Zariski's Main Theorem and some applications

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Abstract

We give an exposition of the various forms of Zariski's Main Theorem, following EGA. Most of the basic machinery (e.g. properties of fpqc morphisms, locally constructible sets) is developed from the beginning. In addition, several applications to the theory of étale and unramified morphisms are included.

1 Introduction

What Zariski's main theorem states is succint: a quasi-finite morphism of finite presentation between separated noetherian schemes factors as a composite of an open immersion and a finite morphism. This is a fairly big deal, as the condition of quasi-finiteness is seemingly rather weak—it is a condition on the fibers—while open immersions and finite morphisms are *very* nice. For instance, Zariski's Main Theorem is used to prove the local structure theory for étale and unramified morphisms of rings. It also immediately implies the theorem of Chevalley that a proper quasi-finite morphism is finite.

In this note, we will give Grothendieck's argument for ZMT. The argument proceeds by reducing the case of a general finitely presented quasi-finite morphism $f : X \to Y$ to the case where Y is a complete local noetherian ring(!). This reduction, which uses a general method developed by Grothendieck of reducing results about finitely presented morphisms to finite type morphisms between noetherian schemes, has many other applications, and fills an entire section of EGA IV. The argument thereafter uses a clever induction on the dimension of the ground scheme Y. In addition to this, we include various applications, such as the theory of unramified and étale morphisms and various functorial characterizations of open and closed immersions.

While it is easy to state, the general form of ZMT is fairly difficult; it is in EGA IV-3. We are going to start with the "baby" version of Zariski's main theorem (in EGA III-4 or Hartshorne), which runs as follows.

Theorem 1.1 (Zariski). *Let $f : X \to Y$ be a birational proper morphism of noetherian integral schemes, where Y is normal. Then the fibers $f^{-1}(y) = X_y, y \in Y$ are all connected.*

A priori, we know that for any open subset $U \subset Y$, the inverse image $f^{-1}(U) \subset X$ is open and thus connected. As the U's shrink towards $y \in Y$, we might expect the "limit" of the $f^{-1}(U)$ to be connected. However, this does not work. The U's that contain y are actually rather large, since we are working with the Zariski topology.

The problem is that Zariski neighborhoods are rather large, and so, intuitively, one might think to consider completions. In fact, this is what we are going to do: we will deduce the result from the so-called "formal function theorem."

2 The formal function theorem

The formal function theorem gives a basic comparison result between the operations of taking cohomology and taking completions.

2.1 Motivation

Let X be a noetherian scheme and $Z \subset X$ a closed subset, defined by a coherent sheaf of ideals $\mathcal{I}$. (So $Z = \operatorname{supp}(\mathcal{O}_X/\mathcal{I})$.) Given a coherent sheaf $\mathcal{F}$ on X, we can consider the sheaves

$$\mathcal{F}_k = \mathcal{F}/\mathcal{I}^k\mathcal{F},$$

each of which is supported on the closed *set* Z. We can thus consider their cohomologies $H^n(X, \mathcal{F}_k) = H^n(Z, \mathcal{F}_k)$; these form an inverse system over k because the $\mathcal{F}_k$ do.

The question arises: what might this inverse system look like when we take the projective limit over k? In particular, can we get it directly from $\mathcal{F}$?

First of all, the morphisms of sheaves $\mathcal{F} \to \mathcal{F}_k$ induce natural maps

$$H^n(X, \mathcal{F}) \to H^n(X, \mathcal{F}_k)$$

for each k, which commute with the maps of the projective system. There is thus a map

$$H^n(X, \mathcal{F}) \to \varprojlim_k H^n(X, \mathcal{F}_k).$$

Now, we cannot expect this map to be an isomorphism. The $\mathcal{F}_k$ are all supported on Z, while $\mathcal{F}$ probably is not.

So let $I \subset \Gamma(X, \mathcal{I})$ be an ideal. We know that $I \subset \Gamma(X, \mathcal{O}_X)$ and I acts by multiplication on the cohomology groups $H^n(X, \mathcal{F})$. Furthermore, the kth element in the I-adic filtration, that is $I^k H^n(X, \mathcal{F})$, maps to zero in $H^n(X, \mathcal{F}_k)$ because it factors

$$I^k H^n(X, \mathcal{F}) \to H^n(X, \mathcal{I}^k\mathcal{F}) \stackrel{0}{\to} H^n(X, \mathcal{F}_k).$$

In particular, for each k, we have maps

$$H^n(X, \mathcal{F})/I^k H^n(X, \mathcal{F}) \to H^n(X, \mathcal{F}_k)$$

which induces a map on the inverse limits

$$\widehat{H^n(X, \mathcal{F})} = \varprojlim H^n(X, \mathcal{F})/I^k H^n(X, \mathcal{F}) \to \varprojlim H^n(X, \mathcal{F}_k)$$

where the first is the *completion* with respect to the I-adic topology. This seems much more reasonable because, intuitively, completing at the I-adic topology is like taking a very small neighborhood of the subset Z.

So we ask, when is *this* an isomorphism? First of all, completion is only really well-behaved for finitely generated modules. So we should have some condition that the cohomology groups are finitely generated. This we can do if there is a noetherian ring A and a morphism $X \to \text{Spec} A$ which is *proper*. In this case, it is a—nontrivial—theorem that the cohomology groups of any coherent sheaf on X are finitely generated A-modules.

In addition, the ideal I should somehow determine the ideal $\mathcal{I}$, since many different $\mathcal{I}$'s could hypothetically have the same I. We could take $\mathcal{I} = f^*(I)$. It turns out that this is what we need.

2.2 The formal function theorem

So, motivated by the previous section, we state:

Let X be a proper scheme over $\text{Spec} A$, for A noetherian. Let $I \subset A$ be an ideal whose pull-back $\mathcal{I}$ is a sheaf of ideals on X. Fix a coherent sheaf $\mathcal{F}$ on X and define as before the sheaves $\mathcal{F}_k = \mathcal{F}/\mathcal{I}^k\mathcal{F}$.

Theorem 2.1 (Formal function theorem). *Hypotheses as above, the natural morphism*

$$\widehat{H^n(X, \mathcal{F})} \to \varprojlim H^n(X, \mathcal{F}_k)$$

is an isomorphism of A-modules, for each n.

Proof. We shall now prove the formal function theorem. Let us fix $A, \mathcal{F}, X$, and n.

Let $k \in \mathbb{Z}_{\geq 0}$. Then we have an exact sequence of sheaves

$$0 \to \mathcal{I}^k\mathcal{F} \to \mathcal{F} \to \mathcal{F}_k \to 0,$$

and in fact, for $k \leq k'$, we can draw a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{k'}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_{k'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}^{k}\mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_{k} & \longrightarrow & 0 \end{array}.$$

From this, we can draw a commutative diagram of long exact sequences in cohomology:

$$\begin{array}{ccccccc} H^n(X, \mathcal{I}^{k'}\mathcal{F}) & \longrightarrow & H^n(X, \mathcal{F}) & \longrightarrow & H^n(X, \mathcal{F}_{k'}) & \longrightarrow & H^{n+1}(X, \mathcal{I}^{k'}\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(X, \mathcal{I}^{k}\mathcal{F}) & \longrightarrow & H^n(X, \mathcal{F}) & \longrightarrow & H^n(X, \mathcal{F}_{k}) & \longrightarrow & H^{n+1}(X, \mathcal{I}^{k}\mathcal{F}) \end{array}.$$

This is a good thing. By a careful analysis of this diagram, we will be able to deduce the formal function theorem. The point is that we are going to take an inverse limit of this system of exact sequences. The first thing we need to do, however, is to make the sequence *short* exact. To do that, we introduce some notation.

Since n is fixed throughout this discussion, we write

$$H = H^n(X, \mathcal{F}), \quad H_k = H^n(X, \mathcal{F}_k)$$

and

$$R_k = \mathrm{im}(H^n(X, \mathcal{I}^k\mathcal{F}) \to H^n(X, \mathcal{F})).$$

Finally, we write Q_k for the image

$$Q_k = \mathrm{im}(H^n(X, \mathcal{F}_k) \to H^{n+1}(X, \mathcal{I}^k\mathcal{F}));$$

this is equivalently a kernel:

$$Q_k = \ker(H^{n+1}(X, \mathcal{I}^k\mathcal{F}) \to H^{n+1}(X, \mathcal{F})).$$

Given this notation, it is clear that we have a family of semi-short exact sequences, which forms an inverse system:

$$0 \to R_k \to H \to H_k \to Q_k \to 0.$$

The goal is to take the inverse limit of this and somehow get the formal function theorem.

We are going to show two things. One is that the $R_k \subset H$ form essentially the I-adic filtration on H. Precisely, the point is that they induce the I-adic topology. This is reasonable, since the R_k's are defined via $H^n(X, \mathcal{I}^k\mathcal{F})$ and this "looks like" multiplication by $\mathcal{I}^k$, although it is in the wrong spot. Second, we are going to show that the morphisms between the Q_k's are eventually zero, so that $\varprojlim Q_k = 0$.

After this, when we take the inverse limit[1] of the system of exact sequences

$$0 \to H/R_k \to H_k \to Q_k \to 0$$

it will follow (since $\varprojlim Q_k = 0$) that $\varprojlim H/R_k = \varprojlim H_k$. Since the R_k form the I-adic topology on H, we will have derived the formal function theorem.

2.2.1 Step one: the R_k

Let us begin by analyzing the $R_k = \mathrm{im}(H^n(X, \mathcal{I}^k\mathcal{F}) \to H^n(X, \mathcal{F}))$. We are going to show that the R_k form a filtration on H that is equivalent to the I-adic one. On the one hand, it is clear that $I^k H \subset R_k$ because if $x \in I^k$, the map

$$x : H \to H$$

factors as

$$H^n(X, \mathcal{F}) \xrightarrow{x} H^n(X, \mathcal{I}^k\mathcal{F}) \to H^n(X, \mathcal{F}).$$

So the R_k are at least as large as the I-adic filtration. For the other inclusion, namely that the R_k are small, we shall need to invoke a big result.

[1] Inverse limits are left-exact, by general categorical nonsense.

2.2.2 Some tools from algebraic geometry

We shall now quote two important coherence theorems in algebraic geometry.

Theorem 2.2 (EGA III.3.2.1). *Let $f : X \to Y$ be a morphism of proper noetherian schemes. Then if $\mathcal{F}$ is a coherent sheaf on X, the higher direct images $R^i f_*(\mathcal{F})$ are coherent on Y.*

This is a relative and scheme-theoretic version of the fact that on a complete (e.g. projective) variety over a field, the cohomology groups are always finite-dimensional. We shall not give the proof.

Actually, however, we need a stronger variant of the above direct image theorem:

Theorem 2.3 (EGA III.3.3.1). *Let $f : X \to Y$ be a morphism of proper noetherian schemes and let $\mathcal{S}$ be a quasi-coherent, finitely generated graded algebra over $\mathcal{O}_Y$. Then if $\mathcal{F}$ is a quasi-coherent sheaf on X which is a finitely generated graded $f^*(\mathcal{S})$-module, the higher direct images $R^i f_*(\mathcal{F})$ are finitely generated $\mathcal{S}$-modules.*

This is a strengthening of the usual proper mapping theorem, but it is in fact not terribly difficult to deduce from it by simply forming the relative Spec of $\mathcal{S}$ and invoking Theorem 2.2. We refer the reader to EGA for the argument.

2.2.3 The filtration of the R_k

Now we are going to return to the case of interest in the formal function theorem. Namely, we have a proper morphism $f : X \to \mathrm{Spec} A$ for A noetherian, $I \subset A$ an ideal, $f^*(I) = \mathcal{I}$ an ideal of $\mathcal{O}_X$, and $\mathcal{F}$ a coherent sheaf on X. We have defined $H = H^n(X, \mathcal{F})$ and $R_k = \mathrm{im}(H^n(X, \mathcal{I}^k \mathcal{F}) \to H^n(X, \mathcal{F}))$; we shall now show that the topology induced by the R_k on H is equivalent to the I-adic topology.

Consider the **blowup algebra** $A \oplus I \oplus I^2 \oplus \dots$, which is a finitely generated A-algebra; it can be considered as a quasicoherent sheaf $\mathcal{S}$ of finitely generated graded $\mathcal{O}_{\mathrm{Spec} A}$ algebras on $\mathrm{Spec} A$ and pulled back to X, when it becomes the $\mathcal{O}_X$-algebra

$$f^*(\mathcal{S}) = \mathcal{O}_X \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots.$$

Now the quasi-coherent sheaf

$$\mathcal{F} \oplus \mathcal{I}\mathcal{F} \oplus \dots$$

is a finitely generated graded sheaf of modules on X over the quasicoherent, finitely generated graded algebra $f^*(\mathcal{S})$. It follows by Theorem 2.3 that the cohomology

$$H^n(X, \mathcal{F}) \oplus H^n(X, \mathcal{I}\mathcal{F}) \oplus \dots$$

is a finitely generated module over the blowup algebra $A \oplus I \oplus I^2 \oplus \dots$.

By finite generation, it follows that the map

$$I \otimes_R H^n(X, \mathcal{I}^k \mathcal{F}) \to H^n(X, \mathcal{I}^{k+1} \mathcal{F})$$

is surjective for large k; this is a familiar argument as in the proof of the Artin-Rees lemma. Let us draw a commutative diagram

$$\begin{array}{ccc} I \otimes_R H^n(X, \mathcal{I}^k\mathcal{F}) & \longrightarrow & H^n(X, \mathcal{I}^{k+1}\mathcal{F}) \\ \downarrow & & \downarrow \\ I \otimes_R H^n(X, \mathcal{F}) & \longrightarrow & H^n(X, \mathcal{F}) \end{array} .$$

If k is large, then the top horizontal map is surjective, and consequently the image of $I \otimes_R H^n(X, \mathcal{I}^k\mathcal{F})$ in $H^n(X, \mathcal{F})$ (namely, IR_k) is the same as that of $H^n(X, \mathcal{I}^{k+1}\mathcal{F})$ (namely, R_{k+1}). This implies that the $\{R_k\}$ form a filtration equivalent to the I-adic one. This is precisely what we wanted to see.

So one step is done. We have analyzed the R_k.

2.3 Step two: the Q_k

Now, we need to analyze the Q_k. Namely, we need to show that the maps $Q_{k'} \to Q_k$ between them are basically zero, for $k' \gg k$.

Recall the definition

$$Q_k = \ker(H^{n+1}(X, \mathcal{I}^k\mathcal{F})) \to H^{n+1}(X, \mathcal{F}))$$

or

$$Q_k = \mathrm{Im}(H^n(X, \mathcal{F}_k) \to H^{n+1}(X, \mathcal{I}^k\mathcal{F})).$$

We will play both off against each other together with the proper coherence theorem. First, we note that since each Q_k is an A-module, and by the first definition there are canonical maps $Q_k \times I \to Q_{k+1}$ given by multiplication, we find that

$$Q = \bigoplus Q_k$$

is a module over the blowup algebra $A \oplus I \oplus \dots$, as before. In fact, we find that Q is a submodule of $\bigoplus H^n(X, \mathcal{I}^{k+1}\mathcal{F})$. Since we know that the module $\bigoplus H^n(X, \mathcal{I}^{k+1}\mathcal{F})$ is finitely generated by Theorem 2.3, we find that Q is finitely generated over the blowup algebra, which is noetherian.

However, it is also true, in view of the second definition, that the A-module Q_k is annihilated by I^k, because $\mathcal{F}_k$ is. It follows by finite generation of Q that there is a high power N such that the ideal $I^N \oplus I^{N+1} \oplus \dots$ in the blowup algebra annihilates the entire module Q.

There are maps $Q_{k+r} \to Q_k$, which are induced by the usual maps $H^n(X, \mathcal{I}^{k+r}\mathcal{F}) \to H^n(X, \mathcal{I}^k\mathcal{F})$, We have a sequence

$$Q_k \otimes I^r \to Q_{k+r} \to Q_k$$

where the first comes from the blowup multiplication and the second is the canonical map just described. The composite is the A-module multiplication in Q_k. The first map is surjective for r by finite generation over the blowup algebra, and the composite is zero for $r \geq N$ by the above reasoning. It follows that

$$Q_{k+r} \to Q_k$$

is zero if $r \gg 0$.

2.3.1 Completion of the proof

So we have established two things. One, that the R_k form basically an I-adic filtration on $H = H^n(X, \mathcal{F})$. Second, that the maps $Q_{k'} \to Q_k$ are zero for $k' \gg k$, and in particular, the inverse limit $\varprojlim Q_k$ is zero.

Now we have an exact sequence for each k,

$$0 \to H/R_k \to H_k \to Q_k \to 0,$$

and we can take the inverse limit, which is a left-exact functor. We find an exact sequence

$$0 \to \widehat{H} \to \varprojlim H_k \to 0, \qquad \square$$

which is precisely the isomorphism that the formal function theorem states.

2.4 A fancier version of the formal function theorem

We shall now restate what we proved earlier in a slightly fancier form (i.e. using the derived functors $R^n f_*$ instead of cohomology).

Previously, we said that if X was a proper scheme over Spec$A$ with structure morphism $f : X \to \mathrm{Spec}A$, and $\mathcal{I} = f^*(I)$ for some ideal $I \subset A$, then there were two constructions one could do on a coherent sheaf $\mathcal{F}$ on X that were in fact the same. Namely, we could complete the cohomology $H^n(X, \mathcal{F})$ with respect to I, and we could take the inverse limit $\varprojlim H^n(X, \mathcal{F}/\mathcal{I}^k\mathcal{F})$. The claim was that the natural map

$$\widehat{H^n(X, \mathcal{F})} \to \varprojlim H^n(X, \mathcal{F}/\mathcal{I}^k\mathcal{F})$$

was in fact an isomorphism. This is a very nontrivial statement, but in fact we saw that a reasonably straightforward proof could be given via diagram-chasing if one appeals to a strong form of the proper mapping theorem.

Now, however, we want to jazz this up a little by replacing cohomology groups with higher direct images. Slightly more is done in EGA, where the two quantities in question are identified with the derived functors of the push-forward on *formal schemes,* but we shall not discuss formal schemes here.

First, however, we need to introduce a notion. Let $Z' \subset Z$ be a closed subscheme of the scheme Z, defined by a sheaf of ideals $\mathcal{J}$. Given a sheaf $\mathcal{F}$ on Z, we can define the **completion along** Z', denoted $\hat{\mathcal{F}}$, to be the inverse limit $\varprojlim \mathcal{F}/\mathcal{J}^k\mathcal{F}$; this is a sheaf supported on the underlying set of Z'. If one is working with noetherian schemes, this depends only on the underlying *set* of Z', since any two such ideals will have comparable powers.

Now we can state the general formal function theorem:

Theorem 2.4. *Let $f : X \to Y$ be a proper morphism of noetherian schemes. Let $Y' \subset Y$ be a closed subscheme whose pull-back to X is defined by the sheaf $\mathcal{I}$ of ideals. If $\mathcal{F}$ is a coherent sheaf on X, then there is a canonical isomorphism*

$$\widehat{R^n f_*(\mathcal{F})} \simeq \varprojlim R^n f_*(\mathcal{F}/\mathcal{I}^k\mathcal{F}). \tag{1}$$

Now one wants to say further that this is isomorphic to $R^n \widehat{f}_*(\widehat{\mathcal{F}})$, where $\widehat{f}$ denotes the morphism on the formal completions. This is also true, but requires a close analysis of exactly how cohomology behaves with respect to projective limits.

Proof. We have essentially already proved this jazzed-up version of the formal function theorem. Indeed, to compute the higher direct images, one just takes ordinary cohomology over inverse images and sheafifies. If one restricts to open affines, one does not even have to sheafify.

So let $U = \text{Spec}A \subset Y$ be an affine over which Y' is given by an ideal $I \subset A$. Then we can evaluate both sides of the above equation at U. On the left, we get the I-adic completion

$$\varprojlim H^n(f^{-1}(U), \mathcal{F})/I^k H^n(f^{-1}(U), \mathcal{F});$$

on the right, we get

$$\varprojlim \Gamma(U, R^n f_*(\mathcal{F}/\mathcal{I}^k\mathcal{F})) = \varprojlim H^n(f^{-1}(U), \mathcal{F}/\mathcal{I}^k\mathcal{F}). \qquad \square$$

Note that taking sections commutes with taking inverse limits of sheaves. We know that these two are canonically isomorphic by the formal function theorem though, as $f^{-1}(U) \to U$ is proper.

2.5 An example of the fft

As an example, let $f : X \to Y$ be a proper morphism of noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on X. Suppose furthermore that $y \in Y$. We will show that there is a canonical isomorphism

$$(\widehat{R^n f_*(\mathcal{F})})_y \simeq \varprojlim H^n(f^{-1}(y), \mathcal{F}/\mathfrak{m}_y^k\mathcal{F}).$$

This is to be interpreted as follows. The first term is the completion of the stalk $(R^n f_*(\mathcal{F}))_y$ at the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_y$. The second is the inverse limit of the cohomologies of the pull-backs of $\mathcal{F}$ to "infinitesimal neighborhoods" of the fiber.

This is, incidentally, the version in which Hartshorne states the formal function theorem.

We can prove this as follows. First, let us make the flat base change $\text{Spec}\mathcal{O}_y \to Y$. The formation of higher direct images commutes with flat base change for separated morphisms, and consequently we can reduce to the case where $Y = \text{Spec}\mathcal{O}_y$. As a result, y can be assumed to be a closed point. In this case, the result is a direct restatement of the formal function theorem. The only thing to note is that $H^n(f^{-1}(y), \mathcal{F}/\mathfrak{m}_y^k\mathcal{F}) = H^n(X, \mathcal{F}/\mathfrak{m}_y^k\mathcal{F})$ because $\{y\}$, and consequently $f^{-1}(y)$, is closed.

Corollary 2.5. *Let $f : X \to Y$ be a proper morphism of noetherian schemes whose fibers are of dimension $\leq r$. Then $R^n f_*(\mathcal{F}) = 0$ for any coherent $\mathcal{F}$ if $n > r$.*

Proof. Indeed, if $\mathcal{F}, n$ are as above, we know that

$$H^n(f^{-1}(y), \mathcal{F}/\mathfrak{m}_y^k\mathcal{F}) = 0 \qquad \square$$

for all k because sheaf cohomology is trivial in dimension higher than the combinatorial dimension, by a theorem of Grothendieck. The above version of the formal function theorem implies that the coherent sheaf the $R^n f_*(\mathcal{F})$ has trivial stalks at each y (since completion is faithful for finite modules), and consequently is trivial.

Example 2.6 (Hartshorne exercise). *Let $f : X \to Y$ be a proper morphism of noetherian schemes. Let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Let us suppose that, for some $y \in Y$, we have $H^n(X_y, \mathcal{F} \otimes k(y)) = 0$; that is, the cohomology is trivial over the fiber at y. We will show more generally that $R^n f_*(\mathcal{F})$ is trivial in a neighborhood of y.*

To do this, it is sufficient to show that the completion of the stalk $(R^n f_*(\mathcal{F}))_y$ vanishes. By the formal function theorem, it is sufficient to see that

$$H^n(X_y, \mathcal{F} \otimes \mathcal{O}_y/\mathfrak{m}_y^k) = 0, \quad , \forall k.$$

It will suffice by the long exact sequence and induction to show that

$$H^n(X_y, \mathcal{F} \otimes \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1}) = 0, \quad , \forall k,$$

since $\mathcal{F}$ is flat. But $\mathfrak{m}_y/\mathfrak{m}_y^{k+1}$ is isomorphic to a direct sum of copies of $\mathcal{O}_y/\mathfrak{m}_y$, so we are done.

3 Zariski's main theorem: preliminary versions

3.1 The connectedness theorem

The baby version of Zariski's main theorem is a connectedness theorem about the fibers of a certain type of morphism.

Theorem 3.1. *Let $f : X \to Y$ be a proper morphism of noetherian schemes. Suppose that the map $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is an isomorphism. Then the fibers of f are connected.*

Proof. In order to prove this, we will apply the formal function theorem in the simplest case: when $\mathcal{F} = \mathcal{O}_X$ and $n = 0$! Namely, by (1), we have that

$$(\widehat{f_*(\mathcal{O}_X)})_y = \varprojlim \Gamma(f^{-1}(y), \mathcal{O}_X/\mathfrak{m}_y^n \mathcal{O}_X).$$

Now the first term is just the completion $\widehat{\mathcal{O}_y}$ by the assumptions. Consequently, it is a complete local ring.

The second term is also a ring, and the isomorphism is clearly one of rings, but if $f^{-1}(y)$ is disconnected into pieces $X_1 \cup X_2$, then the latter splits into two nontrivial pieces (the first piece generated by the section 1 on X_1 and 0 on X_2, and the reversal for the second). But a local ring is never decomposable as a module over itself. If it were, there would proper ideals $I_1, I_2 \subset \widehat{\mathcal{O}_y}$ that summed to the entire ring, which is a contradiction.□

From this the version of Zariski's Main Theorem stated in [Har77], for instance, can be easily deduced.

Corollary 3.2 (Baby Zariski). *Let $f : X \to Y$ be a proper birational morphism between noetherian schemes. Suppose Y is normal. Then the fibers $f^{-1}(y)$ are connected.*

Proof. Indeed, the claim is that in this case, $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. But $f_*(\mathcal{O}_X)$ is a finite $\mathcal{O}_Y$-algebra (finite as a module, in view of Theorem 2.2) and by birationality both have the same quotient field at each point, so normality implies that we have equality $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Now the above theorem gives the result. □

3.2 Stein factorization

Let $f : X \to Y$ be a proper morphism of noetherian schemes. We are going to define a *factorization* of f into the composite of a map which always has connected fibers, and a finite morphism.

To do this, note that $f_*(\mathcal{O}_X)$ is a finite $\mathcal{O}_Y$-algebra, by Theorem 2.2. By the universal property of the relative **Spec**,[2] there is a factorization of f:

$$X \xrightarrow{f'} \mathbf{Spec} f_*(\mathcal{O}_X) \xrightarrow{g} Y,$$

where the second morphism is finite because $f_*(\mathcal{O}_X)$ is coherent.

Consider now the morphism $f' : X \to Z = \mathbf{Spec} f_*(\mathcal{O}_X)$. It is easy to check that $f'_*(\mathcal{O}_X) = \mathcal{O}_Z$ since the push-forwards to Y are the same (namely, $f_*(\mathcal{O}_X)$) and $Z \to Y$ is finite. In particular, the fibers of f' are connected by Theorem 3.1.

We have now proved:

Theorem 3.3 (Stein factorization). *Hypotheses as above, there is a factorization $f = g \circ f'$ where $g : Z \to Y$ is finite and $f' : X \to Z$ has connected fibers. Further $g_*(\mathcal{O}_Z)$ is isomorphic to $f_*(\mathcal{O}_X)$, and $f'_*(\mathcal{O}_X)$ is isomorphic to $\mathcal{O}_Z$.*

3.3 The connected components of the fiber

Via Stein factorization, we can now get a good picture of the connected components of the fiber of f, only assuming f is proper and all schemes are noetherian.

Corollary 3.4. *The connected components of $f^{-1}(y)$ are in bijection with the maximal ideals of $(f_*(\mathcal{O}_X))_y$.*

Proof. Write $f = g \circ f'$ as above, where g is finite and f' has connected fibers with $g_*(\mathcal{O}_Z) = f_*(\mathcal{O}_X)$. We know that the connected components of $f^{-1}(y)$ are in bijection with the points in the fiber $g^{-1}(y)$, because the fibers of f' are connected, and those of g are discrete, and we have $f^{-1}(y) = f'^{-1}(g^{-1}(y))$.

But the points of Z lying above y under g are in bijection with the maximal ideals of $(g_*(\mathcal{O}_Z))_y = (f_*(\mathcal{O}_X))_y$. This follows from the next lemma, since Z is a relative **Spec**. □

Lemma 3.5. *Let X be a scheme, $\mathcal{C}$ a finite $\mathcal{O}_X$-algebra. Then the fiber over $x \in X$ of the natural map $\mathbf{Spec}\mathcal{C} \to X$ consists of the maximal ideals in $\mathcal{C}_x$.*

[2] Cf. EGA II.1.

Proof. Indeed, by definition the fiber is $\mathbf{Spec}\mathcal{C} \times_X k(x) = \mathrm{Spec}\mathcal{C}_x \otimes k(x)$ for $k(x)$ the residue field of x; the points of this are in bijection with the prime ideals of **Spec**, which are all maximal by finiteness. By Nakayama's lemma, the maximal ideals of $\mathcal{C}_x \otimes k(x)$ are the same as those of $\mathcal{C}_x$.[3] □

In view of the above corollary, we can generalize the inital version of Zariski's Mian Theorem to give an upper bound on the number of connected components in the fiber of a proper map.

Corollary 3.6. *Let $f : X \to Y$ be a dominant proper morphism of integral noetherian schemes. Then $k(Y)$ is naturally a subfield of $k(X)$. If $y \in Y$, the number of connected components of $f^{-1}(y)$ is at most the number of maximal ideals of the integral closure of $\mathcal{O}_y$ in $k(X)$.*

As a result of this corollary, we can in fact strengthen the baby form of Zariski's main theorem. In fact, let us say that a local domain is **unibranch** if its integral closure is also local. We can say that a point of a scheme is unibranch if its local ring is. Then, the above corollary implies that if $f : X \to Y$ is a *birational* proper morphism of noetherian schemes, for any unibranch $y \in Y$, the fiber $f^{-1}(y)$ is connected.

Proof. As before, we saw that the connected components of the fiber are in bijection with the maximal ideals of $(f_*(\mathcal{O}_X))_y$. However, this is a finite $\mathcal{O}_y$-module because $f_*(\mathcal{O}_X)$ is coherent. In particular, it is contained in the integral closure R_y of $\mathcal{O}_y$ in $k(Y)$. Now every maximal ideal in the ring $(f_*(\mathcal{O}_X))_y$ lifts to a different maximal ideal in R_y by the lying over theorem. So the number of maximal ideals in $(f_*(\mathcal{O}_X))_y$ is at most the number of maximal ideals in R_y. This is what we wanted to prove. □

There is another version of Zariski's main theorem, intermediate between the baby one and the hard quasi-finite one. It basically is the big Zariski theorem but for quasi-finite *and* quasi-projective morphisms. In the next few subsections, we shall explain this.

3.4 Points isolated in their fibers

As before, let $f : X \to Y$ be a proper morphism of noetherian schemes. We are now interested in the question of when some $x \in X$ is isolated in the fiber $f^{-1}(f(x)) \subset X$.

Proposition 3.7. *If f is proper, then the set of $x \in X$ isolated in their fiber $f^{-1}(f(x))$ is open.*

Proof. To see this, we shall use the Stein factorization. As before, we can write

$$f = g \circ f' : X \xrightarrow{f'} Z \xrightarrow{g} Y$$

where g is finite and f' has connected fibers. The fibers of g are discrete, and $f^{-1}(f(x)) = f'^{-1}(g^{-1}(f(x)))$. The connected component of x in $f^{-1}(f(x))$ is thus $f'^{-1}(f'(x))$.

[3] Any maximal ideal I of $\mathcal{C}_x$ must contain $\mathfrak{m}_x\mathcal{C}_x$ if $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_x$, or one could add $\mathfrak{m}_x\mathcal{C}_x$ to it. Note that if $I \neq \mathcal{C}_x$, then $I + \mathfrak{m}_x\mathcal{C}_x \neq \mathcal{C}_x$ by Nakayama.

We thus find that x is isolated in its fiber if and only if $f'^{-1}(f'(x)) = x$. So we may look at the corresponding set for f'; it suffices to prove the theorem for f'. That is, we may prove the proposition *under the assumption* that the fibers are all connected, and furthermore that the pushforward of the structure sheaf is the structure sheaf (since that is true of f').

Let us now drop the notation f', and assume that $f : X \to Y$ itself has connected fibers and satisfies $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. We need to show that the set of x such that $f^{-1}(f(x)) = \{x\}$ is open.

We shall prove more generally:

Proposition 3.8. *Let f be a proper morphism of noetherian schemes satisfying $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. If T is the set of $x \in X$ which are isolated in their fibers (or equivalently such that $f^{-1}(f(x)) = x$ by the connectedness principle), then T is open and $f|_T$ is an open immersion.*

Proof. Let us first check that T is open. (This is all we need for Proposition 3.7.)

If $x \in T$, we claim that $(f_*(\mathcal{O}_X))_{f(x)} = \mathcal{O}_{f(x)}$. This follows because as U becomes a small neighborhood of $f(x)$, then $f^{-1}(U)$ becomes a small neighborhood of $x = f^{-1}(f(x))$ as f is closed. Thus, since $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, we find that the natural map

$$\mathcal{O}_{X,x} \simeq \mathcal{O}_{Y,f(x)} \qquad \square$$

is an isomorphism. However, since f is of finite type, this means that there are small neighborhoods U, V of $x, f(x)$ which are *isomorphic* under f. By choosing V small, we can just take $U = f^{-1}(V)$. It follows then that every point in U is isolated in its fiber. This proves that T is open.

Now let us check that $f|_T$ is an open immersion. If $x \in T$, then we have seen that there are neighborhoods U of x and V of $f(x)$, with $U = f^{-1}(V)$, such that $f|_U : U \to V$ is an isomorphism. If we take the inverse $g_V : V \to U$, and do the same for every such pair U, V that occurs around *any* $x \in U$, then the various inverses g_V have to glue, as $f|_T$ is one-to-one. It follows that that $g|_T$ is an isomorphism onto an open subset of Y.

3.5 A useful corollary

The following special case is frequently used.

Corollary 3.9. *Let $f : X \to Y$ be a birational, bijective, proper morphism of noetherian integral schemes. Suppose Y is normal. Then f is an isomorphism.*

Proof. Indeed, as before we find that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Since every point of x is isolated in its fiber $f^{-1}(f(x))$ by assumption, we find by Proposition 3.8 that f is an open immersion. But f is proper and must have closed image, which must be all of Y as Y is irreducible. $\square$

3.6 The Zariski theorem

Finally, we can state the intermediate version of the Zariski theorem.

Theorem 3.10 (Zariski's Main Theorem). *Let $f : X \to Y$ be a quasiprojective morphism of noetherian schemes. Let U be the subset of X consisting of points which are isolated in their fibers. Then $f|_U$ factors as a composite of an open immersion and a finite morphism.*

Proof. To see this, first note that by definition of quasiprojective, f factors as a composite of an open immersion and a projective morphism. Consequently we can just assume f is projective. In this case, we have the Stein factorization $f = g \circ f'$ where g is finite. But on the set of points which are isolated in their fibers, we know that f' is actually an open immersion by Proposition 3.8. So we find

$$f|_U = g \circ f'|_U$$ □

is the requisite factorization.

The harder form of Zariski's Main Theorem deals with the structure of quasi-finite morphisms. Recall that a morphism of schemes is *quasi-finite* if it is of finite type and the fibers are finite. If we knew that every quasi-finite morphism was quasiprojective (which turns out to be true, but we shall only prove it later, in Theorem 8.4), or more generally factored as an open immersion and a *proper morphism*, then we could argue similarly as above to prove the quasi-finite version of ZMT. However, we have not shown that.

Nonetheless, we can prove:

Theorem 3.11 (Chevalley). *A proper, quasi-finite morphism $f : X \to Y$ of noetherian schemes is finite.*

Proof. We can appeal to the Stein factorization theorem to get $f = g \circ f'$ where g is finite and f' is an open immersion, since every point is isolated in its fiber. However, f is proper, and g is separated, so by the cancellation property, we find that f' is itself proper. This means that it must be a *closed* immersion as well, and consequently finite. It follows that f is finite. □

3.7 Application: a functorial characterization of closed immersions

Many characterizations of morphisms in algebraic geometry have purely functorial characterizations. We use the above machinery to give one for closed immersions. Later, we shall give one for open immersions (Theorem 4.31, in view of the characterization of étale morphisms in Theorem 4.35).

Theorem 3.12. *Let $f : X \to Y$ be morphism of finite type between locally noetherian schemes. Then f is a closed immersion if and only if it is a proper monomorphism.*

The condition of being a monomorphism is obviously a purely functorial one. Moreover, by the valuative criterion for properness, so is that of being a proper morphism. In [GD], this is extended to morphisms of finite presentation, where the noetherian hypothesis is dropped.

Proof. It is easy to see that a closed immersion is a proper monomorphism.

Conversely, suppose f is a proper monomorphism. Then f is radicial, so it is clearly quasi-finite; by Theorem 3.11, it is finite.

Now for each $y \in Y$, the base-change $X_y \to \mathrm{Spec} k(y)$ is a monomorphism. Since X_y is a one-point scheme (f being radicial), it is affine, say $X_y = \mathrm{Spec} A_y$. Since $X_y \to \mathrm{Spec} k(y)$ is a monomorphism, the diagonal embedding

$$X_y \to X_y \times_{k(y)} X_y \tag{2}$$

is an isomorphism.[4]

So, in view of (2), the associated multiplication morphism

$$A_y \otimes_{k(y)} A_y \to A_y$$

is an isomorphism of algebras. As A_y is finite over $k(y)$, this implies that $A_y = k(y)$ or A_y is the zero ring (in which case the fiber is empty).

Thus f is a finite morphism such that each fiber X_y is either empty or isomorphic to $\mathrm{Spec} k(y)$. The claim is that this implies that f is a closed immersion. Indeed, there is a coherent sheaf $\mathcal{A}$ of $\mathcal{O}_Y$-algebras such that $X = \mathrm{Spec}\mathcal{A}$ and f can be represented as the natural projection $\mathrm{Spec}\mathcal{A} \to Y$. The associated morphism $\mathcal{O}_X \to \mathcal{A}$ is surjective on the fibers by the above argument, which means that $\mathcal{A}$ is $\mathcal{O}_X/\mathcal{I}$ for some quasi-coherent sheaf of ideals; in particular, f is a closed immersion. □

3.8 The algebraic form of ZMT

We can now state a purely algebraic result, which is a form of Zariski's Main theorem. In fact, it is a direct restatement of what we have already argued, though it is slightly less elegant in the language of commutative rings (rather than schemes).

Theorem 3.13 (Zariski's Main Theorem). *Let A be a noetherian ring, B a finitely generated A-algebra. Suppose $\mathfrak{q} \in \mathrm{Spec} B$ lies over the prime ideal $\mathfrak{p} \in \mathrm{Spec} A$ and $\mathfrak{q}$ is isolated in its fiber: that is, it is simultaneously minimal and maximal for primes lying above $\mathfrak{q}$.*

Then there is $g \notin \mathfrak{q}$, a finite[5] A-algebra C with $\mathfrak{q}' \in \mathrm{Spec} C$ mapping to $\mathfrak{p}$, $h \in C - \mathfrak{q}'$ such that

$$B_g \simeq C_h$$

as A-algebras.

Proof. Indeed, the morphism $\mathrm{Spec} B \to \mathrm{Spec} A$ is quasiprojective (it is affine of finite type), and it is quasi-finite at $\mathfrak{q}$. In particular, by Theorem 3.10, it is quasi-finite in some neighborhood of $\mathfrak{q}$; in fact, there is $g \notin \mathfrak{q}$ such that the morphism

$$\mathrm{Spec} B_g \to \mathrm{Spec} A$$

[4] In a category with fibered products, a morphism $X \to Y$ is a monomorphism if and only if the diagonal $X \to X \times_Y X$ is an isomorphism. This is easy to check by considering the associated hom functors.

[5] Finite as in a finitely generated module.

is quasi-finite, and furthermore factors as an open immersion $\mathrm{Spec} B_g \hookrightarrow Z'$ and a finite morphism $Z' \to \mathrm{Spec} A$. Here Z' is necessarily affine, say $Z' = \mathrm{Spec} C$ for C a finite A-module.

Let $\mathfrak{q}'$ be the image of $\mathfrak{q} \in \mathrm{Spec} B_g$ in $\mathrm{Spec} C$. Then B_g near $\mathfrak{q}$ and C near $\mathfrak{q}'$ are locally isomorphic because $\mathrm{Spec} B_g \to \mathrm{Spec} C$ is an open immersion. This is the meaning of the theorem. □

4 Application: unramified and étale morphisms

As an application of the algebraic version of ZMT, we shall prove the basic results, in particular the local structure theorem, about unramified and étale morphisms of rings.

Throughout this section, we shall assume that all rings are noetherian.[6]

4.1 Unramified morphisms

The notion of an unramified extension generalizes the familiar notion from algebraic number theory, though it is not immediately obvious how.

Definition 4.1. *A morphism $\phi : R \to S$ of rings is* ***unramified*** *if it is of finite type and $\Omega_{S/R} = 0$.*

It is in fact possible to reformulate the final condition in the above definition in terms of an infinitesimal lifting property. Namely, a morphism $R \to S$ of finite type is unramified if and only if for every R-algebra T and ideal $I \subset T$ satisfying $I^2 = 0$, the natural map

$$\hom_R(S, T) \to \hom_R(S, T/I)$$

is injective. In other words, in a diagram of R-algebras

$$\begin{array}{ccc} & & T \\ & \nearrow & \downarrow \\ S & \longrightarrow & T/I \end{array} , \qquad (3)$$

there is at most one dotted arrow making the diagram commute. If one does not assume the finite type hypothesis, the morphism is called *formally unramified.* In fact, we can state this precisely:

Proposition 4.2. *The following are equivalent for a R-algebra S:*

1. $\Omega_{S/R} = 0$.
2. *In every diagram* (3)*, there is at most one lifting $S \to T$.*

Proof. Suppose first $\Omega_{S/R} = 0$. This is equivalent to the statement that *any* R-derivation of S into an S-module is trivial. If given an R-algebra T with an ideal $I \subset T$ of square zero and a morphism

$$S \to T/I,$$

□

[6]This should be fixed at some point.

and two liftings $f, g : S \to T$, then we find that $f - g$ maps S into I. Since T/I is naturally an S-algebra, it is easy to see (since I has square zero) that I is naturally an S-module and $f - g$ is an R-derivation $S \to I$. Thus $f - g \equiv 0$ and $f = g$.

Conversely, suppose S has the property that liftings in (3) are unique. Consider the S-module $T = S \oplus \Omega_{S/R}$ with the multiplicative structure $(a, a')(b, b') = (ab, ab' + a'b)$ that makes it into an algebra. (This is a general construction one can do with an S-module M: $S \oplus M$ is an algebra where M becomes an ideal of square zero.)

Consider the ideal $\Omega_{S/R} \subset T$, which has square zero; the quotient is S. We will find two liftings of the identity $S \to S$. For the first, define $S \to T$ sending $s \to (s, 0)$. For the second, define $S \to T$ sending $s \to (s, ds)$; the derivation property of b shows that this is a morphism of algebras.

By the lifting property, the two morphisms $S \to T$ are equal. In particular, the map $S \to \Omega_{S/R}$ sending $s \to ds$ is trivial. This implies that $\Omega_{S/R} = 0$.

We shall now give the typical list of properties ("le sorite") of unramified morphisms.

Proposition 4.3. *Any map $R \to R_f$ for $f \in R$ is unramified.*

More generally, a map from a ring to any localization is *formally* unramified, but not necessarily unramified.

Proof. Indeed, we know that $\Omega_{R/R} = 0$ and $\Omega_{R_f/R} = (\Omega_{R/R})_f = 0$, and the amp is clearly of finite type. □

Proposition 4.4. *A surjection of rings is unramified.*

Proof. Obvious. □

In the language of schemes, one can define the notion of an *unramified morphism,* and it then follows that any immersion is unramified.

Proposition 4.5. *If $R \to S$ and $S \to T$ are unramified, so is $R \to T$.*

Proof. It is clear that $R \to T$ is of finite type. We need to check that $\Omega_{T/R} = 0$. However, we have a standard exact sequence (cf. [Eis95])

$$\Omega_{S/R} \otimes_S T \to \Omega_{T/R} \to \Omega_{T/S} \to 0,$$

and since $\Omega_{S/R} = 0, \Omega_{T/S} = 0$, we find that $\Omega_{T/R} = 0$. □

Proposition 4.6. *If $R \to S$ is unramified, so is $R' \to S' = S \otimes_R R'$ for any R-algebra R'.*

Proof. This follows from the fact that $\Omega_{S'/R'} = \Omega_{S/R} \otimes_S S'$. Alternatively, it can be checked easily using the lifting criterion. □

In fact, the question of what unramified morphisms look like can be reduced to the case where the ground ring is a *field* in view of the previous and the following result.

Given $\mathfrak{p} \in \operatorname{Spec} R$, we let $k(\mathfrak{p})$ to be the residue field of $R_{\mathfrak{p}}$.

Corollary 4.7. *Let $\phi : R \to S$ be a morphism of finite type. Then ϕ is unramified if and only if for every $\mathfrak{p} \in \mathrm{Spec} R$, we have*

$$k(\mathfrak{p}) \to S \otimes_R k(\mathfrak{p})$$

unramified.

Proof. One direction is clear by Proposition 4.6. For the other, suppose $k(\mathfrak{p}) \to S \otimes_R k(\mathfrak{p})$ unramified for all $\mathfrak{p} \subset R$. We then know that

$$\Omega_{S/R} \otimes_R k(\mathfrak{p}) = \Omega_{S \otimes_R k(\mathfrak{p})/k(\mathfrak{p})} = 0$$

for all such $\mathfrak{p}$. By localization, it follows that

$$\mathfrak{p}\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{p}}} = \Omega_{S_{\mathfrak{q}}/R_{\mathfrak{p}}} = \Omega_{S_{\mathfrak{q}}/R} \tag{4}$$

for any $\mathfrak{q} \in \mathrm{Spec} S$ lying over $\mathfrak{p}$.

Let $\mathfrak{q} \in \mathrm{Spec} S$. We will now show that $(\Omega_{S/R})_{\mathfrak{q}} = 0$. Given this, we will find that $\Omega_{S/R} = 0$, which will prove the assertion of the corollary.

Indeed, let $\mathfrak{p} \in \mathrm{Spec} R$ be the image of $\mathfrak{q}$, so that there is a *local* homomorphism $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$. By Eq. (4), we find that

$$\mathfrak{q}\Omega_{S_{\mathfrak{q}}/R} = \Omega_{S_{\mathfrak{q}}/R}.$$

and since $\Omega_{S_{\mathfrak{q}}/R}$ is a finite $S_{\mathfrak{q}}$-module, Nakayama's lemma now implies that $\Omega_{S_{\mathfrak{q}}/R} = 0$, proving what we wanted. □

4.2 Unramified morphisms and the diagonal

We shall now obtain a succint criterion for unramifiedness: the diagonal morphism of associated schemes is an open immersion. This will be useful in classifying unramified morphisms out of a field.

Lemma 4.8. *Let A be a ring, $I \subset A$ a finitely generated nonzero ideal such that $I = I^2$. Then A has nontrivial idempotents, and in fact A contains A/I as a direct factor. In fact, the morphism*

$$\mathrm{Spec} A/I \to \mathrm{Spec} A$$

is an open immersion.

Proof. Consider the imbedding $\mathrm{Spec} A/I \to \mathrm{Spec} A$. We are going to show that it is an open immersion. Let $\mathcal{I}$ be the sheaf of ideals defined by the closed subscheme; then we see that $\mathcal{I}_{\mathfrak{p}} = \mathcal{I}_{\mathfrak{p}}^2$ for each prime $\mathfrak{p}$, By Nakayama's lemma, we find that $I_{\mathfrak{p}} = I_{\mathfrak{p}} = 0$ for every prime containing I. So the sheaf of ideals is zero on each point in the image of $\mathrm{Spec} A/I$ in $\mathrm{Spec} A$. It follows that the map

$$\mathrm{Spec} A/I \to \mathrm{Spec} A$$

is an open immersion, whose image must thus be a union of components of $\mathrm{Spec} A$. The result is now clear. □

Corollary 4.9. *Suppose $A \to B$ is an unramified morphism of rings. Let $X = \mathrm{Spec} B, Y = \mathrm{Spec} A$. Then the diagonal map*

$$X \to X \times_Y X$$

is an open immersion, and conversely (if $A \to B$ is of finite type).

Proof. This is straightforward from the basic properties of Kähler differentials. We know that $\Omega_{B/A}$ can be described in an alternative way. Namely, consider the surjection $B \otimes_A B \to B$ given by multiplication, and let $I \subset B \otimes_A B$ be the kernel. Then we have $\Omega_{B/A} \simeq I/I^2$, so if $A \to B$ is unramified we must have $I = I^2$. However, X is a closed subscheme of $X \times_Y X$ under the diagonal embedding, and I is the ideal cutting it out. By Lemma 4.8, it follows that the diagonal embedding is an open immersion.

Now suppose $X \to X \times_Y X$ is an open immersion. It is a closed immersion in any case, so it embeds X as a union of components of $X \times_Y X$. If I is the kernel of $B \otimes_A B \to B$, as before, then I is the ideal defining the image of X. But the image of X is a union of components. It follows that for each $\mathfrak{p} \in \mathrm{Spec} A$, we have $(I/I^2)_{\mathfrak{p}} = 0$, as the stalk of the sheaf of ideals cutting out the image of X is either the full ring or zero. Hence $I/I^2 = 0$ is true globally, and B/A is an unramified extension of rings. □

4.3 Unramified extensions of fields

Motivated by Corollary 4.7, we classify unramified morphisms out of a field. Let us first consider the case when the field is *algebraically closed.*

Proposition 4.10. *Suppose k is algebraically closed. If A is an unramified k-algebra, then A is a product of copies of k.*

Proof. Let us show first that A is necessarily finite-dimensional. Let $X = \mathrm{Spec} A$. We have a morphism

$$X \to X \times_k X,$$

which we know from Corollary 4.9 to be an open immersion (and also a closed one). Call its image Δ. Let $x \in X$ be a k-valued point; there is then a morphism (id, x) from $X \to X \times_k X$, such that the inverse image of Δ is $\{x\}$.

But since Δ is open, this means that $\{x\}$ is open. In particular, every k-valued point in X (that is, every closed point) is also open, and consequently has no nontrivial generization. But any point in X has a specialization which is closed. We find in total that every point in X is closed. This implies that A is finite-dimensional.

So let us now assume that A is finite-dimensional over k, hence *artinian.* Then A is a direct product of artinian local k-algebras. Each of these is unramified over k. So we need to study what local, artinian, unramified extensions of k look like; we shall show that any such is isomorphic to k with:

Lemma 4.11. *A finite-dimensional, local k-algebra which is unramified over k (for k algebraically closed) is isomorphic to k.*

Proof. First, if $\mathfrak{m} \subset A$ is the maximal ideal, then $\mathfrak{m}$ is nilpotent, and $A/\mathfrak{m} \simeq k$ by the Hilbert Nullstellensatz. Thus the ideal $\mathfrak{M} = \mathfrak{m} \otimes A + A \otimes \mathfrak{m} \subset A \otimes_k A$ is nilpotent and

$(A \otimes_k A)/\mathfrak{M} = k \otimes_k k = k$. In particular, $\mathfrak{M}$ is maximal and $A \otimes_k A$ is also local. (We could see this as follows: A is associated to a one-point variety, so the fibered product $\mathrm{Spec} A \times_k \mathrm{Spec} A$ is also associated to a one-point variety. It really does matter that we are working over an algebraically closed field here!)

By assumption, $\Omega_{A/k} = 0$. So if $I = \ker(A \otimes_k A \to A)$, then $I = I^2$. But from Lemma 4.8, we find that if we had $I \neq 0$, then $\mathrm{Spec} A \otimes_k A$ would be disconnected. This is clearly false (a local ring has no nontrivial idempotents), so $I = 0$ and $A \otimes_k A \simeq A$. Since A is finite-dimensional over k, necessarily $A \simeq k$. □

Now let us drop the assumption of algebraic closedness to get:

Theorem 4.12. *An unramified k-algebra for k any field is isomorphic to a product $\prod k_i$ of finite separable extensions k_i of k.*

Proof. Let k be a field, and $\overline{k}$ its algebraic closure. Let A be an unramified k-algebra. Then $A \otimes_k \overline{k}$ is an unramified $\overline{k}$-algebra by Proposition 4.6, so is a finite product of copies of $\overline{k}$. It is thus natural that we need to study tensor products of fields to understand this problem.

Lemma 4.13. *Let E/k be a finite extension, and L/k any extension. If E/k is separable, then $L \otimes_k E$ is isomorphic (as a L-algebra) to a product of copies of separable extensions of L.*

Proof. By the primitive element theorem, we have $E = k(\alpha)$ for some $\alpha \in E$ satisfying a separable irreducible polynomial $P \in k[X]$. Thus

$$E = k[X]/(P),$$

so

$$E \otimes_k L = L[X]/(P).$$

But P splits into several irreducible factors $\{P_i\}$ in $L[X]$, no two of which are the same by separability. Thus by the Chinese remainder theorem,

$$E \otimes_k L = L(X)/(\prod P_i) = \prod L[X]/(P_i),$$

and each $L[X]/(P_i)$ is a finite separable extension of L. □

As a result of this, we can easily deduce that any k-algebra of the form $A = \prod k_i$ for the k_i separable over k is unramified. Indeed, we have

$$\Omega_{A/k} \otimes_k \overline{k} = \Omega_{A \otimes_k \overline{k}/\overline{k}},$$

so it suffices to prove that $A \otimes_k \overline{k}$ is unramified over $\overline{k}$. However, from Lemma 4.13, $A \otimes_k \overline{k}$ is isomorphic as a $\overline{k}$-algebra to a product of copies of $\overline{k}$. Thus $A \otimes_k \overline{k}$ is obviously unramified over $\overline{k}$.

On the other hand, suppose A/k is unramified. We shall show it is of the form given as in the theorem. Then $A \otimes_k \overline{k}$ is unramified over $\overline{k}$, so it follows by Proposition 4.10

that A is finite-dimensional over k. In particular, A is *artinian*, and thus decomposes as a product of finite-dimensional unramified k-algebras.

We are thus reduced to showing that a local, finite-dimensional k-algebra that is unramified is a separable extension of k. Let A be one such. Then A can have no nilpotents because then $A \otimes_k \overline{k}$ would have nilpotents, and could not be isomorphic to a product of copies of $\overline{k}$. Thus the unique maximal ideal of A is zero, and A is a field. We need only show that A is separable over k. This is accomplished by:

Lemma 4.14. *Let E/k be a finite inseparable extension. Then $E \otimes_k \overline{k}$ contains nonzero nilpotents.*

Proof. There exists an $\alpha \in E$ which is inseparable over k, i.e. whose minimal polynomial has multiple roots. Let $E' = k(\alpha)$. We will show that $E' \otimes_k \overline{k}$ has nonzero nilpotents; since the map $E' \otimes_k \overline{k} \to E \otimes_k \overline{k}$ is an injection, we will be done. Let P be the minimal polynomial of α, so that $E' = k[X]/(P)$. Let $P = \prod P_i^{e_i}$ be the factorization of P in $\overline{k}$ for the $P_i \in \overline{k}[X]$ irreducible (i.e. linear). By assumption, one of the e_i is greater than one. It follows that

$$E' \otimes_k \overline{k} = \overline{k}[X]/(P) = \prod \overline{k}[X]/(P_i^{e_i})$$

has nilpotents corresponding to the e_i's that are greater than one. ☐

We now come to the result that explains why the present theory is connected with Zariski's Main Theorem.

Corollary 4.15. *An unramified morphism $A \to B$ is quasi-finite.*

Proof. Recall that a morphism of rings is *quasi-finite* if the associated map on spectra is. Equivalently, the morphism must be of finite type and have finite fibers. But by assumption $A \to B$ is of finite type. Moreover, if $\mathfrak{p} \in \mathrm{Spec} A$ and $k(\mathfrak{p})$ is the residue field, then $k(\mathfrak{p}) \to B \otimes_A k(\mathfrak{p})$ is *finite* by the above results, so the fibers are finite. ☐

Corollary 4.16. *Let $(A, \mathfrak{m})$ be a local ring with residue field k and B an A-algebra of finite type. Then $A \to B$ is unramified if and only if $B/\mathfrak{m}B$ is a finite product of separable extensions of k*

Proof. If $A \to B$ is unramified, then $k \to B/\mathfrak{m}B$ is too, so by the previous results $B/\mathfrak{m}B$ is a finite product of separable extensions.

Suppose conversely that $B/\mathfrak{m}B$ is unramified over k. IDK if this is actually true ☐

4.4 Étale morphisms

An étale morphism is one that is supposed to look like an isomorphism on the tangent spaces, if one is working with smooth varieties over an algebraically closed field.

Definition 4.17. *A morphism of rings $R \to S$ is **étale** if it is flat and unramified, and of finite presentation.*

We give two important examples.

Example 4.18. *The map $R \to R_f$ is always étale, as it is of finite presentation (as $R_f = R[X]/(fX-1)$), flat (as localization is exact), and unramified by Proposition 4.3.*

Example 4.19. *Let R be a ring, $P \in R[X]$ a polynomial. Suppose $Q \in R[X]/P$ is such that in the localization $(R[X]/P)_Q$, the image of the derivative $P' \in R[X]$ is a unit. Then the map*

$$R \to (R[X]/P)_Q$$

*is called a **standard étale morphism.***

The name is justified by:

Proposition 4.20. *A standard étale morphism is étale.*

Proof. It is sufficient to check the condition on the Kähler differentials, since a standard étale morphism is evidently flat and of finite presentation. Indeed, we have that

$$\Omega_{(R[X]/P)_Q/R} = Q^{-1}\Omega_{(R[X]/P)/R} = Q^{-1}\frac{R[X]}{(P'(X), P(X))R[X]}$$

by basic properties of Kähler differentials. Since P' is a unit after localization at Q, this last object is clearly zero. ☐

Proposition 4.21. *Étale morphisms are preserved under composites and base change.*

Proof. This is true individually for unramified morphisms, flat morphisms, and morphisms of finite presentation. ☐

4.5 The local structure theory for unramified morphisms

We know two easy ways of getting an unramified morphism out of a ring R. First, we can take a standard étale morphism, which is necessarily unramified; next we can take a quotient of that. The local structure theory states that this is all we can have, locally.

For this we introduce a definition.

Definition 4.22. *Let R be a commutative ring, S an R-algebra of finite type. Let $\mathfrak{q} \in \mathrm{Spec} S$ and $\mathfrak{p} \in \mathrm{Spec} R$ be the image. Then S is called **unramified at** $\mathfrak{q}$ (resp. **étale at** $\mathfrak{p}$) if $\Omega_{S_\mathfrak{q}/R_\mathfrak{p}} = 0$ (resp. that and $S_\mathfrak{q}$ is $R_\mathfrak{p}$-flat).*

Now when works with finitely generated algebras, the module of Kähler differentials is always finitely generated over the top ring. In particular, if $\Omega_{S_\mathfrak{q}/R_\mathfrak{p}} = (\Omega_{S/R})_\mathfrak{q} = 0$, then there is $f \in S - \mathfrak{q}$ with $\Omega_{S_f/R} = 0$. So being unramified at $\mathfrak{q}$ is equivalent to the existence of $f \in S - \mathfrak{q}$ such that S_f is unramified over R. Clearly if S is unramified over R, then it is unramified at all primes, and conversely.

Theorem 4.23. *Let $\phi : R \to S$ be morphism of finite type, and $\mathfrak{q} \subset S$ prime with $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Suppose ϕ is unramified at $\mathfrak{q}$. Then there is $f \in R - \mathfrak{p}$ and $g \in S - \mathfrak{q}$ (divisible by $\phi(f)$) such that the morphism*

$$R_f \to S_g$$

factors as a composite

$$R_f \to (R_f[x]/P)_h \twoheadrightarrow S_g$$

where the first is a standard étale morphism and the second is a surjection. Moreover, we can arrange things such that the fibers above $\mathfrak{p}$ are isomorphic.j

Proof. We shall assume that R is *local* with maximal ideal $\mathfrak{p}$. Then the question reduces to finding $g \in S$ such that S_g is a quotient of an algebra standard étale over R. This reduction is justified by the following argument: if R is not necessarily local, then the morphism $R_\mathfrak{p} \to S_\mathfrak{p}$ is still unramified. If we can show that there is $g \in S_\mathfrak{p} - \mathfrak{q}S_\mathfrak{p}$ such that $(S_\mathfrak{p})_g$ is a quotient of a standard étale $R_\mathfrak{p}$-algebra, it will follow that there is $f \notin \mathfrak{p}$ such that the same works with $R_f \to S_{gf}$.

We shall now reduce to the case where S is a finite R-algebra. Let R be local, and let $R \to S$ be unramified at $\mathfrak{q}$. By assumption, S is finitely generated over R. We have seen by Corollary 4.15 that S is quasi-finite over R at $\mathfrak{q}$. By Zariski's Main Theorem (Theorem 3.13), there is a finite R-algebra S' and $\mathfrak{q}' \in \mathrm{Spec} S'$ such that S near $\mathfrak{q}$ and S' near $\mathfrak{q}'$ are isomorphic (in the sense that there are $g \in S - \mathfrak{q}$, $h \in S' - \mathfrak{q}'$ with $S_g \simeq S'_h$). Since S' must be unramified at $\mathfrak{q}'$, we can assume at the outset, by replacing S by S', that $R \to S$ is finite and unramified at $\mathfrak{q}$.

We shall now reduce to the case where S is generated by one element as R-algebra. This will occupy us for a few paragraphs.

We have assumed that R is a local ring with maximal ideal $\mathfrak{p} \subset R$; the maximal ideals of S are finite, say, $\mathfrak{q}, \mathfrak{q}_1, \dots, \mathfrak{q}_r$ because S is finite over R; these all contain $\mathfrak{p}$ by Nakayama. These are no inclusion relations among $\mathfrak{q}$ and the $\mathfrak{q}_i$ as $S/\mathfrak{p}S$ is an artinian ring.

Now $S/\mathfrak{q}$ is a finite separable field extension of $R/\mathfrak{p}$ by Theorem 4.12; indeed, the morphism $R/\mathfrak{p} \to S/\mathfrak{p}S \to S/\mathfrak{q}$ is a composite of unramified extensions and is thus unramified. In particular, by the primitive element theorem, there is $x \in S$ such that x is a generator of the field extension $R/\mathfrak{p} \to S/\mathfrak{q}$. We can also choose x to lie in the other $\mathfrak{q}_i$ by the Chinese remainder theorem. Consider the subring $C = R[x] \subset S$. It has a maximal ideal $\mathfrak{s}$ which is the intersection of $\mathfrak{q}$ with C. We are going to show that locally, C and S look the same.

Lemma 4.24. *Let $(R, \mathfrak{p})$ be a local ring and S a finite R-algebra. Let $\mathfrak{q}, \mathfrak{q}_1, \dots, \mathfrak{q}_r \in \mathrm{Spec} S$ be the prime ideals lying above $\mathfrak{p}$. Suppose S is unramified at $\mathfrak{q}$.*

Then there is $x \in S$ such that the rings $R[x] \subset S$ and S are isomorphic near $\mathfrak{q}$: more precisely, there is $g \in R[x] - \mathfrak{q}$ with $R[x]_g = S_g$.

Proof. Choose x as in the paragraph preceding the statement of the lemma. We have morphisms

$$R \to C_\mathfrak{s} \to S_\mathfrak{s}$$

where $S_\mathfrak{s}$ denotes S localized at $C - \mathfrak{s}$, as usual. The second morphism here is finite. However, we claim that $S_\mathfrak{s}$ is in fact a local ring with maximal ideal $\mathfrak{q}S_\mathfrak{s}$; in particular, $S_\mathfrak{s} = S_\mathfrak{q}$. Indeed, S can have no maximal ideals other than $\mathfrak{q}$ lying above $\mathfrak{s}$; for, if $\mathfrak{q}_i$ lay over $\mathfrak{s}$ for some i, then $x \in \mathfrak{q}_i \cap C = \mathfrak{s}$. But $x \notin \mathfrak{s}$ because x is not zero in $S/\mathfrak{q}$.

It thus follows that $S_\mathfrak{s}$ is a local ring with maximal ideal $\mathfrak{q}S_\mathfrak{s}$. In particular, it is equal to $S_\mathfrak{q}$, which is a localization of $S_\mathfrak{s}$ at the maximal ideal. In particular, the morphism

$$C_\mathfrak{s} \to S_\mathfrak{s} = S_\mathfrak{q}$$

is finite. Moreover, we have $\mathfrak{s}S_{\mathfrak{q}} = \mathfrak{q}S_{\mathfrak{q}}$ by unramifiedness of $R \to S$. So since the residue fields are the same by choice of x, we have $\mathfrak{s}S_{\mathfrak{q}} + C_{\mathfrak{s}} = S_{\mathfrak{q}}$. Thus by Nakyama's lemma, we find that $S_{\mathfrak{s}} = S_{\mathfrak{q}} = C_{\mathfrak{s}}$.

There is thus an element $g \in C - \mathfrak{r}$ such that $S_g = C_g$. In particular, S and C are isomorphic near $\mathfrak{q}$. □

We can thus replace S by C and assume that C has one generator.

With this reduction now made, we proceed. We are now considering the case where S is generated by one element, so a quotient $S = R[X]$ for some monic polynomial P. Now $\overline{S} = S/\mathfrak{p}S$ is thus a quotient of $k[X]$, where $k = R/\mathfrak{p}$ is the residue field. It thus follows that

$$\overline{S} = k[X]/(\overline{P})$$

for $\overline{P}$ a monic polynomial, as $\overline{S}$ is a finite k-vector space.

Suppose $\overline{P}$ has degree n. Let $x \in S$ be a generator of S/R. We know that $1, x, \dots, x^{n-1}$ has reductions that form a k-basis for $S \otimes_R k$, so by Nakayama they generate S as an R-module. In particular, we can find a monic polynomial P of degree n such that $P(x) = 0$. It follows that the reduction of P is necessarily $\overline{P}$. So we have a surjection

$$R[X]/(P) \twoheadrightarrow S$$

which induces an isomorphism modulo $\mathfrak{p}$ (i.e. on the fiber).

Finally, we claim that we can modify $R[X]/P$ to make a standard étale algebra. Now, if we let $\mathfrak{q}'$ be the preimage of $\mathfrak{q}$ in $R[X]/P$, then we have morphisms of local rings

$$R \to (R[X]/P)_{\mathfrak{q}'} \to S_{\mathfrak{q}}.$$

The claim is that $R[X]/(P)$ is unramified over R at $\mathfrak{q}'$.

To see this, let $T = (R[X]/P)_{\mathfrak{q}'}$. Then, since the fibers of T and $S_{\mathfrak{q}}$ are the same at $\mathfrak{p}$, we have that

$$\Omega_{T/R} \otimes_R k(\mathfrak{p}) = \Omega_{T \otimes_R k(\mathfrak{p})/k(\mathfrak{p})} = \Omega_{(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})/k(\mathfrak{p})} = 0$$

as S is R-unramified at $\mathfrak{q}$. It follows that $\Omega_{T/R} = \mathfrak{p}\Omega_{T/R}$, so a fortiori $\Omega_{T/R} = \mathfrak{q}\Omega_{T/R}$; since this is a finitely generated T-module, Nakayama's lemma implies that is zero. We conclude that $R[X]/P$ is unramified at $\mathfrak{q}'$; in particular, by the Kähler differential criterion, the image of the derivative P' is not in $\mathfrak{q}'$. If we localize at the image of P', we then get what we wanted in the theorem. □

4.6 Local structure theory for étale morphisms

We now want to deduce a corresponding (stronger) result for *étale* morphisms. Indeed, we prove:

Theorem 4.25. *If $R \to S$ is étale at $\mathfrak{q} \in \mathrm{Spec} S$ (lying over $\mathfrak{p} \in \mathrm{Spec} R$), then there are $f \in R - \mathfrak{p}, g \in S - \mathfrak{q}$ such that the morphism $R_f \to S_g$ is a standard étale morphism.*

Proof. By localizing suitably, we can assume that $(R, \mathfrak{p})$ is local, and (in view of **??**), $R \to S$ is a quotient of a standard étale morphism

$$(R[X]/P)_h \twoheadrightarrow S$$

with the kernel some ideal I. We may assume that the surjection is an isomorphism modulo $\mathfrak{p}$, moreover. By localizing S enough[7] we may suppose that S is a *flat* R-module as well.

Consider the exact sequence of $(R[X]/P)_h$-modules

$$0 \to I \to (R[X]/P)_h/I \to S \to 0.$$

Let $\mathfrak{q}'$ be the image of $\mathfrak{q}$ in $\mathrm{Spec}(R[X]/P)_h$. We are going to show that the first term vanishes upon localization at $\mathfrak{q}'$. Since everything here is finitely generated, it will follow that after further localization by some element in $(R[X]/P)_h - \mathfrak{q}'$, the first term will vanish. In particular, we will then be done.

Everything here is a module over $(R[X]/P)_h$, and certainly a module over R. Let us tensor everything over R with $R/\mathfrak{p}$; we find an exact sequence

$$I \to S/\mathfrak{p}S \to S/\mathfrak{p}S \to 0;$$

we have used the fact that the morphism $(R[X]/P)_h \to S$ was assumed to induce an isomorphism modulo $\mathfrak{p}$.

However, by étaleness we assumed that S was R-*flat*, so we find that exactness holds at the left too. It follows that

$$I = \mathfrak{p}I,$$

so a fortiori

$$I = \mathfrak{q}'I,$$

which implies by Nakayama that $I_{\mathfrak{q}'} = 0$. Localizing at a further element of $(R[X]/P)_h - \mathfrak{q}'$, we can assume that $I = 0$; after this localization, we find that S looks *precisely* a standard étale algebra. ☐

4.7 Morphisms of schemes

We now globalize the earlier theory to schemes.

Definition 4.26. *A morphism $f : X \to Y$ of schemes of finite type is* ***unramified*** *if there is a pair of affine open coverings $\{U_\alpha = \mathrm{Spec} B_\alpha\}, \{V_\alpha = \mathrm{Spec} A_\alpha\}$ of X and Y such that $f(U_\alpha) \subset V_\alpha$ and such that the induced morphism*

$$A_\alpha \to B_\alpha$$

is unramified.

We have a very simple global criterion:

Proposition 4.27. *The following are equivalent for a morphism $f : X \to Y$ of finite type:*

1. *f is unramified.*
2. *The sheaf $\Omega_{X/Y}$ of relative Kähler differentials vanishes.*

[7] We are not assuming S finite over R here,

3. The diagonal map $\Delta : X \to X \times_Y X$ is an open immersion.

Proof. The equivalence of 1 and 2 is basically the definition, as $\Omega_{X/Y}$ can be computed locally (i.e. it requires no global data). Now the rest is just a globalization of Corollary 4.9. □

Proposition 4.28 (Le sorite). *An immersion is unramified. Unramified morphisms are closed under base change and composition.*

Definition 4.29. *A morphism of schemes of finite type is* ***étale*** *if it is flat and unramified.*

Proposition 4.30 (Le sorite). *An open immersion is étale. Étale morphisms are closed under base change and composition.*

4.8 The fundamental property of étale morphisms

Theorem 4.31. *An étale, radicial morphism is an open immersion.*

Grothendieck calls this the "fundamental property" of étale morphisms, and we follow him in [Gro71] in the proof.

Proof. Let $f : X \to Y$ be étale and radicial. Then f, being flat and of finite presentation, is open (**??**), so let us replace Y by the image of f and assume f surjective. In this case, we shall prove f is even an *isomorphism.*

So suppose f étale, radicial, and surjective. Since f is open and bijective (by radicialness), it is a homeomorphism. Since any base change of f is also étale, radicial, and surjective, any base change of f is a homeomorphism. Then f is universally closed, hence *proper* because any radicial morphism is separated.[8] It is also quasi-finite (being étale, hence unramified), so f is finite by Theorem 3.11.

In particular, the morphism $f : X \to Y$ can be represented as the projection $\mathbf{Spec}(\mathcal{B}) \to Y$ for a coherent sheaf of algebras $\mathcal{B}$ on Y. Since f is flat, it follows that $\mathcal{B}$ must be flat, hence locally free. However, since the fibers are of cardinality one, $\mathcal{B}$ is locally free of rank one as a $\mathcal{O}_Y$-algebra. In particular, the morphism $\mathcal{O}_Y \to \mathcal{B}$ is an isomorphism (since it is an isomorphism on the fibers), and thus $\mathcal{B} = \mathcal{O}_Y$, so f is an isomorphism. □

4.9 The infinitesimal lifting property

We begin with a standard cancellation argument.

Proposition 4.32. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms of finite type. If $g \circ f$ is étale and g is unramified, then f is étale.*

Proof. Since the diagonal of an unramified morphism is an open immersion (which is étale), this follows from standard arguments. □

[8] A radicial morphism has the property that the diagonal morphism is surjective. This is actually a characterization. See EGA I.

Let S be a scheme, and X an unramified S-scheme. Then any S-*section* $S \to X$ is, by Proposition 4.32, étale. Moreover, since the composite $S \to X \to S$ is the identity and thus radicial, it follows that $S \to X$ itself is radicial.[9] If, furthermore, X is *separated* over S, then the same cancellation argument shows that the section is *proper.* By Theorem 4.31, S is an open immersion—thus an isomorphism of S onto a union of components of X.

We conclude:

Proposition 4.33. *If $X \to S$ is a separated and unramified morphism for S connected, then the sections $S \to X$ are naturally in bijection with the set of components of X such that the projection to S is an isomorphism.*

In fact, it is possible to prove the aforementioned fact, that a section of an unramified morphism is an open immersion in a much more elementary manner. Consider the class P of open immersions of schemes; this is stable under composition and base-change. An unramified morphism is in $\mathrm{diag}(P)$; thus in a composition

$$X \to Y \to Z$$

with the composite an open immersion and $Y \to Z$ unramified, we can conclude that $X \to Y$ is unramified. We thus deduce a simpler proof of Proposition 4.33.

We are now able to enunciate the *infinitesimal lifting property.* Let S be a scheme, X, Y S-schemes. Let S_0 be a subscheme of S, and let $X_0 = X \times_S S_0$ and $Y = Y \times_S S_0$. Given a morphism $X \to Y$, "restriction" (that is, taking products) gives a morphism $X_0 \to Y_0$. In general, the associated map

$$\hom_S(X, Y) \to \hom_S(X_0, Y_0) \tag{5}$$

is neither injective nor surjective.

Theorem 4.34. *Suppose Y is étale over S and the immersion $S_0 \to S$ is surjective. Then* (5) *is an isomorphism.*

Proof. The S-morphisms $X \to Y$ (or $X_0 \to Y_0$) are in bijection with the sections of $X \times_S Y \to X$ (resp. $X_0 \times_{S_0} Y_0 \to Y_0$). Since these morphisms are étale by base change, we find that it is sufficient to prove the theorem in the case where $X = S$, in particular, when X is separated and the question boils down to finding S-sections. So let us take $X = S$. Furthermore, we may assume S, and thus S_0, connected, as S_0 and S have the same topological space.

Let $S_0 \to X_0$ be a section, or equivalent a S-morphism $S_0 \to X$. Then $S_0 \to X_0$ is by **??** an isomorphism onto a connected component X_0' of X_0, which corresponds to a connected component $X' \subset X$. The map $X' \to S$ is such that the base-change to S_0 is an isomorphism (i.e. $X_0' \to S_0$). The claim is that $X' \to S$ is itself an isomorphism, so that we can get a section $S \to X$. But $X' \to S$ is clearly étale and surjective; since the

[9] There is a general principle, which one may enunciate as follows. Let $\mathcal{C}$ be a category with fibered products. Suppose P is a class of morphisms in $\mathcal{C}$ closed under composition and base-change. Say that a map $X \to Y$ is in $\mathrm{diag}P$ if $X \to X \times_Y X$ is in P. Then if $X \to Y \to Z$ is a composite with $X \to Z$ in P and $Y \to Z$ in $\mathrm{diag}P$, then one can "cancel" and conclude that $X \to Y$ is in P. See [**?**]. This applies in the present case because any diagonal morphism of schemes is an immersion, thus radicial.

base-change to S_0 is a universal homeomorphism, $X' \to S$ is a universal homeomorphism as well, and in particular radicial. Theorem 4.31 shows that $X' \to S$ is an isomorphism.

Conversely, we have to show that any section $S_0 \to X_0$ can be lifted to X in at most one way. But this is clear from the "topological" description of sections. If two maps $S \rightrightarrows X$ lift a section $S_0 \to X_0$, then the corresponding connected components of X must be the same, so the two maps $S \rightrightarrows X$ are the same. □

In fact, we can *characterize* étale morphisms of schemes by such a lifting property. We thus get a purely *functorial* description of étale morphisms.

Theorem 4.35. *Let $f : X \to S$ be a morphism of finite presentation. Then f is étale if and only if it has the following property. For all S-schemes T, and subschemes T_0 whose underlying space is all of T, the morphism*

$$\hom_S(T, X) \to \hom_S(T_0, X)$$

is bijective.

In other words, given a diagram

$$\begin{array}{ccc} T_0 & & \\ \downarrow & \searrow & \\ T & \dashrightarrow & X \end{array}$$

a dotted arrow can be found, and found *uniquely.* It suffices to check this on affine schemes.

Proof. Let us first suppose X étale over S. Then the S-morphisms $T \to X$ are in bijection with the T-morphisms $T \to T \times_S X$; similarly the S-morphisms $T_0 \to X$ are in bijection with the T_0-morphisms $T_0 \to T_0 \times_S X$.

In other words, we are comparing sections $T \to T \times_S X$ and sections $T_0 \to T_0 \times_S X$. Since $T \times_S X$ is étale over T, we find the required bijection by Theorem 4.34.

Now let us suppose the lifting property holds. (to be added)

4.10 Another functorial characterization of closed immersions

Theorem 4.36. *A morphism $f : X \to Y$ of finite type between locally noetherian schemes is a closed immersion if and only if it is proper, unramified, and radicial.*

This is a legitimate functorial characterization, because properness can be checked via the valuative criterion, radicialness of f is equivalent to the statement that $\hom(\mathrm{Spec} K, X) \to \hom(\mathrm{Spec} K, Y)$ is injective (as a map of sets) for K a field, and unramifiedness is equivalent to a lifting property (by **??**—not added yet!).

Proof. Clearly a closed immersion has those three properties.

Suppose f is proper, unramified, and radicial. Then f is quasi-finite (being radicial), so by Theorem 3.11 is finite. In particular, f can be represented as the natural projection

$$\mathrm{Spec}\mathcal{A} \to Y$$

for $\mathcal{A}$ a coherent sheaf of $\mathcal{O}_Y$-algebras. We need to show that the natural map $\mathcal{O}_Y \to \mathcal{A}$ is a surjection. To do this, it suffices by Nakayama to check on the fibers.

But each fiber X_y is unramified and radicial over $\mathrm{Spec} k(y)$ (for $y \in Y$). The second means that $X_y = \mathrm{Spec} K$ for K a purely inseparable extension of $k(y)$; by the first, it follows that $K = k(y)$. Consequently each fiber X_y is either $\mathrm{Spec} k(y)$ or empty. It thus follows that the map $\mathcal{O}_Y \to \mathcal{A}$ is a surjection, so we are thus done. □

5 The generalized Chevalley theorem

5.1 Preliminary remarks

In classical algebraic geometry, one defines a subset of a variety over an algebraically closed field to be *constructible* if it is a union of locally closed subsets (in the Zariski topology). One of the basic results that one proves, which can be called "elimination theory" and is due to Chevalley, states that constructible sets are preserved under taking images: if $f : X \to Y$ is a regular map and $C \subset X$ is constructible, then so is $f(C)$. In general, this is the best one can say: even very nice subsets of X (e.g. X itself) need not have open or closed (or even locally closed) images.

In the theory of schemes, one can formulate a similar result. A morphism of finite type between noetherian schemes sends constructible sets into constructibles. One proves this result by making a sequence of reductions to considering the case of two integral affine schemes, and then using a general fact from commutative algebra.

It turns out, however, that there is a more general form of the Chevalley theorem:

Theorem 5.1. *Let $f : X \to Y$ be a finitely presented morphism of schemes. Then if $C \subset X$ is locally constructible, so is $f(C)$.*

We will explain how one deduces this more general fact from the specific case of noetherian schemes. This will highlight a useful fact: oftentimes, general facts in algebraic geometry can be reduced to the noetherian case since, for instance, every ring is an inductive limit of noetherian rings. This can be developed systematically, as is done in EGA IV-8, but we shall wait until Section **??** to see that.

5.2 Constructible sets

Nonetheless, there are several caveats. The first is that the notion of constructibility needs to be modified for general (non-noetherian) schemes. Also, we have to explain what "locally constructible" means.

So, first, let us give the modified definition of constructibility.

Definition 5.2. *If X is a topological space and $U \subset X$ an open subset, then U is called* ***retrocompact*** *if the inclusion $U \hookrightarrow X$ is a quasi-compact morphism. That is, U is retrocompact if whenever $V \subset X$ is a quasicompact open set, so is $U \cap V$.*

Classically, the algebra of constructible sets is defined to be that generated by the open sets. In general, we just work with retrocompact open sets.

Definition 5.3. *Let X be a topological space. The algebra of* ***constructible sets*** *is that generated by the retrocompact open sets. So a set is constructible if and only if it is expressible as*

$$\bigcup_{i=1}^{n} (U_i - V_i)$$

for the U_i, V_i retrocompact in X.

If X is *noetherian*, then every open set is retrocompact, so this notion agrees with the usual one for noetherian spaces.

Naturally, we can define what **locally constructible** means: a subset T is locally constructible if there is an open covering by open sets U such that $T \cap U$ is constructible in U.

5.3 First properties of constructible sets

We want to show that constructible sets behave well under morphisms of schemes. To do this, let us first show that *always,* they are closed under taking inverse images.

Proposition 5.4. *If $f : X \to Y$ is a morphism of schemes, and $T \subset Y$ is constructible, so is $f^{-1}(T)$.*

Proof. It is sufficient to show that retrocompact open sets pull back to retrocompact ones, because constructible sets are obtained from them by taking finite unions and differences.

So let $U \subset Y$ be retrocompact. This is equivalent to saying that the morphism of schemes $U \to Y$ is quasicompact. But the morphism $f^{-1}(U) \to X$ is the pull-back of $U \to Y$ via $X \to Y$, so it is also quasicompact. Thus $f^{-1}(U)$ is retrocompact in X, and we have proved the claim.

The result now implies that the same is true for *local* constructibility:

Corollary 5.5. *If $f : X \to Y$ is a morphism of schemes, and $T \subset Y$ is* ***locally*** *constructible, so is $f^{-1}(T)$.*

We are interested in when the *image* of a constructible set is constructible. To start with, we will handle the case of a quasi-compact open immersion.

Proposition 5.6. *Let $U \hookrightarrow X$ be a quasi-compact open immersion. Then the image of a constructible $C \subset U$ is constructible in X.*

Proof. This will follow at once if we show that if $V \subset U$ is retrocompact open, then V is constructible in X. Note that any constructible set is a finite union of differences of sets of this form, so it is sufficient to handle the case of a retrocompact open.

If $V \subset U$ is retrocompact, then $V \hookrightarrow U$ is quasicompact. But by assumption $U \hookrightarrow X$ is quasicompact, so the composite $V \hookrightarrow X$ is quasicompact as well. This means that V is constructible in X. ∎

Quasicompact and quasiseparated schemes are nice. In there, *every* quasicompact open set is retrocompact, and conversely. This is the definition of quasiseparatedness: that the intersection of two quasicompacts be quasicompact. So, motivated by this, we finally include a result that gives a characterization of what it means to be locally constructible.

Proposition 5.7. *If X is a quasicompact, quasiseparated scheme and $T \subset X$ is locally constructible, it is globally constructible.*

Proof. By definition, there is a finite cover of X by small neighborhoods $U_i \subset X$, which by shrinking we may take affine (since shrinking preserves constructibility thanks to Proposition 5.4), such that $T \cap U_i \subset U_i$ is constructible for each i. However, the inclusion $U_i \to X$ is quasi-compact by quasiseparatedness. Consequently, in view of Proposition 5.6, the image $T \cap U_i$ is constructible in X. It follows that the union

$$T = \bigcup T \cap U_i$$

is constructible in X. □

In particular, since any affine is quasicompact and quasiseparated, we find:

Corollary 5.8. *Let X be a scheme. Then $T \subset X$ is locally constructible if and only if $T \cap U$ is constructible in U for any open affine U.*

5.4 A converse to the Chevalley theorem

Now we are interested in proving the Chevalley theorem. First, however, to make things clear, we prove a type of converse, which will also be useful in proving the theorem.

Proposition 5.9. *If X is a quasicompact and quasiseparated scheme and $T \subset X$ a constructible subset, there is an affine scheme X' and a morphism $X' \to X$, quasicompact and of finite presentation, whose image is T.*

Recall that a morphism of schemes is of *finite presentation* if it is quasicompact, quasiseparated, and locally of finite presentation.

Proof. To do this, we start by covering X by open affine subsets $X_i, 1 \leq i \leq n$. Since X is quasicompact, we only need finitely many of them. The morphism

$$\sqcup X_i \to X$$

is quasi-compact because X is quasiseparated, and it is also surjective. Thus it is of finite presentation. Now the inverse image of T is constructible in $\sqcup X_i$ (by Proposition 5.4). As a result, by reducing to the case of the inverse image of T in $\sqcup X_i$, we may suppose that *X itself is affine.*

T is constructible, so we can write T as a finite union $\bigcup_j (U_j - V_j)$ for $U_j, V_j \subset X$ quasicompact.[10] By taking finite coproducts, we see that it is enough to handle the case

[10] The retrocompact open sets in a quasicompact, quasiseparated scheme are precisely the quasicompact ones.

of $T = U - V$ for U, V quasicompact, and to show that such a T arises as the image of an affine scheme by a morphism of finite presentation.

By assumption, each of U, V is a finite union of basic open sets X_{g_a} for each $g_a \in \Gamma(X, \mathcal{O}_X)$. By taking finite unions, as earlier, we can reduce to the case

$$U = X_g, \quad V = X_{f_1} \cup \cdots \cup X_{f_r}. \qquad \square$$

The claim is that $U - V$ arises as the image of an affine scheme, of finite presentation over X.

Suppose $X = \mathrm{Spec} R$. Then we can take the natural map $\mathrm{Spec} R_g/(f_1, \dots, f_r) \to \mathrm{Spec} R$ whose image is $U - V$; moreover, it is immediate that this is of finite presentation, from the explicit form $R_g/(f_1, \dots, f_r) = R[X]/(1 - gX, f_1, \dots, f_r)$.

5.5 The proof of the Chevalley theorem

Finally, we are now ready to prove the general Chevalley theorem on constructible sets:

Theorem 5.10 (Chevalley). *Let $f : X \to Y$ be a quasicompact morphism, locally of finite presentation. Then if $C \subset X$ is locally constructible, so is $f(C) \subset Y$.*

Interestingly, it turns out that Grothendieck does not (in EGA IV-1) state the theorem in maximal generality. Grothendieck assumes that f is of finite presentation, so he assumes in addition that f is quasiseparated. The more general form of the theorem is in the [dJea10].

So let us prove this fact.

5.5.1 Step 1: Reduction to X, Y affine

The first step is to make some reductions. Since we are trying to prove an assertion which is local on Y, we may assume that Y is itself affine, and consequently that X is quasi-compact. Let us now keep the assumption that Y is affine.

It follows that X is a finite union $\bigcup X_i$, where each X_i is affine (and thus quasicompact and quasiseparated). In particular, $C \cap X_i$ is not only locally constructible, but constructible, by Corollary 5.8. The morphisms $X_i \to X \to Y$ are quasicompact for each i as Y, X_i are both affine, and they are locally of finite presentation.

If we show (assuming Y affine, as usual) that $f(C \cap X_i)$ is constructible for each i, then we will be done. The upshot is that we can reduce to the case of Y affine *and* X affine. In particular, we need to prove:

Lemma 5.11. *Let $f : X \to Y$ be a morphism of affine schemes which is of finite presentation. If $C \subset X$ is constructible, so is $f(C)$.*

But in fact we can make a further reduction. We know that there is an affine scheme X' and a morphism of finite presentation $X' \to X$ such that C is the image of X', by Proposition 5.9. So we are reduced to something even more concrete:

Lemma 5.12. *Let $f : X \to Y$ be a morphism of affine schemes which is of finite presentation. Then $f(X)$ is constructible.*

5.5.2 Step 2: Reduction to the noetherian case

Finally, it is here that we introduce the "finite presentation" trick most clearly. Consider a map

$$\mathrm{Spec} S \to \mathrm{Spec} R$$

coming from a finitely presented morphism of rings

$$R \to S.$$

The claim is that the image $\mathrm{Spec} S$ in $\mathrm{Spec} R$ is constructible.

Now the key observation is that we *know* this, when R is noetherian, because then it is the usual Chevalley theorem:

Theorem 5.13 (Chevalley). *Let $f : X \to Y$ be a morphism of finite type between noetherian schemes. Then if $C \subset X$ is constructible, so is $f(C)$.*

We shall not give the proof of this result. (Reference to be added.)

We are not necessarily working with noetherian rings. However, in general, R is a filtered colimit of noetherian rings: namely, of its finitely generated $\mathbb{Z}$-subalgebras. The claim is that we can reduce to this case by:

Lemma 5.14. *Let $R \to S$ be a finitely presented morphism of rings. Then there is a finitely generated subring $R' \subset R$ and a morphism of finite presentation $R' \to S'$ such that $R \to S$ is a base-change of $R' \to S'$.*

So visually, this looks like a cocartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \uparrow & & \uparrow \\ R' & \longrightarrow & S' \end{array} .$$

Let us assume this lemma for the moment and see how it implies the big theorem. Recall that our goal is to show that the image of $\mathrm{Spec} S \to \mathrm{Spec} R$ is constructible. But we have a cartesian diagram of schemes:

$$\begin{array}{ccc} \mathrm{Spec} S & \longrightarrow & \mathrm{Spec} R \\ \downarrow & & \downarrow \\ \mathrm{Spec} S' & \longrightarrow & \mathrm{Spec} R' \end{array} .$$

Consequently, we know that the image of $\mathrm{Spec} S \to \mathrm{Spec} R$ is the preimage of the image of $\mathrm{Spec} S' \to \mathrm{Spec} R'$, by basic properties of fibered products. But $\mathrm{Spec} S' \to \mathrm{Spec} R'$ has constructible image by noetherianness, so the same is true for the top map, as inverse images preserve constructibility. We are thus done, modulo the lemma.

5.5.3 Step 3: the finite presentation lemma

Finally, we are left with the lemma, Lemma 5.14. This lemma is the crucial step in the above approach, and is one of the reasons that "finite presentation" results can often be reduced to noetherian ones. There are more results on this theme, which essentially boil down to the idea that in the category of rings, finitely presented objects are *compact.*

So we have a finitely presented morphism $R \to S$. This means that we can write $S = R[x_1, \dots, x_n]/(f_1, \dots, f_k)$, so S is a quotient of a finitely generated polynomial ring over S by a finitely generated ideal. However, the coefficients in the finitely many f_i all lie in some finitely generated subring R'. This means that the ring

$$S' = R'[x_1, \dots, x_n]/(f_1, \dots, f_k)$$

makes sense, and is a finitely presented R'-algebra. However, it is also easy to see that $R \to S$ is a base-change of this. This proves the lemma.

5.6 Pro-constructible and ind-constructible sets

Finally, we would like to generalize the above results to the case where the morphism $X \to Y$ is not even assumed to be of finite presentation. In this case, we can say much less in general. We will now have to introduce a looser class of sets that will be preserved under images.

Definition 5.15. *Let X be a topological space, and $T \subset X$.*

1. *T is called* ***pro-constructible*** *if it locally an intersection of constructible sets.*
2. *T is called* ***ind-constructible*** *if it locally a union of constructible sets.*

So in the first case, if T is pro-constructible, then each $x \in X$ has a neighborhood U such that $U \cap T$ is an intersection of sets constructible in U. The other case is similar, with unions replacing intersections. The "generic point" of the affine line $\mathbb{A}^1$ is an example of a pro-constructible set which is not constructible (and not ind-constructible).

We would now like to show that pro- and ind-constructible sets behave reasonably well with respect to morphisms of schemes. First, we handle the case of inverse images.

Proposition 5.16. *Let $f : X \to Y$ be a morphism of schemes, and let $T \subset Y$ be pro-constructible (resp. ind-constructible). Then $f^{-1}(T)$ is pro-constructible (resp. ind-constructible).*

Proof. We handle the first case; the second is similar. It also follows from the simple observation that a set is pro-constructible if and only if its complement is ind-constructible.

Fix $x \in X$. There is a neighborhood $U \subset Y$ containing $f(x)$ such that $T \cap U$ can be written as $\bigcap T_\alpha$ for the $T_\alpha \subset U$ constructible. Now consider $f^{-1}(T \cap U) \subset f^{-1}(U)$; it follows that

$$f^{-1}(T \cap U) = \bigcap f^{-1}(T_\alpha),$$

where each $f^{-1}(T_\alpha)$ is constructible in $f^{-1}(U)$ by Proposition 5.4. □

This is reasonable and expected. With it, we want to get a slightly better characterization of pro- and ind-constructible sets. Namely, the condition of locality in the definition is slightly annoying, and we want to get a better estimate. We will now see that any open affine will do for those "neighborhoods" in the definition.

Proposition 5.17. *Let X be a quasi-compact, quasi-separated scheme. Then $T \subset X$ is pro-constructible (resp. ind-constructible) in X if and only if it is a intersection (resp. union) of constructible sets.*

Proof. We will prove the result for pro-constructible sets; it is clear that this is sufficient by taking complements.

It is clear that intersections of constructible sets are pro-constructible. Conversely, let $T \subset X$ be pro-constructible. Then there is a finite open cover $\{U_\alpha\}$ of X such that $T \cap U_\alpha$ is an intersection of constructibles $\bigcap T_\alpha^\beta$ (for the $\left\{T_\alpha^\beta\right\}$ constructible subsets of U_α) for each α.

By shrinking (thanks to Proposition 5.16), we may assume that each U_α is affine, so that the morphism $U_\alpha \to X$ is quasicompact. In particular, each T_α^β is constructible in X. We now need to write T as an intersection of globally constructible sets. Let the α's indexing the open covering $\{U_\alpha\}$ range from 1 to n, and suppose that for each $\alpha \in [1, n]$, the relevant β's (yielding the T_α^β) range over a set B_α.

The claim is that

$$T = \bigcap_{\beta_1 \in B_1, \dots \beta_n \in B_n} T_1^{\beta_1} \cup \dots \cup T_n^{\beta_n}, \tag{6}$$

where the intersection runs over all sequences $\beta_1, \dots, \beta_n$. If this is proved, then it is clear that T is an intersection of constructible sets.

This is easy to check. If $x \in T$, then x belongs to some U_α; it follows that for this α, and *every* $\beta \in B_\alpha$, we have $x \in T_\alpha^\beta$. So x lies in the big intersection in Eq. (6). Conversely, suppose $x \notin T$. Then for each α, there is $\beta_\alpha \in B_\alpha$ such that $x \notin T_\alpha^{\beta_\alpha}$. It follows that

$$x \notin B_1^{\beta_1} \cup \dots \cup B_n^{\beta_n}, \qquad \square$$

so it does not lie in the left-hand-side of Eq. (6).

As we saw for constructible sets, it is now clear that:

Corollary 5.18. *Let X be a scheme. Then $T \subset X$ is pro-constructible (resp. ind-constructible) if and only if for every open affine $U \subset X$, we have that $T \cap U$ is an intersection (resp. union) of constructible subsets of U.*

5.7 A characterization of pro-constructible sets

We now want to imitate the arguments in the previous sections to show that pro-constructible sets are closed under push-forwards under reasonable circumstances. To start with, as before, we prove the converse: we can always get a pro-constructible set as an image of a quasi-compact morphism.

Lemma 5.19. *If $U \hookrightarrow X$ is a quasicompact open immersion, then the image of a proconstructible set in U is proconstructible in X.*

Proof. We may assume that X is affine, and thus U quasicompact and quasiseparated, by the local nature of proconstructibility. In this case, any constructible set $T \subset U$ is an intersection $T = \bigcap T_\alpha$ of constructible subsets $\{T_\alpha\}$ in U. However, each of these $\{T_\alpha\}$ are constructible in X by Proposition 3.8, so we are done. $\square$

Proposition 5.20. *Let X be a quasicompact, quasiseparated scheme and $T \subset X$ proconstructible. Then there is an affine scheme X' and a quasicompact morphism $f : X' \to X$ such that $T = f(X')$.*

This result is thus analogous to Proposition 5.9.

Proof. Indeed, as usual we may reduce to X affine, for we can cover X with finitely many open affines $\{X_i\}$. If we prove the result for each set in this open cover, then we need only take the coproduct of the various X''s that occur. Note that the morphism $\sqcup X_i \to X$ is quasicompact by quasiseparatedness. So we may just assume at the outset that X is affine.

In this case, we know (Proposition 5.17) that T is an intersection $\bigcap T_\alpha$ of constructible sets T_α. We also know that each T_α occurs as the image of an affine scheme $\mathrm{Spec} R_\alpha$ via a morphism

$$\mathrm{Spec} R_\alpha \to X,$$

which is evidently quasi-compact, thanks to Proposition 5.9, which is the counterpart of the present result for constructible sets. We need to make the *intersection* of all the R_α's occur as an image.

To do this, we first note that any *finite* intersection of the T_α's occurs as an image in a natural way. Say $X = \mathrm{Spec} R$; then for every finite set F, consider the morphism

$$g_F : \mathrm{Spec} \bigotimes_{\alpha \in F} R_\alpha \to \mathrm{Spec} R.$$

Here $\mathrm{Spec} \bigotimes_{\alpha \in F} R_\alpha$ is the fibered product of the maps $\mathrm{Spec} R_\alpha \to \mathrm{Spec} R$ for $\alpha \in F$. Consequently, the image of g_F is $\bigcap_F T_\alpha$.

We now want to move from getting finite intersections to the whole intersection. However, the g_F's form a *projective system* (where F ranges over finite subsets of the indexing set of α), and it turns out that in such instances we can always do it. We formalize this in the following lemma.

Lemma 5.21. *Let R be a ring and $R_\beta, \beta \in B$ be a inductive system of R-algebras over a directed set B. There is consequently a projective system $\mathrm{Spec} R_\beta$ of schemes.*

Then the natural morphism $R \to \varinjlim R_\beta$ induces a morphism

$$\mathrm{Spec} \varinjlim R_\beta \to \mathrm{Spec} R$$

whose image is the intersection of the images of all the $\mathrm{Spec} R_\beta$.

Once we prove this lemma, the proposition will be clear.

Proof. First, the morphism $\mathrm{Spec} \varinjlim R_\beta \to \mathrm{Spec}R$ factors through each morphism $\mathrm{Spec}R_\beta \to \mathrm{Spec}R$, so the image is clearly contained in the intersection of the images $\mathrm{im}(\mathrm{Spec}R_\beta \to \mathrm{Spec}R)$. We need to show the converse inclusion.

Suppose $y \in \mathrm{Spec}R$ occurs in the image of $\mathrm{Spec}R_\beta \to \mathrm{Spec}R$ for all β. We have to show that there is $y' \in \mathrm{Spec} \varinjlim R_\beta$ mapping to y. In other words, if $k(y)$ is the quotient field, we must show that the fiber $\mathrm{Spec} \varinjlim R_\beta \otimes_R k(y)$ is nontrivial.

This, however, is equivalent to the statement that $\varinjlim R_\beta \otimes_R k(y) \neq 0$. Since colimits commute with tensor products, this in turn is equivalent to $\varinjlim (R_\beta \otimes_R k(y)) \neq 0$. We know that $R_\beta \otimes_R k(y) \neq 0$ for each β. But a filtered colimit of nonzero rings $\{S_\alpha\}$ can never be the zero ring. For if $1 = 0$ in $\varinjlim S_\alpha$ then the definition of a filtered colimit shows that $1 = 0$ in some S_γ. This shows that $\mathrm{Spec} \varinjlim R_\beta \otimes_R k(y) \neq 0$ and proves the lemma.□

6 Faithfully flat morphisms

Recall that a morphism of schemes $f : X \to Y$ is called *flat* if for each $x \in X$, $\mathcal{O}_x$ is a flat algebra over $\mathcal{O}_{f(x)}$.

Definition 6.1. *f is called **faithfully flat** if it is flat and surjective.*

It is straightforward to check (EGA IV.2) that:

1. Faithfully flat morphisms are closed under base-change and composition.
2. A morphism of affine schemes $\mathrm{Spec}S \to \mathrm{Spec}R$ is faithfully flat if and only if $R \to S$ is faithfully flat.

We omit the proofs of these facts, and focus on proving the following principle:

> **Principle:** Let $f : X \to Y$ be a morphism of S-schemes for some base scheme S. Suppose $S' \to S$ is a faithfully flat and quasi-compact morphism and $f_{S'} = f \times_S S' : X \times_S S' \to Y \times_S S'$ has some property. Then f does too.

Let us introduce the notation $X_{S'} = X \times_S S', Y_{S'} = Y \times_S S'$.

This is intended to be a heuristic principle, which is not valid in *every* case,[11] but it is valid in many interesting ones. In general, it will always be true that if f satisfies a property, so does $f \times_S S'$, even without faithfully flat hypothesis. The converse is the interesting part. There are many properties of morphisms in which such a "descent" is possible:

1. Finite
2. Proper
3. Separated
4. Open (resp. closed, resp. locally closed) immersions

[11]For instance, it is not valid for the property of being a projective morphism.

6.1 Fpqc morphisms induce the quotient topology

Morphisms of schemes which are simultaneously faithfully flat and quasi-compact (which we call **fpqc**[12]) are epimorphisms in the category of schemes in a very strong sense. We shall not enter into the theory of descent (explained in [Vis08], for instance), whereby given a fpqc morphism $X' \to X$ quasicoherent sheaves on X' with suitable "descent data" lead to quasicoherent sheaves on X. However, we shall at least show that such fpqc morphisms induce the quotient topology.

Theorem 6.2. *Let $f : X' \to X$ be a fpqc morphism of schemes. Let $Z \subset X$ be a subset. Then Z is closed if and only if $f^{-1}(Z)$ is closed.*

Proof. One direction is obvious, f being continuous. Let us prove the other direction.

By base-changing by the morphism $\overline{Z} \to X$ (where $\overline{Z}$ is given the reduced induced subscheme structure), we may assume that Z is *dense* in X. In particular, we have to prove that if Z is *dense* in X and $f^{-1}(Z)$ is closed, then Z is all of X. It will suffice to show that $f^{-1}(Z)$ is dense in X', for then it will be equal to X'.

Suppose $Z \subset X$ is dense and is such that $f^{-1}(Z)$ is closed. Then $f^{-1}(Z)$ is proconstructible (by **??**), so $Z = f(f^{-1}(Z))$ is proconstructible as f is quasi-compact (by **??**). We want to show that $f^{-1}(Z)$ is dense in X'.

Thus we are reduced to proving:

Lemma 6.3. *Let $f : X' \to X$ be a fpqc morphism. Suppose Z is a proconstructible dense subset. Then $f^{-1}(Z)$ is dense.*

Proof. The result is local on X, so we may assume X affine; X' is then quasi-compact. We know that there is an affine scheme Y and a morphism $Y \to X$ whose image is Z. We are going to show that $Y \times_X X' \to X'$ has dense image. It suffices to show that every maximal point is in the image. So let ξ be a maximal point in X'. To show that ξ is in the image of $Y \times_X X' \to X$, it suffices to show that $f(\xi)$ is in the image of $Y \to X$, i.e. is in T.

But f is flat, so f sends maximal points into maximal points as it is dominant. Thus $f(\xi)$ is a maximal point in X. In addition, as $Y \to X$ is quasi-compact and has dense image, every maximal point of Y is the image of a point in Y, so every maximal point in Y lies in T. This shows that $f(\xi)$ is in T. This concludes the proof. □

After this, the following preliminary "descent" result will be straightforward.

Proposition 6.4. *Let $f : X \to Y$ be a morphism of S-schemes, and let $S' \to S$ be a fpqc morphism. Then f is open (resp. closed) if $f_{S'}$ is.*

Note that this is technically an exception to the philosophy outlined earlier, in that the implication is only one-way! The problem is that, unlike most properties of interest in algebraic geometry, being open (or closed) is not generally stable under base-change.

[12] Short for fidèlement plat et quasi-compact.

In proving such a result, we will *always reduce to* $Y = S$. We can do this, because in general the diagram

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y \times_S S' & \longrightarrow & Y \end{array}$$

is cartesian, and $Y \times_S S' \to Y$ is fpqc (as a base-change of $S' \to S$). In short, we can always assume the base-change happens with f going to the "base." We will make this assumption in all such future arguments, and the preceding discussion shows that it is reasonable.

Proof. As usual, we may assume $Y = S$. If f is universally open or universally closed, any base change, whether fpqc or not, clearly has the same property.

Suppose $f_{S'}$ is open. In the diagram

$$\begin{array}{ccc} X_{S'} & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f_{S'}} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array},$$

where we have named the horizontal morphisms for convenience, fix an open subset $U \subset X$. The claim is that $f(U)$ is open. For this, it suffices to show that $g^{-1}(f(U))$ is open. However, by basic properties of fibered products, we know that (EGA I.??)

$$g^{-1}(f(U)) = f_{S'}(g'^{-1}(U)),$$

and the latter is open because $f_{S'}$ is open. Since U was arbitrary, we find that f is open.

The proof for closed morphisms is the same. ☐

6.2 Examples of descent

Armed with the initial result (Theorem **??**) on fpqc morphisms, we shall now prove that wide varieties of properties of morphisms can "descend" under fpqc base change.

Proposition 6.5. *Let $f : X \to Y$ be a morphism of S-schemes, and let $S' \to S$ be a fpqc morphism. Then f is separated if and only if $f_{S'}$ is.*

Proof. As usual, we may assume $Y = S$.

Separatedness is preserved under base-change, so one direction is clear. Suppose now that $f_{S'}$ is separated. We need to see that f is separated.

Consider the diagonal map

$$\Delta : X \to X \times_S X$$

and its base-change

$$\Delta_{S'} : X_{S'} \to X_{S'} \times_{S'} X_{S'};$$

by assumption, the latter is closed (indeed, a closed immersion). It follows that Δ itself is closed, so f is separated. ☐

6.3 Immersions

With the above result in mind, we shall show that the notion of "immersion" descends under faithfully flat base change.

Proposition 6.6. *Let $f : X \to Y$ be a morphism of S-schemes, and let $S' \to S$ be a fpqc morphism. Then f is an immersion (resp. an open immersion, a closed immersion) if and only if $f_{S'}$ is.*

7 The finite presentation argument

Following EGA IV-8, we shall now discuss a general argument that enables one to make reductions about statements about finitely presented morphisms to the case of noetherian schemes.

The essential idea of the approach is the following.

(TO BE ADDED)

8 The quasi-finite form of ZMT

8.1 A lemma on complete local rings

Paradoxically, a key lemma we shall use in proving ZMT is the following:

Lemma 8.1. *Let A be a noetherian local complete ring, with $y \in \mathrm{Spec} A$ denoting the maximal ideal. Let $X \to \mathrm{Spec} A$ be a quasi-finite separated morphism. Then there is a decomposition $X = X_0 \sqcup X_1$ where X_0 is finite over $\mathrm{Spec} A$ and the image of X_1 does not contain y.*

Proof. Suppose we have $f^{-1}(y) = \{x_1, \dots, x_n\}$. We pick one of the points $x \in f^{-1}(y)$ and will induct on n. As always, there is a morphism

$$\mathrm{Spec}\mathcal{O}_x \to X,$$

which is an imbedding on the underlying topological spaces. By assumption, f is quasi-finite, so that $f^{-1}(y)$ is quasi-finite over $k(y)$; it is thus a discrete set.[13] In particular, in the composite

$$\mathrm{Spec}\mathcal{O}_x \to \mathrm{Spec} A,$$

the fiber above y consists just of x, as x is isolated in its f-fiber.

The claim is that $\mathcal{O}_x$ is a finite A-module. For the moment, let us assume this. In that case, the composite

$$\mathrm{Spec}\mathcal{O}_x \to X \to \mathrm{Spec} A$$

is finite, and since $X \to \mathrm{Spec} A$ is separated, it follows by standard cancellation arguments (cf. []), that $\mathrm{Spec}\mathcal{O}_x \to X$ is finite. It is clear that the map induces an isomorphism on the local rings at x. Given a morphism of finite type that induces an isomorphism of a pair of local rings, it is a local isomorphism by straightforward arguments, cf. [].

[13] Proof?

It follows that $\mathrm{Spec}\mathcal{O}_x \to X$ is a local isomorphism near x, so maps $\mathrm{Spec}\mathcal{O}_x$ isomorphically onto an open neighborhood of X because any open set containing the maximal ideal in $\mathrm{Spec}\mathcal{O}_x$ contains the whole scheme. Since this map is proper, it is closed, and we find that $X = X' \sqcup X''$ where $X' = \mathrm{Spec}\mathcal{O}_x$ is finite over $\mathrm{Spec}A$ and X'' has one fewer point in the fiber over y. We can thus repeat this procedure to split of chunks of X'' until nothing in the fiber over y is left; this must terminate as X is noetherian. Thus, modulo the claim about $\mathcal{O}_x$, we are done.

We now prove the claim. Let $\mathfrak{m} \subset A$ be the maximal ideal (corresponding to the point $y \in \mathrm{Spec}A$), and let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal. By assumption, $\mathfrak{m}$ is mapped into $\mathfrak{m}_x$ under the map $A \to \mathcal{O}_x$.

We know that $\mathcal{O}_x$ is noetherian, because $X \to \mathrm{Spec}A$ is of finite type, and consequently by the Krull intersection theorem, we have

$$\bigcap \mathfrak{m}^n \mathcal{O}_x \subset \bigcap \mathfrak{m}_x^n \mathcal{O}_x = \{0\}.$$

The following bit of commutative algebra will imply that $\mathcal{O}_x$ is finite over A.

Lemma 8.2. *Let A be a complete local ring with maximal ideal $\mathfrak{m}$, M an A-module. Suppose $M/\mathfrak{m}M$ is finitely generated and $\bigcap \mathfrak{m}^n M = 0$. Then M is finitely generated.*

8.2 Quasi-finite proper morphisms

The argument for Zariski's Main Theorem is still somewhat complex, so we "warm up" to it by proving:

Theorem 8.3. *A proper quasi-finite and finitely presented morphism $f : X \to Y$ is finite.*

We have already proved it in the noetherian case, but here we give a new proof of that too, one independent of cohomology.

Proof. We shall make a series of reductions.

8.2.1 Step 1: Reduce to Y affine

The properties of properness, quasi-finiteness, and finite presentation are all preserved under base change. Moreover, they are all local on the base. So we can assume Y affine.

8.2.2 Step 2: Reduce to Y affine and noetherian

Let us suppose $Y = \mathrm{Spec}A$, and $X \to A$ is finitely presented. We know that A is the direct limit of its finitely generated subrings. We know, furthermore, that the scheme X "descends" to a finitely generated subring: that is, there exists $A_0 \subset A$ finitely generated, and a finite type map $X_0 \to \mathrm{Spec}A_0$ such that $X \to \mathrm{Spec}A$ is the base-change of $X_0 \to \mathrm{Spec}A_0$.

By choosing A_0 large enough, we can moreover assume (**??**), that $X_0 \to \mathrm{Spec}A_0$ is quasi-finite and proper. If we show that $X_0 \to \mathrm{Spec}A_0$ is finite, then the same will be true for X as finiteness is preserved under base change.

So it is sufficient to prove the theorem for $X_0 \to \mathrm{Spec}A_0$. In particular, we can reduce to the noetherian case. Here we could just quote Theorem 3.11, but we prefer not to, as it will illustrate important ideas.

8.2.3 Step 3: Reduction to Y affine, noetherian, and local

We have seen that it is sufficient to prove the theorem for Y of the form $\mathrm{Spec}A$, with A noetherian. We now want to go further and assume A local noetherian.

To do this, recall the discussion in **??**. We know that it is sufficient to find an open cover $\{U_\alpha\}$ of Y such that $f^{-1}(U_\alpha) \to U_\alpha$ is finite for each α. That is, given $y \in Y$, we must find a small neighborhood U of y—which we may assume affine—such that $f^{-1}(U) \to U$ is finite.

However, we have seen that the local scheme $\mathrm{Spec}\mathcal{O}_y$ is the projective limit of the small open affine neighborhoods of y. In particular, if we show that

$$X \times_Y \mathrm{Spec}\mathcal{O}_y \to \mathrm{Spec}\mathcal{O}_y$$

is finite, we will see that there is such an open neighborhood U. Since $\mathcal{O}_y$ is local noetherian, we see it is enough to handle that case.

8.2.4 Step 4: Reduction to Y affine, noetherian, local, and complete

We have now seen that it is sufficient to show that if $f : X \to Y$ is a quasi-finite proper morphism with $Y = \mathrm{Spec}A$, for A noetherian local, then f is finite. We now claim that it is sufficient to do the same when A is, moreover, *complete*. This will be the final reduction, after which we will be able to apply Lemma 8.1.

Let $\hat{A}$ be the completion of A at its maximal ideal. This is a noetherian local ring, and we have a faithfully flat morphism $A \to \hat{A}$ by elementary commutative algebra (cf. []). In the cartesian diagram

$$\begin{array}{ccc} X \times_A \hat{A} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}\hat{A} & \longrightarrow & \mathrm{Spec}A \end{array} ,$$

we see that it is sufficient to show that the proper quasi-finite morphism $X \times_A \hat{A} \to \mathrm{Spec}\hat{A}$ is finite by "descent," as in Proposition **??**.

8.2.5 Step 5: Completion of the argument

After this long series of reductions, we have found that it is enough to prove that if $f : X \to Y$ is a quasi-finite proper morphism for $Y = \mathrm{Spec}A$, with A noetherian, local, and complete, then f is finite.

This is now easy. We know (Lemma 8.1) that $X = X_0 \sqcup X_1$ where $X_0 \to \mathrm{Spec}A$ is finite and the image of X_1 does not meet the closed point of $\mathrm{Spec}A$. But $X_1 \to X$ is a closed immersion, so $X_1 \to \mathrm{Spec}A$ is proper, and in particular closed; if its image does not meet the only closed point, then we find that $X_1 = \emptyset$.

Thus $X = X_0$ is finite over $\mathrm{Spec}A$. □

8.3 Compactifiability of quasi-finite morphisms

We shall now prove the following basic result.

Theorem 8.4. *A quasi-finite morphism of finite presentation is quasi-affine.*

Recall that a morphism of schemes $f : X \to Y$ is **quasi-affine** if $\mathcal{O}_X$ is an f-ample line bundle. This is equivalent to saying that f factors as an open immersion and an affine morphism.

Proof. Let $f : X \to Y$ be a quasi-finite morphism of finite presentation. We begin with the usual steps.

It suffices to assume Y affine, because all properties here (quasi-finiteness, morphisms of finite presentation, quasi-affineness) are local on the base.[14] By the same "finite presentation" argument as before, we may reduce to the case that Y is of finite type over $\mathbb{Z}$, and in particular of *finite dimension.* So we are reduced to proving the result in the case of noetherian schemes of finite dimension.

Let us now induct on $\dim Y$. When $\dim Y = 0$, the result is trivial. Suppose $\dim Y > 0$ and the result proved for noetherian schemes Y' with $\dim Y' < \dim Y$. We will show for each $y \in Y$, the morphism $X \times_Y \mathrm{Spec}\mathcal{O}_y \to \mathrm{Spec}\mathcal{O}_y$ is quasi-affine. This will imply, by the usual argument, that each point $y \in Y$ has a small affine neighborhood U_y such that $X \times_Y U_y \to U_y$ is quasi-affine. Since quasi-affineness is local on the base, this will prove that $X \to Y$ is quasi-affine.

So, let $Y = \mathrm{Spec}A$, for A a local noetherian ring, and assume that for all Y' noetherian of *finite dimension* (not necessarily affine), the result is true. We want to show that $X \to Y$ is quasi-affine. By making the fpqc base change $\mathrm{Spec}\hat{A} \to \mathrm{Spec}A$, we may assume that A is in fact *complete.* (Recall that completion does not affect the dimension, by Hilbert polynomial arguments, cf [Eis95].)

But then $X \to \mathrm{Spec}A$ is a quasi-finite morphism to the Spec of a complete local noetherian ring. Thus, we can apply Lemma 8.1. We find that X decomposes as $X = X_1 \sqcup X_2$ where X_1 is finite over $\mathrm{Spec}A$ and the image of X_2 does not intersect the maximal ideal of $\hat{A}$.

Then $X_1 \to \mathrm{Spec}A$ is finite, hence affine, so it is clearly quasi-affine. The map $X_2 \to \mathrm{Spec}A - \{y\}$ for y the maximal ideal in $\mathrm{Spec}A$ is also quasi-finite, but the dimension of the space $\mathrm{Spec}A - \{y\}$ is strictly less than that of $\mathrm{Spec}A$. By the inductive hypothesis, we find that $X_2 \to \mathrm{Spec}A$ is quasi-affine. Piecing both parts together, $X \to \mathrm{Spec}A$ is quasi-affine. □

One real bit of cleverness in the above argument—which did not appear in the proof that a proper quasi-finite morphism is finite—is the induction on the dimension of Y (after the reduction to the noetherian case!). This is a geometric, and not an algebraic argument.

8.4 The final version of Zariski's Main Theorem

We can now state:

Theorem 8.5 (Zariski's Main Theorem, final version). *A quasi-finite morphism between noetherian schemes factors as an open immersion and a finite morphism.*

[14] I don't think quasi-projectiveness is!

Proof. By Theorem 3.10 (that is, the cohomological theory), this is true for quasiprojective quasi-finite morphisms. However, we have just seen in Theorem 8.4 that any quasi-finite morphism of finite presentation (and since we are in the noetherian case, this applies to us) is quasi-affine, hence quasi-projective. Thus, we are done. □

We have not stated Zariski's Main Theorem in the greatest possible generality (in which it is true for finitely presented morphisms between possibly non-noetherian schemes). We have avoided doing so because the argument in [GD], though mostly similar in spirit to the arguments here, involves a technical fact about excellent rings and integral closure, that we have not entered into here.

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