is easy to check now that the matrices associated to e_1, e_2, e_3, e_4 anticommute with $\alpha \otimes \beta$; note while verifying that α, β are *antilinear* and *anticommute* with multiplication by i.

Proof of the lemma. □

We have now verified that there are $\mathrm{Spin}(n)$-equivariant quaternionic structures J in Δ_4^+, Δ_4^- which anticommute with Clifford multiplication. Because of the equivariance, these become *parallel* quaternionic structure on S^+, S^-. Since the structures *anti*commute with Clifford multiplication, it follows from the definition of D^+ that

$$JD^+ = -JD^-,$$

and consequently $\ker D^+$ is a quaternionic vector space. Similarly, $\ker D^-$ is a quaternionic vector space. This means that the index, or $\dim \ker D^+ - \dim \ker D^-$, is divisible by two. By the previous discussion, this completes the proof.

□

Here is another example of an "integrality" theorem.

Theorem 14 (Bott)**.** The nth Chern class $c_n(E)$ of any complex vector bundle E over the sphere S^{2n} is divisible by $(n-1)!$.

Proof. It suffices to show that the Chern character $\mathrm{ch}(E)$ is an integer (that is, when evaluated on the fundamental class of S^{2n}). We will show that $\mathrm{ch}(E)[S^{2n}]$ is actually the index of an elliptic operator.

Consider the Dirac operator $D : S^+ \to S^-$ on S^{2n}. It has a symbol $s \in K(TS^{2n})$ such that $\mathrm{ch}(s)\tau(TM \otimes_{\mathbb{R}} \mathbb{C}) = \hat{A}(S^{2n}) = 1$ (this refers to the $\hat{A}$-class, not the $\hat{A}$-genus; when evaluated on $[S^{2n}]$ this clearly gives zero!). Informally, we can form an operator $D \otimes 1$ with symbol $s \otimes 1_E$ by "replacing partial differentiation with covariant differentiation." (In fact, we can get any element in K-theory as the symbol of an elliptic operator.) Then the index theorem gives

$$\mathrm{index}(D \otimes 1) = \mathrm{ch}(E)[S^{2n}],$$

which is therefore an integer. □

4.2. **Lichnerowicz's theorem.** In this section, we prove a result where the analysis is genuinely necessary.

Theorem 15 (Lichnerowicz, Hitchin)**.** Let M be a spin-manifold of positive scalar curvature. Then $\hat{\mathbf{A}}(M) = 0$.

There are a number of "global" obstructions to a manifold's admitting a metric of positive curvature. For instance, if the Ricci curvature is bounded away from zero (e.g., for a compact semisimple Lie group), the manifold has finite fundamental group; see [6], for instance. This is another global obstruction in the case of positive *scalar* curvature, valid for spin-manifolds.

Example. This obstruction is only valid for spin-manifolds. For instance, in dimension 4, the $\hat{\mathbf{A}}$-genus is a multiple of the signature, so this implies that the signature vanishes. The manifold $\mathbb{CP}^2$ with the Fubini-Study metric has signature 1, even though it has positive sectional curvature.

The work of the previous section shows that $\hat{\mathbf{A}}(M)$ can be interpreted as the index of the Dirac operator. In fact, Lichnerowicz showed that there were no harmonic spinors when the scalar curvature is positive, so $\ker D^+ = \ker D^- = 0$. This clearly implies the result.

The basic idea behind this theorem is "Bochner's method." Namely, the point is that D is supposed intuitively to be a square root of a Laplacian-type operator such as $\nabla^*\nabla$ (for ∇ the connection on spinor fields). This is not quite true, but D^2 and $\nabla^*\nabla$ turn out to differ by only first-order terms, which are functions of the curvature. When the curvature is positive, this provides a lower bound for D^2, which is enough to show that it is a positive operator. (In his complex analysis class, Y. T. Siu refers to this method as "completion of squares.") For instance, a classical example is the following result.

Theorem 16 (Bochner)**.** Let M be a manifold with Ricci curvature positive-definite everywhere. Then $H^1(M; \mathbb{R}) = 0$.

Bochner's theorem can be proved by bounding below the Hodge Laplacian $\Delta = dd^* + d^*d$ on 1-forms; Hodge theory allows one to identify $H^1(M; \mathbb{R})$ with the kernel of Δ on 1-forms. Namely, one proves the Weitzenböck formula

$$\Delta = \nabla^*\nabla + \mathrm{Ric}$$

on 1-forms, and if Ric is positive, this forces Δ to be a positive-definite operator, implying that its kernel is trivial. This does not give the stronger result (finiteness of π_1), but it is a useful technique.

The analog of the Weitzenböck identity is furnished by the following:

Theorem 17 (Schrödinger-Lichnerowicz formula)**.** Let M be an even-dimensional spin-manifold. Then we have

$$D^2 = \nabla^* \nabla + \frac{\mathrm{scal}}{4}$$

as operators $S \to S$, where $\nabla : S \to T^*M \otimes S$ is the connection and ∇^* its formal adjoint.

This is valid when S is replaced by any other vector bundle derived from a representation of $\mathrm{Cliff}(\mathbb{R}^n)$.

Proof. The proof will be a local calculation at a given point $p \in M$. Namely, we will verify it point by point, by taking an especially good frame specifically at that point. This is a useful trick which simplifies the computations; the fact that the Dirac operator does not depend on the choice of frame turns out to be helpful here.

Choose an orthonormal frame of vector fields $e_1, \dots, e_n$ near p; we can assume $\nabla_{e_i} e_j(p) = 0$ for ∇ also used to denote the Levi-Civita connection. This implies that $[e_i, e_j](p) = 0$. We then have, for any spinor field s defined near p:

$$\begin{aligned}
D^2 s(p) &= \sum e_i . \nabla_{e_i} \left(\sum e_j . \nabla_{e_j} s \right)(p) \\
&= \sum_{i,j} e_i e_j \nabla_{e_i} \nabla_{e_j s}(p) \quad \text{because } \nabla \text{ is a derivation for Clifford multiplication} \\
&= -\sum_i \nabla^2_{e_i} s(p) + \sum_{i<j} e_i e_j (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) s(p) \quad \text{by anticommutation} \\
&= \nabla^* \nabla s(p) + \sum_{i<j} e_i e_j R_S(e_i, e_j) s(p).
\end{aligned}$$

Here $R_S(\cdot, \cdot)$ denotes the curvature tensor of the connection ∇ on S. By definition, it takes in two tangent vectors and outputs a transformation of the fiber S_x to itself.

The key observation is that the connection ∇ on S is derived from the connection on TX. In fact, unwinding definitions, the connection on TX (that is, the Levi-Civita connection) is defined in a local frame by a $\mathfrak{so}(n)$-valued 1-form. The connection on S is given in a local frame by a $\mathfrak{so}(2^{n/2})$-valued n-form. These are closely related. A local section for the $\mathrm{Spin}(n)$-bundle P giving the spin structure leads to a local frame TX and for S. The connection on S is obtained by taking the $\mathfrak{so}(n)$-valued 1-form ω defining the Levi-Civita connection, applying the isomorphism $\mathfrak{so}(n) \simeq \mathfrak{spin}(n)$, and then using the homomorphism $\mathfrak{spin}(n) \to \mathfrak{so}(2^{n/2})$. This defines a local $\mathfrak{so}(2^{n/2})$-valued 1-form. The same holds for the curvature, except we have 2-forms.

Let us recall the isomorphism $\mathfrak{so}(n) \simeq \mathfrak{spin}(n)$. The Lie algebra $\mathfrak{spin}(n)$ can be identified with the subspace of $\mathrm{Cliff}(\mathbb{R}^n)$ spanned by products $v_1 v_2$ for $v_1, v_2 \in \mathbb{R}^n$ perpendicular. If V is a $\mathrm{Cliff}(\mathbb{R}^n)$-module, regarded as a $\mathrm{Spin}(n)$-representation, then we can compute the action of $v_1 v_2 \in \mathfrak{spin}(n)$ on V just by Clifford multiplication by $v_1 v_2$. If an orthonormal basis $\{e_i\}$ is fixed, the product $\frac{1}{2} e_i e_j$ corresponds to the matrix M_{ij} with 1 in (i, j) and -1 in (j, i). This is worked out in, for instance, [7] or [5].

Consequently, suppose locally[3] the curvature $R(\cdot,\cdot)$ of the Levi-Civita connection is represented by a $\mathfrak{so}(n)$-valued 2-form $\Omega = \sum_{k<l} \omega_{k,l} M_{k,l}$ (where the $\omega_{k,l}$ are orindary 2-forms). Then the curvature $R(\cdot,\cdot)$ of the connection of S is given by the $\mathfrak{so}(2^{n/2})$-valued 2-form $\widetilde{\Omega} = \frac{1}{2}\sum_{k<l} \omega_{k,l}(e_k e_l)$. Here $e_k e_l$ is regarded as an operator on $\mathbb{C}^{2^{n/2}}$ by Clifford multiplication in the spin representation.

In particular, note that, by definition,

$$\omega_{k,l}(e_i, e_j) = (R(e_i, e_j)e_k, e_l).$$

This means that

$$\widetilde{\Omega}(e_i, e_j)s = \frac{1}{2}\sum_{k<l}(R(e_i, e_j)e_k, e_l)e_k e_l s,$$

where we are using Clifford multiplication. Combining this with the previous analysis at p yields

$$D^2 s(p) = \nabla^*\nabla s(p) + \frac{1}{2}\sum_{i<j}\sum_{k<l} R_{ijkl} e_i e_j e_k e_l s(p) = \nabla^*\nabla s(p) + \frac{1}{8}\sum_{i,j,k,l} R_{ijkl} e_i e_j e_k e_l s(p). \tag{18}$$

Here we have used the notation $R_{ijkl} = (R(e_i, e_j)e_k, e_l)$.

Now we will simplify (18). First, the terms for i, j, k all distinct all cancel by the Bianchi identity, which states that R_{ijkl} cyclically permuted in i, j, k sums to zero (and the fact that cyclic permutation of i, j, k does not change $e_i e_j e_k$). So the only terms come from either $i = j$ or $j = k$ or $i = k$. The case $i = j$ contributes nothing as R_{ijkl} is antisymmetric in the first two indices.

So we need to compute $\sum_{i,j,l} R_{ijjl} e_i e_j e_j e_l$ and $\sum_{i,j,l} R_{ijil} e_i e_j e_i e_l$. Both sums are equal by the symmetry identities for the curvature. So we will just compute the first one. We have

$$\sum_{i,j,l} R_{ijjl} e_i e_j e_j e_l = -\sum_{i,j,l} R_{ijjl} e_i e_l = \sum_{i,j} R_{ijji}$$

by symmetry (all the terms where $i \neq l$ cancel), and this is precisely the scalar curvature. Plugging this back into (18) shows that the last term on the right hand side is precisely $\frac{1}{4}$scal, as desired. □

Proof of the Lichnerowicz-Hitchin theorem. If a spin-manifold M has scal > 0 everywhere, then any harmonic spinor field s (that is, a spinor satisfying $Ds = 0$) satisfies

$$0 = D^2 s = \nabla^*\nabla s + (\text{scal})s.$$

Taking the inner product with s now shows that $s \equiv 0$, because $(\nabla^*\nabla s, s) = (\nabla s, \nabla s) \geq 0$. Thus there are no harmonic spinors, and the $\hat{\mathbf{A}}$-genus vanishes. □

We also observe that if scal ≥ 0, then the same argument shows that any harmonic spinor s is *parallel*—that is, $\nabla s = 0$. This implies that s cannot vanish at even one point without vanishing everywhere. Consequently, if M admits a metric of nonnegative scalar curvature which is positive at one point, then the same conclusion holds.

Remark. The Schrödinger-Lichnerowicz formula implies moreover that the eigenvalues λ of the Dirac operator on a spin-manifold are bounded below by $\lambda^2 \geq \frac{\kappa}{4}$ for $\kappa = \min_{p\in M} \text{scal}(p)$. This can be improved slightly to $\lambda^2 \geq \frac{\kappa}{4}\frac{n}{n-1}$; see [5].

[3]That is, in a local frame represented by a local section of the principal bundle P.

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