DESCENT AND NILPOTENCE IN ALGEBRAIC K-THEORY

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Abstract. Let $A \to B$ be a G-Galois extension of ring spectra in the sense of Rognes. One can ask how close the map $K(A) \to K(B)^{hG}$ is to being an equivalence, i.e., how close algebraic K-theory is to satisfying Galois descent. In joint work with Clausen, Naumann, and Noel, we prove that this map is often an equivalence after telescopic localization. The purpose of these lectures is to outline some of the context of this problem and the ingredients that go into our results.

In the first lecture, we discuss the descent problem broadly and its origin in the classical work of Thomason. In the second lecture, we review the classical Dress induction theorem. We discuss the formalism of G-spectra or spectral Mackey functors, and then explain a homotopical version of the induction theorem. We explore the resulting idea of "derived defect bases" in the third lecture and in particular give a "rational to L_n-local" transfer principle for understanding the derived defect base. In the final lecture, we explain how these ideas can be applied to various cases of the descent problem in algebraic K-theory.

Contents

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1. The descent problem

The purpose of these lectures is to describe joint work with Clausen, Naumann, and Noel on descent theorems in the telescopically localized algebraic K-theory of structured ring spectra and derived schemes. Along the way, we will highlight some ideas in equivariant stable homotopy theory. These talks will primarily draw on material from the following three preprints:

- Descent in algebraic K-theory and a conjecture of Ausoni-Rognes [12], by D. Clausen, A. Mathew, N. Naumann, and J. Noel. arXiv:1606.03328.
- Nilpotence and descent in equivariant stable homotopy theory [21], by A. Mathew, N. Naumann, and J. Noel. arXiv:1507.06869.
- Derived induction and restriction theory [20], by A. Mathew, N. Naumann, and J. Noel. arXiv:1507.06867.

Further background on the theory of "nilpotence" used in this work can be found in the following two papers.

- The Galois group of a stable homotopy theory [19], by A. Mathew. arXiv:1404.2156.
- A thick subcategory theorem for modules over certain ring spectra [18], by A. Mathew. arXiv:1311.3940.

Our work is inspired by the classical work of Thomason [35] and Thomason-Trobaugh [36], and we begin by reviewing the context.

1.1. Context.

Let X be a (suitably nice) scheme, for example a quasi-projective variety over $\mathbb{C}$. A basic invariant of X is the Grothendieck group $K_0(X)$ of perfect complexes on X (for appropriate X, one can work with vector bundles on X).

$K_0(X)$ refines to a more sophisticated invariant. To X, we can associate the *algebraic K-theory* spectrum $K(X)$, such that $\pi_0 K(X) \simeq K_0(X)$. The association $X \mapsto K(X)$ defines a contravariant functor from schemes to spectra, introduced by Quillen [29] and Thomason-Trobaugh [36]. We have groups $K_i(X) = \pi_i K(X)$ for $i \geq 0$, which are more mysterious in general. There is, nonetheless, an analogy with algebraic topology that might give one hope for computing these groups.

We should think of the algebraic K-theory of a *scheme* as somehow analogous to *topological* K-theory KU (of complex topological vector bundles), which defines a cohomology theory on the category of topological spaces [2]. Given a topological space Y, the topological K-theory $KU^*(Y)$ can be computed via a "local-to-global" principle, i.e., the Atiyah-Hirzebruch spectral sequence

$$H^*(Y;\mathbb{Z}) \implies KU^*(Y).$$

Remark 1.1. We explain briefly why the AHSS can be thought of as a "local-to-global" principle. Let Y be a compact manifold. To obtain the Atiyah-Hirzebruch spectral sequence for $KU^*(Y)$, choose a finite open cover $Y = U_1 \cup \cdots \cup U_n$ such that any nonempty finite intersection of the U_i's is either empty or contractible (e.g., use geodesically convex open subsets). One can compute $KU^*(Y)$ using the Mayer-Vietoris spectral sequence for this cover; this yields the AHSS.

One can ask to what extent *algebraic* K-theory of schemes behaves like a cohomology theory, and how computable it is. Algebraic K-theory satisfies a local-to-global principle for the Zariski (or even Nisnevich) topology: this amounts to a sort of Mayer-Vietoris property. We have the classical localization theorem of Thomason-Trobaugh. In the regular case, this result is due to Quillen [29].

Theorem 1.2 (Thomason-Trobaugh [36], Quillen [29]). *Let X be a quasi-compact and quasi-separated scheme and let $U, V \subset X$ be quasi-compact open subsets that cover X. Then the diagram of connective spectra*

$$\begin{array}{ccc} K(X) & \longrightarrow & K(U) \\ \downarrow & & \downarrow \\ K(V) & \longrightarrow & K(U \cap V) \end{array}$$

is homotopy cartesian in $\mathrm{Sp}_{\geq 0}$.

Remark 1.3. In fancier language, algebraic K-theory is a sheaf of connective spectra on the Zariski site of X.

The above result enables one to obtain an analog of the Mayer-Vietoris sequence in algebraic K-theory. One might draw further hope from the rigidity theorem of Suslin [32] identifying the l-adic completion $\widehat{K(F)}_l$ for any algebraically closed field F of characteristic $\neq l$ via

$$\widehat{K(F)}_l \simeq \widehat{ku}_l.$$

Rings and schemes are unlike topological spaces, however, in the following sense: a one-point scheme (e.g., the spectrum of a field) may be arithmetically very interesting, and the K-theory of fields is quite complicated. Stated another way, the Zariski topology is too coarse to relate well to algebraic topology. The appropriate replacement is the *étale topology* on a scheme.

Algebraic K-theory does *not*, however, satisfy the local-to-global principle for étale covers, e.g., Galois extensions. In particular, if one has a G-Galois extension of fields $E \subset E'$, one obtains a natural map

$$K(E) \to K(E')^{hG} \tag{1}$$

and this *need not* be an equivalence of spectra.[1] Although one has Galois descent for vector spaces and modules, that procedure cannot be taken inside algebraic K-theory. From this point of view, algebraic K-theory doesn't behave like a cohomology theory: it is not a sheaf for the étale topology.

These lectures will focus on variants of (1) and how close these maps are to being equivalences.

1.2. **Baby Thomason.** A basic observation here is that when one rationalizes, the problem goes away.

Proposition 1.4 ("Baby Thomason" [35, Th. 2.15])**.** *Let $E \subset E'$ be a G-Galois extension of fields. Then the natural map $K(E)_{\mathbb{Q}} \to (K(E')_{\mathbb{Q}})^{hG}$ is an equivalence.*

Note that the homotopy fixed points of a rational spectrum are computed at the level of homotopy groups: that is, if X is a rational spectrum with a G-action, we have $\pi_* X^{hG} \simeq (\pi_* X)^G$. A careful examination of the argument will show that it does not matter if we rationalize or take homotopy fixed points first, i.e., $(K(E')^{hG})_{\mathbb{Q}} \simeq (K(E')_{\mathbb{Q}})^{hG}$. In addition, rather than rationalizing, it suffices to invert $|G|$.

Proof. The key to this argument, and to all the fancier results, is that K-theory has another functoriality in the opposite direction.

Construction 1.5. Given a map of rings $A \to B$ such that B is a finite projective A-module, we have a forgetful functor from finite projective B-modules to finite projective A-modules. Algebraic K-theory is an invariant defined from the category of finitely generated projective modules.[2] Taking K-theory of this forgetful functor, this defines a map of K-theory spectra

$$K(B) \to K(A).$$

This is not a map of ring spectra, but it is a map of $K(A)$-module spectra (the projection formula).

Returning to the proof of Proposition 1.4, consider now the G-Galois extension of fields $E \subset E'$. Let's consider all the structure that exists on $K(E), K(E')$.

(1) There is a natural map $i^* : K(E) \to K(E')$ given by extension of scalars.
(2) There is a G-action on E', which induces a G-action on the category of E'-modules and thus a G-action on $K(E')$.
(3) There is a natural map $i_* : K(E') \to K(E)$ which comes from restriction of scalars as above.
(4) Both the maps i_*, i^* are G-equivariant, where G acts trivially on $K(E)$ and in the way described above on $K(E')$. This follows because $E \subset E'$ is G-equivariant.
(5) We compute the compositions in both directions:

$$i_* i^* = |G| : K(E) \to K(E) \tag{2}$$

$$i^* i_* = \sum_{g \in G} g : K(E') \to K(E'). \tag{3}$$

This follows from two computations:

[1]Part of the issue is that $K(E')^{hG}$ need not be connective, while $K(E)$ always is by definition. But the map need not even be an equivalence on connective covers. For fields of finite l-cohomological dimension, the map is an l-equivalence in high enough degrees as a consequence of the Bloch-Kato conjecture. We will not get into this here though.

[2]When we work with ring spectra, we will need to work with perfect complexes.

(a) If V is an E-vector space, then there is an isomorphism of E-vector spaces $V \otimes_E E' \simeq \bigoplus_{|G|} V$.

(b) If W is an E'-vector space, there is an isomorphism of E'-vector spaces $W \otimes_E E' \simeq (W \otimes_{E'} (E' \otimes_E E')) \simeq \bigoplus_{g \in G} W^g$. Here $W \otimes_E E'$ is an E'-vector space by the action on the second factor and W^g refers to the E'-vector space with the same underlying set as W but with the E'-module action twisted by the automorphism g of E'.

A straightforward diagram chase involving (2) and (3) shows that the map

$$i^* : K(E)_{\mathbb{Q}} \to (K(E')_{\mathbb{Q}})^{hG}$$

is an isomorphism, with inverse given by $i_*/|G|$. □

Of course, in stable homotopy theory one is usually more interested in integral or torsion phenomena rather than rational phenomena. The above argument provides no information about those. Thomason's main theorem shows that a large piece of the torsion information can be controlled though.

1.3. **A primer on telescopy.** Throughout these lectures, we will need a little background from chromatic homotopy theory. We review here what we need.

Let $P_\bullet$ be a perfect complex over $\mathbb{Z}$, i.e., a bounded complex of finitely generated projectives. Let $\phi : \Sigma^d P_\bullet \to P_\bullet$ be a self-map of some nonzero degree $d > 0$. Some composite of ϕ is null for degree reasons.

The analog of this phenomenon completely fails in the world of stable homotopy theory. That is, there are lots of examples of non-nilpotent self-maps of finite spectra[3] of positive degree. This was first observed by Adams.

Example 1.6. Suppose for simplicity ℓ is odd. Then there is an Adams self-map $v_1 : \Sigma^{2\ell-2}(S^0/\ell) \to S^0/\ell$ (cf. [1, Th. 1.7]) which induces an isomorphism in KU_*-homology. Let $T(1)$ be the homotopy colimit

$$T(1) = \operatorname{hocolim}\left(S^0/\ell \xrightarrow{v_1} \Sigma^{-(2\ell-2)}(S^0/\ell) \xrightarrow{v_1} \dots\right).$$

At $p = 2$, one has a map $\Sigma^8(S^0/2) \to S^0/2$ with the same property and we define $T(1)$ at the prime 2 by inverting this map.

Definition 1.7. A *telescope* T is the homotopy colimit of a self-map $\phi : \Sigma^d F \to F$ of non-zero degree $d > 0$, where F is a finite spectrum.

Telescopes exist in abundance, thanks to the technology of Hopkins-Smith and in particular their periodicity theorem [17, Th. 9]. The Adams construction $T(1)$ is only the first of an infinite family.

Construction 1.8. Fix a prime number p. There exists a sequence (F_n, ϕ_n) of finite p-torsion spectra F_n of a non-nilpotent *central* self-maps $\phi_n : \Sigma^{d_n} F_n \to F_n$ for some $d_n > 0$ with the following properties.

(1) $F_1 = S^0/p$ is the Moore spectrum mod p.
(2) F_{n+1} be the cofiber of the self-map $\phi_n : \Sigma^{d_n} F_n \to F_n$.

[3]Finite spectra are the compact objects in spectra; similarly, perfect complexes are the compact objects in the derived category of $\mathbb{Z}$.

The inductive existence of the non-nilpotent central self-map of F_{n+1} (and its uniqueness up to taking powers) is part of the machinery of [17].

The telescope $F_n[\phi_n^{-1}]$ of the map $\phi_n : \Sigma^{d_n} F_n \to F_n$ will be called a *height n telescope*. Although the homotopy type of this construction depended on some choices, the Bousfield class is independent of them.

Given any telescope T as in Definition 1.7, the p-localization $T_{(p)}$, if it is nonzero, has the property that $T_{(p)} \wedge F_n[\phi_n^{-1}] \neq 0$ for exactly one n. This n is called the *height* of T at the prime p.

1.4. **Thomason's results.** In [35, 36], Thomason shows that the failure of étale descent (for instance, the failure of (1) to be an equivalence) can be fixed if one smashes K-theory with the telescope $T(1)$ of Example 1.6. Fix a prime number ℓ.

We assume various technical hypotheses on X:[4]

(1) ℓ is invertible on X.
(2) X is noetherian and has finite Krull dimension.
(3) If $\ell = 2$, we have $\sqrt{-1} \in \Gamma(X, \mathcal{O}_X)$.
(4) The residue fields have uniformly bounded étale ℓ-cohomological dimension and admit a "Tate-Tsen filtration" (cf. [35, Th. 2.43]).

In this case, Thomason is able to show that maps of the form (1) are equivalences after one smashes with $T(1)$ (considered at the prime ℓ). In fact, Thomason obtains an analog of the Atiyah-Hirzebruch spectral sequence.

Theorem 1.9 (Thomason [35, 36])**.** *Under the above hypotheses on X, there is a descent spectral sequence*

$$E_2^{s,t} \simeq H^s_{et}(X, \mathbb{Z}/\ell(t/2)) \implies \pi_{t-s}(K(X) \wedge T(1)).$$

Here $\mathbb{Z}/\ell(t/2)$ denotes the sheaf $\mu_\ell^{\otimes(t/2)}$ (where μ_ℓ denotes the sheaf of ℓth roots of unity) for t even and is zero for t odd.

In particular, it follows that $X \mapsto K(X) \wedge T(1)$ is an *étale sheaf of spectra.*

Optional Remark 1.10. It is also known that $K(X) \wedge T(1)$ can be obtained by inverting a so-called *Bott element,* or by $K(1)$-localizing $K(X)/\ell$. We refer to the treatment of Mitchell [27] for a detailed survey.

Optional Remark 1.11. Thomason's result shows that $K(\cdot) \wedge T(1)$ is an actually a *hypercomplete* étale sheaf, which leads to the above descent spectral sequence once the stalks are identified using Gabber-Suslin rigidity.

Optional Remark 1.12. Using the tools of the Bloch-Kato conjecture (proved by Voevodsky-Rost), it is possible to say much more, but we won't get into any of this here.

1.5. **Galois extensions.** The goal of these talks is to prove a special case of Thomason's results[5] where instead of studying the K-theory of ordinary rings and schemes, we study the K-theory of *structured ring spectra*, following the philosophy of "brave new rings" of Waldhausen. Instead of taking the algebraic K-theory of a ring, we take the algebraic K-theory of an $\mathbb{E}_\infty$-ring R, which produces a new $\mathbb{E}_\infty$-ring $K(R)$. There are numerous computational tools for approaching $\pi_* K(R)$; we refer to [31] for a survey of recent work.

[4] These types of conditions will play no role in the remainder of these notes as we use very different methods from Thomason.

[5] Specifically, we are studying analogs of étale *descent* whereas Thomason is exploring *hyperdescent.*

The idea is to carry over concepts from classical algebra and algebraic geometry to the setting of ring spectra. For instance, the notion of the étale site has an analog in the world of ring spectra.

Definition 1.13. A morphism $R \to R'$ of $\mathbb{E}_\infty$-rings is *étale* if the following two conditions happen:

(1) $\pi_0 R \to \pi_0 R'$ is an étale morphism of ordinary rings.
(2) The natural map $\pi_* R \otimes_{\pi_0 R} \pi_0 R' \to \pi_* R'$ is an isomorphism.

Given this, one can investigate whether the algebraic K-theory of ring spectra is, after some periodic localization, a sheaf of spectra on the étale site. The above definition gives a good analog of the classical theory of étaleness for connective $\mathbb{E}_\infty$-ring spectra. In the world of nonconnective ring spectra, however, there are additional morphisms of $\mathbb{E}_\infty$-ring spectra that have many of the formal properties of étale morphisms. These were introduced by Rognes under the name *Galois extensions.*

Definition 1.14 (Rognes [30])**.** Let G be a finite group. An extension $R \to R'$ of $\mathbb{E}_\infty$-rings together with a G-action on R' in $\mathbb{E}_\infty$-R-algebras is said to be ***G*-Galois** if:

(1) $R \to R'^{hG}$ is an equivalence.
(2) The natural map $R' \otimes_R R' \to \prod_{g \in G} R'$ is an equivalence.

A Galois extension $R \to R'$ is said to be *faithful* when tensoring with R' is conservative as a functor $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$.

Optional Remark 1.15. The theory of *faithful Galois extensions* fits into Grothendieck's axiomatic Galois theory and one can obtain a Galois group. We refer to [19] for more details.

Example 1.16. Let R be an $\mathbb{E}_\infty$-ring spectrum and let R'_0 be a G-Galois extension of the commutative ring $\pi_0 R$ (in the classical sense). Then one has a canonically determined G-Galois extension $R \to R'$ of $\mathbb{E}_\infty$-rings such that $\pi_0 R' \simeq R'_0$ compatible with the G-action.

Example 1.17. The basic example of a Galois extension of $\mathbb{E}_\infty$-rings is the map $KO \to KU$ from periodic real to complex K-theory. This is a C_2-Galois extension. Note that $\pi_*(KO)$ is much more complicated homologically than $\pi_*(KU)$ (e.g., it has nilpotent elements).

Example 1.18. There are more "chromatic" examples. Given any finite subgroup H of the Morava stabilizer group $\mathbb{G}_n$ at height n, one has an H-Galois extension $E_n^{hH} \to E_n$.

Example 1.19. Let R be a discrete commutative ring where p is nilpotent and let G be a p-group. Then the extension $F(BG_+; R) \to R$ is a G-Galois extension.

The idea is that a Galois extension of $\mathbb{E}_\infty$-ring spectra is supposed to be a "brave new" analog of a Galois extension of fields or of commutative rings. For example, there is a notion of "Galois descent" of modules, at least for faithful Galois extensions. Galois extensions are useful in studying invariants such as Picard groups of $\mathbb{E}_\infty$-rings. We refer to [23, 15] for some applications to calculating Picard groups.

1.6. **Galois descent.** We can try to use Galois theory to study algebraic K-theory as well. This leads to the following question.

Question 1.20. Suppose $A \to B$ is a G-Galois extension of $\mathbb{E}_\infty$-ring spectra. One obtains a map $K(A) \to K(B)^{hG}$. How close is this map to being an equivalence?

In particular, Ausoni-Rognes conjecture in [3]:

Conjecture (Ausoni-Rognes)**.** *If $A \to B$ is a $K(n)$-local G-Galois extension, and T is a height $n+1$ telescope, then the map $T \wedge K(A) \to T \wedge K(B)^{hG}$ is an equivalence.*

In the setup of Thomason's work, for Galois descent for discrete rings, one has descent after smashing with the height one telescope $T(1)$. One could ask about telescopes at higher heights, but a theorem of Mitchell [26] implies that if A is a discrete ring and T is a telescope of height > 1, then $K(A) \wedge T$ is contractible.

In the setup of structured ring spectra, one obtains in general more interesting chromatic phenomena. To understand this, we will use the notion of L_n-localization, which is localization with respect to Morava E-theory.

Our main result is the following.

Theorem 1.21 (Clausen, M., Naumann, Noel [12])**.** *Suppose $A \to B$ is a G-Galois extension of $\mathbb{E}_\infty$-ring spectra. Suppose we have an equality $[B] = |G|$ in $K_0(B) \otimes \mathbb{Q}$.*[6] *Let T be any telescope. Then the maps*

$$T \wedge K(A) \to T \wedge K(B)^{hG} \to (T \wedge K(B))^{hG}$$

are equivalences. This result holds with K-theory replaced by THH,TC, etc.

This result applies to all the Galois extensions listed above.

Example 1.22. For example, it applies to $KO \to KU$ using the Wood equivalence $KU \simeq KO \wedge C\eta$. This shows that the class of the KO-module KU is equal to 2 in $K_0(KO)$.

We have the following general criterion for the condition to be satisfied.

Theorem 1.23 ([12])**.** *Suppose R has the property that $\pi_0 R$ has no nontrivial idempotents and $K_0(R) \otimes \mathbb{Q}$ has no nontrivial idempotents. Then the above result holds for every Galois extension of R.*

As a corollary, we can prove the analog of étale descent in localized algebraic K-theory, using the definition of étale from Definition 1.13.

Corollary 1.24. *Let R be an $\mathbb{E}_\infty$-ring and let T be any telescope. Then the association $R' \mapsto T \wedge K(R)$ is a sheaf of spectra on the étale site of R.*

Rationalized algebraic K-theory of discrete rings satisfies finite flat descent as well. We prove the following result, which generalizes the above to the non-Galois case.

Theorem 1.25 ([12])**.** *Let $A \to B$ be a morphism of $\mathbb{E}_\infty$-rings such that B is perfect as an A-module. Suppose that the induced wrong-way map $K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ is surjective. Then for any telescope T, the natural map*

$$T \wedge K(A) \to T \wedge \mathrm{Tot}(K(B) \rightrightarrows K(B \otimes_A B) \substack{\to\\\to\\\to} \dots)$$

is an equivalence.

Question 1.26. Is there an example of a Galois extension which fails to satisfy the hypotheses of our result?

[6]Note that B is a perfect A-module, so $[B]$ defines a class in $K_0(B)$.

2. Derived induction and restriction theory

Let $A \to B$ be a G-Galois extension of $\mathbb{E}_\infty$-rings. Our goal is to analyze the descent comparison map $K(A) \to K(B)^{hG}$. An idea of Carlsson suggests that the comparison map $K(A) \to K(B)^{hG}$ should be analogous to the map $X^G \to X^{hG}$ from fixed points to homotopy fixed points, when X is a G-space or a G-spectrum. We will discuss this story more in the final lecture. Before that, we will shift gears away from algebraic K-theory to discuss some tools for analyzing the map $X^G \to X^{hG}$.

The next goal of these lectures is to develop the theory of $\mathcal{F}$-nilpotent G-spectra. The theory of $\mathcal{F}$-nilpotence gives tools for proving that such comparison maps are equivalences. The main references for $\mathcal{F}$-nilpotence are [21, 20]. Classical induction and restriction theory, with which we shall start, is of course much older.

Throughout, let G be a finite group.

2.1. Coefficient systems and families.

In this subsection, we introduce the basic problem and set up some notation.

Definition 2.1. Let $\mathcal{O}(G)$ denote the *orbit category* of G: that is, the category of G-sets of the form $\{G/H, H \leq G\}$ and G-maps between them.

Example 2.2. Suppose $G = C_2$. Then $\mathcal{O}(G)$ has two objects, C_2/e and C_2/C_2. The object C_2/C_2 is terminal. We have $\mathrm{Hom}_{\mathcal{O}(C_2)}(C_2/e, C_2/e) \simeq C_2$ and $\mathrm{Hom}_{\mathcal{O}(C_2)}(C_2/C_2, C_2/e) = \emptyset$.

Definition 2.3. A functor $\mathcal{O}(G)^{op} \to \mathrm{Ab}$ is called a *coefficient system*.

Remark 2.4. Let M be a coefficient system. Then for each $H \leq G$, we have an abelian group $M(G/H)$. We have the following functoriality.

(1) Whenever $H' \leq H$, we obtain a *restriction* map
$$\mathrm{Res}^H_{H'} : M(G/H) \to M(G/H')$$
induced by the natural map of G-sets $G/H' \to G/H$.

(2) Whenever H, H' are conjugate by an element $g \in G$ so that $H' = g^{-1}Hg$, we have a conjugation isomorphism $G/H \simeq G/H'$ given by multiplication on the *right* by g. This induces an isomorphism
$$c_g : M(G/H) \simeq M(G/H').$$

(3) These satisfy some natural relations which we will not write out explicitly.

The following example gives a large source of coefficient systems for *any* finite group.

Example 2.5. Let $\mathrm{Gp}^{\mathrm{inj}}$ denote the category of finite groups and *injections* between them. Suppose $F : (\mathrm{Gp}^{\mathrm{inj}})^{op} \to \mathrm{Ab}$ is a functor. Suppose that for any group K, the inner automorphisms of K act trivially on $F(K)$.

Then for any group G, we obtain a coefficient system on G which sends $G/H \mapsto F(H)$. The hypothesis that inner automorphisms of K act trivially on $F(K)$ is necessary to obtain a well-defined functor.

Here are two basic examples:

(1) Fix a field k. Group cohomology with coefficients in k gives an example of a coefficient system on G. That is, there is a natural functor
$$\mathcal{O}(G)^{op} \to \mathrm{Ring}_k, \quad G/H \mapsto H^*(H; k).$$

(2) Given a finite group K, let $R(K)$ denote the complex *representation ring* of K, i.e., the Grothendieck group of finite-dimensional complex representations of K. For any finite group G, we obtain a coefficient system from the representation rings of the subgroups. There is a natural functor

$$\mathcal{O}(G)^{op} \to \mathrm{Ring}, \quad G/H \mapsto R(H).$$

We now include an example of a coefficient system that will be important for the descent questions.

Example 2.6. Let $E \subset F$ be a G-Galois extension of fields and let $i \in \mathbb{Z}_{\geq 0}$. Then we have a coefficient system given by

$$G/H \mapsto K_i(F^H).$$

In fact, we have a functor $\mathcal{O}(G)^{op} \to \mathrm{Ring}$ sending $G/H \mapsto F^H$ and we can compose this functor with $K_i : \mathrm{Ring} \to \mathrm{Ab}$.

Definition 2.7. A **family of subgroups** $\mathcal{F}$ is a collection of subgroups of G which is closed under subconjugation. Given a family $\mathcal{F}$, we let $\mathcal{O}_{\mathcal{F}}(G)$ denote the subcategory of $\mathcal{O}(G)$ spanned by the G-sets $\{G/H, H \in \mathcal{F}\}$.

Example 2.8. Common examples of families of subgroups include the p-subgroups, the abelian subgroups, the cyclic subgroups, etc. If H is a normal subgroup of G, then the subgroups contained in H form a family of subgroups of G.

Let M be a coefficient system. Our setup is that we are interested in $M(G/G)$, which might be hard to compute, while we have $M(G/H)$ for $H \in \mathcal{F}$ for some family $\mathcal{F}$. We obtain a basic comparison map

$$M(G/G) \to \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} M(G/H). \tag{4}$$

Question 2.9. How close is the comparison map (4) to being an isomorphism?

Example 2.10. Suppose $\mathcal{F} = \{(1)\}$ is the family consisting only of the trivial subgroup. Let us unwind what happens here. We have a G-action on the G-set G/e given by *right* multiplication. Our comparison map becomes

$$M(G/G) \to M(G/e)^G,$$

For example, in the setting of the G-Galois extension $E \subset F$, we obtain the map $K_i(E) \to K_i(F)^G$. We saw earlier that these maps are *rational* isomorphisms.

Example 2.11 (Cartan-Eilenberg)**.** Let G be a finite group and let $\mathcal{F}$ be the family of p-subgroups. Let k be a field of characteristic p. Then the natural comparison map

$$H^*(G; k) \to \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} H^*(H; k)$$

is an isomorphism.

We next include the examples involving representation rings. Note that $R(G)$, as a ring, is easy to determine when G is abelian: in this case, we have $R(G) \simeq \mathbb{Z}[G^\vee]$ where $G^\vee = \mathrm{Hom}(G, \mathbb{C}^\times)$ is the group of characters of G.

Example 2.12 (Artin induction)**.** The map

$$R(G)[1/|G|] \to \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} R(H)[1/|G|]$$

is an isomorphism when $\mathcal{F}$ is the family of cyclic subgroups. Note that when we tensor with $\mathbb{C}$ (instead of inverting $|G|$), we even have an isomorphism

$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathrm{Fun}(G_{\mathrm{conj}}, \mathbb{C}),$$

where G_{conj} denotes the set of conjugacy classes of elements in G.

When we do not invert the order of G, one has to choose a larger family of subgroups.

Example 2.13 (Brauer induction)**.** The map

$$R(G) \to \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} R(H)$$

is an isomorphism when $\mathcal{F}$ is the family of *Brauer elementary* subgroups. A group H is called *Brauer elementary* if $H \simeq P \times P'$ where P is a p-group and P' is cyclic.

2.2. **Mackey functors and Dress induction.** To prove results such as the inverse limit decompositions of Artin and Brauer induction (and treat more general questions involving (4)), it is fundamental that the coefficient systems in question have an additional functoriality in the other direction. Instead of being simply coefficient systems, they are *Mackey functors.*

Remarkably, it is the induction homomorphisms that enable one to prove that comparison maps (4) are equivalences.

Definition 2.14. We define the *effective Burnside category*[7] Burn_G as follows.

(1) The objects are given by the finite G-sets.
(2) Given finite G-sets S and T, the maps $\mathrm{Hom}_{\mathrm{Burn}_G}(S, T)$ is given by isomorphism classes of *spans*

$$S \leftarrow A \rightarrow T \, .$$

(3) To compose spans

$$S \leftarrow A \rightarrow T \, , \qquad T \leftarrow B \rightarrow U \, ,$$

we form the span

$$S \leftarrow A \times_T B \rightarrow U \, .$$

The effective Burnside category is a pre-additive category: to add spans, we can take their disjoint union.

[7]The terminology "effective" is due to Clark Barwick [7].

Definition 2.15. A *Mackey functor* is a functor $M : \mathrm{Burn}_G^{op} \to \mathrm{Ab}$ which carries disjoint unions of finite G-sets to direct sums of abelian groups. (In particular, M is determined by its values on the G-orbits.)

Construction 2.16. Let FinSet_G be the category of finite G-sets. There is a natural functor

$$\mathrm{FinSet}_G \to \mathrm{Burn}_G$$

which is the identity on objects, and which carries a morphism $f : S \to T$ to the correspondence

$$\begin{array}{ccc} & S & \\ {\scriptstyle \mathrm{id}_S} \swarrow & & \searrow {\scriptstyle f} \\ S & & T \end{array} .$$

Similarly, one constructs a functor $\mathrm{FinSet}_G^{op} \to \mathrm{Burn}_G$.

Let M be a Mackey functor. From the above two functors, we obtain the following data. For any map of finite G-sets $f : S \to T$, we obtain homomorphisms

$$f^* : M(T) \to M(S), \quad f_* : M(S) \to M(T).$$

These have the following properties:

(1) f^*, f_* are naturally functorial (in opposite directions) in the morphism $f : S \to T$. Note in particular that M restricts to a *coefficient system* (by considering the functoriality along f^*).
(2) Given a pullback diagram of finite G-sets

$$\begin{array}{ccc} T' & \xrightarrow{f'} & T \\ \downarrow {\scriptstyle p'} & & \downarrow {\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array} ,$$

we have an equality

$$p^* f_* = f'_* p'^* : M(S') \to M(T). \tag{5}$$

Conversely, given an association $T \mapsto M(T)$ from finite G-sets to abelian groups together with functorialities $f \mapsto f^*, f_*$ satisfying the base-change relation (5), one obtains a Mackey functor.

Definition 2.17. Let M be a Mackey functor for the group G. Given subgroups $H' \leq H \leq G$, then we have a natural map $f : G/H' \to G/H$. We write

$$\mathrm{Res}^H_{H'} = f^* : M(G/H) \to M(G/H'), \tag{6}$$

$$\mathrm{Ind}^H_{H'} = f_* : M(G/H') \to M(G/H). \tag{7}$$

Definition 2.18. A *Green functor* is a Mackey functor M together with the following structure:

(1) For each finite G-set S, $M(S)$ has the structure of a commutative ring.
(2) For each map $f : S \to T$, $f^* : M(T) \to M(S)$ is a map of commutative rings.
(3) For each map $f : S \to T$, $f_* : M(S) \to M(T)$ is a map of $M(T)$-modules (where $M(S)$ is given the $M(T)$-module structure via f^*). This is called the *projection formula.*

Example 2.19. Let $H' \leq H$ be subgroups of a finite group G. We then have a homomorphism

$$\mathrm{I}_{H'}^{H} : R(H') \to R(H)$$

obtained by inducing H'-representations to H-representations. Using the $\mathrm{Ind}_{\cdot}^{\cdot}$ maps, we can construct a functor

$$\mathcal{O}(G) \to \mathrm{Ab}, \quad G/H \mapsto R(H)$$

but where the functoriality comes from the induction homomorphisms. This combines with the representation ring coefficient system to yield a representation ring Green functor. That is, there is a Green functor M such that:

(1) For $H \leq G$, we have $M(G/H) = R(H)$.
(2) For a map $f : G/H' \to G/H$, we have that $f^* : R(H) \to R(H')$ is given by restricting representations and $f_* : R(H') \to R(H')$ is $\mathrm{I}_{H'}^{H}$.

One can make similar constructions in group cohomology (using the "corestriction" functoriality).

2.3. **The Dress induction theorem.**

Definition 2.20. We say that a Green functor M is *induced* from a family $\mathcal{F}$ if the induction homomorphism

$$\mathrm{Ind} : \bigoplus_{H \in \mathcal{F}} M(G/H) \to M(G/G)$$

has image containing the unit, or equivalently (by the projection formula) if the map is surjective.

When a Green functor is induced from a family, then one gets a decomposition of $M(G/G)$ by the following fundamental result of Dress.

Theorem 2.21 (Dress). *Suppose M is a Green functor for the finite group G which is induced from $\mathcal{F}$. Then the restriction map*

$$M(G/G) \to \varprojlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} M(G/H)$$

is an isomorphism.

Example 2.22. The classical theorems of Artin and Brauer state that the representation ring Green functor is induced from the families of cyclic and Brauer elementary subgroups, respectively.

That is, let G be a finite group. Then:

(1) There exist classes $x_C \in R(C)[1/|G|]$ for each cyclic subgroup $C \subset G$ such that we have

$$1 = \sum_{C \subset G \text{ cyclic}} \mathrm{Ind}_C^G(x_C) \in R(G)[1/|G|].$$

(2) There exist classes $y_H \in R(H)$ for each Brauer elementary subgroup $H \subset G$ such that we have

$$1 = \sum_{H \subset G \text{ elementary}} \mathrm{Ind}_H^G(x_H) \in R(G).$$

Example 2.23. In group cohomology, one observes that if G is a finite group, $P \subset G$ a p-Sylow subgroup, and k a field of characteristic p, then the map $H^*(P; k) \to H^*(G; k)$ given by *corestriction* (i.e., the induction in the Mackey functor) carries 1 to $[G : P]$, which is a unit in k.

2.4. **G-spectra.** Mackey functors arise in "nature" as the homotopy groups of a G-spectra. Our goal is to describe an induction and restriction theory that applies directly to the G-spectra, rather than to their homotopy groups.

It is not within the scope of these lectures to cover the foundations of G-spectra. There are several basic facts about them to know.

(1) Let Sp_G denote the category of G-spectra and let $\mathcal{S}_G$ denote the category of G-spaces. Given any two G-spectra E and F, there is a natural *spectrum* of maps $\mathrm{Hom}_{\mathrm{Sp}_G}(E, F)$.

(2) There is an adjunction
$$(\Sigma^\infty_+, \Omega^\infty) : \mathcal{S}_G \rightleftarrows \mathrm{Sp}_G.$$
Any G-spectrum E defines a "cohomology theory" on G-spaces via
$$E^k(X) = \pi_{-k}\mathrm{Hom}_{\mathrm{Sp}_G}(\Sigma^\infty_+ X, E),$$
for any $k \in \mathbb{Z}$. This cohomology theory satisfies analogs of the usual Eilenberg-Steenrod axioms for a cohomology theory on topological spaces.

(3) There are G-spectra representing many naturally occurring cohomology theories on G-spaces. For example, there is a G-spectrum $\underline{H\mathbb{F}_p} \in \mathrm{Sp}_G$ with the property that if X is a G-space, then we have a natural isomorphism
$$\underline{H\mathbb{F}_p}^k(X) = H^k(X_{hG}; \mathbb{F}_p),$$
i.e., $\underline{H\mathbb{F}_p}$ represents Borel-equivariant cohomology. Here $X_{hG} \simeq (X \times EG)/G$ is the homotopy orbits or Borel construction for G acting on X.

Similarly, we have G-spectra KO_G and KU_G representing real and complex equivariant K-theory.

(4) Given a G-spectrum E, we write
$$\pi_*^H E \simeq \mathrm{Hom}_{\mathrm{Sp}_G}(\Sigma^\infty_+ G/H, E) = E^*(G/H),$$
and this clearly defines a coefficient system with values in graded abelian groups. This actually extends naturally to a *Mackey functor*. In fact, this follows from the fact that there are more maps between $\Sigma^\infty_+ G/H$ and $\Sigma^\infty_+ G/K$ in Sp_G than in $\mathcal{S}_G$ or $\mathcal{O}(G)$. We actually have
$$\pi_0\mathrm{Hom}_{\mathrm{Sp}_G}(\Sigma^\infty_+ G/H, \Sigma^\infty_+ G/K) \simeq K_0(\mathrm{Hom}_{\mathrm{Burn}_G}(G/H, G/K)),$$
where K_0 denotes taking group completion of the abelian monoid $\mathrm{Hom}_{\mathrm{Burn}_G}(G/H, G/K)$.

(5) There is a natural smash product on Sp_G. This leads to the notion of a *G-ring spectrum*: that is, an associative algebra up to homotopy. It also leads to more highly structured notions. Given a G-ring spectrum, then the homotopy groups are naturally arranged into a Green functor.

Let E be a genuine G-spectrum. As we saw, the homotopy groups of E naturally form a Mackey functor, given by
$$G/H \mapsto \pi_*^H E = \mathrm{Hom}_{\mathrm{Sp}_G}(G/H_+, E).$$
In fact, it is possible (following Guillou, May, and Barwick) to set up the theory of G-spectra via "spectrum-valued" Mackey functors. This makes it very transparent that the homotopy groups should take values in Mackey functors.

Definition 2.24. Given a G-spectrum E, for every $H \leq G$, we obtain a spectrum
$$E^H \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathrm{Sp}_G}(G/H_+, E),$$

where we consider the spectrum of maps from G/H_+ into E. Since this is defined as maps out of G/H, one sees that $G/H \mapsto E^H$ defines a functor $\mathcal{O}(G)^{op} \to \mathrm{Sp}$.

Definition 2.25. A *spectral coefficient system* is a functor $\mathcal{O}(G)^{op} \to \mathrm{Sp}$.

Therefore, given any G-spectrum E, we obtain a spectral coefficient system $G/H \mapsto E^H$. A key observation is that there is more structure on the fixed points of a G-spectrum.[8]

For this, we will need to modify our definition of the effective Burnside category. We should instead work with the *effective Burnside 2-category* $\mathbf{Burn}_G$, which is a $(2,1)$-category (or category weakly enriched in groupoids) such that:

1. The objects, as before, are given by the finite G-sets.
2. Given finite G-sets S, T, we have that $\mathrm{Hom}_{\mathbf{Burn}_G}(S,T)$ is the *groupoid* of correspondences of finite G-sets from S to T and isomorphisms between them.

Definition 2.26. A *spectral Mackey functor* is a functor $F : \mathbf{Burn}_G^{op} \to \mathrm{Sp}$ with the property that if S and T are finite G-sets, then the natural map $F(S \sqcup T) \to F(S) \times F(T)$ is a homotopy equivalence. Given a spectral Mackey functor F and a map of finite G-sets $f : S \to T$, we obtain maps $f^* : M(T) \to M(S)$ and $f_* : M(S) \to M(T)$ which satisfy a base-change relation (5) up to coherent homotopy.

In other words, one simply reproduces the definition of a Mackey functor but replaces Ab with Sp. Since Sp is an $(\infty,1)$-category rather than a 1-category, a functor $\mathbf{Burn}_G^{op} \to \mathrm{Sp}$ generally does not factor through Burn_G (while a functor to abelian groups does).

Construction 2.27. Given a G-spectrum E, there is a spectral Mackey functor $F_E : \mathbf{Burn}_G^{op} \to \mathrm{Sp}$ with the properties:

1. If T is a finite G-set, we have an equivalence $F_E(T) \simeq \mathrm{Hom}_{\mathrm{Sp}_G}(\Sigma^\infty_+ T, E)$.
2. If $f : T \to T'$ is a morphism of finite G-sets, we obtain a morphism $F_E(T') \xrightarrow{f^*} F_E(T)$ given by precomposing with f.
3. If $f : T \to T'$ is a morphism of finite G-sets, we obtain a morphism $F_E(T) \xrightarrow{f_*} F_E(T')$ as follows. A basic feature of Sp_G is that $\Sigma^\infty_+ T, \Sigma^\infty_+ T'$ are *self-dual* with respect to the smash product. In particular, we also have an identification
$$F_E(T) \simeq \mathrm{Hom}_{\mathrm{Sp}_G}(\Sigma^\infty_+ T, E) \simeq \mathrm{Hom}_{\mathrm{Sp}_G}(S^0, E \wedge \Sigma^\infty_+ T) = \left(E \wedge \Sigma^\infty_+ T\right)^G.$$
Using this identification, the map $f_* : F_E(T) \to F_E(T')$ comes from taking fixed points in the natural map $E \wedge \Sigma^\infty_+ T \to E \wedge \Sigma^\infty_+ T'$.

Theorem 2.28 (Guillou-May; Barwick [14])**.** *There is an equivalence of homotopy theories (e.g., model categories or ∞-categories) between* Sp_G *and the theory of spectral Mackey functors.*

In particular, a G-spectrum is specified *precisely* by its fixed point data, provided that we remember all the relations between them that come from the *Burnside* (and not simply the orbit) category.

[8] By contrast, the G-homotopy type of a G-space is entirely encoded by a functor $\mathcal{O}(G)^{op} \to \mathcal{S}$ by Elmendorf's theorem. A *naive G-spectrum* is equivalent data to a spectral coefficient system.

2.5. **Derived induction and restriction.** We can consider derived analogs of the above comparison maps.

Definition 2.29. Let $\mathcal{F}$ be a family of subgroups of G. Let E be a G-spectrum. Then we have a comparison map

$$E^G \to \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} E^H. \tag{8}$$

The right-hand-side can be identified with $\mathrm{Hom}_{\mathrm{Sp}_G}(E\mathcal{F}_+, E) = F(E\mathcal{F}_+, E)^G$ where $E\mathcal{F}$ is the classifying space of the family and $F(\cdot, \cdot)$ denotes the function spectrum. Here all the maps between the fixed points are the restriction maps. Note that the limit is replaced by a *homotopy limit.*

Similarly, we have a comparison (induction) map

$$\mathrm{hocolim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} E^H \to E^G. \tag{9}$$

In the language of equivariant stable homotopy theory, the left-hand-side can be identified with $(E\mathcal{F}_+ \wedge E)^G$ and the map is induced by the nontrivial map $E\mathcal{F}_+ \to S^0$.

We have the following "derived" analog of the Dress induction theorem. In some form, it goes back to Thomason [35], at least when $\mathcal{F} = \{(1)\}$. It is stated (again for $\mathcal{F}$ the trivial family) in [16, Sec. 10].

Theorem 2.30 (Thomason machine; Hill-Hopkins-Ravenel [16]; Mathew-Naumann-Noel [20])**.** *Suppose that E is a G-ring spectrum and $\mathcal{F}$ is a family of subgroups. Suppose that the map* (9) *has image including $1 \in \pi_0 E^G$. Then the restriction map* (8) *is an equivalence, as is* (9)*.*

Definition 2.31. We will say that a G-ring spectrum is *induced* from $\mathcal{F}$ if the hypothesis of the above result hold.

Remark 2.32. In (9), we can identify the left-hand-side with the fixed points $(E\mathcal{F}_+ \wedge E)^G$. In other words, a G-ring spectrum is induced from $\mathcal{F}$ if and only if the map $E\mathcal{F}_+ \wedge E \to E$ has image in homotopy (on G-fixed points) including the unit.

Example 2.33. Suppose for example that $\mathcal{F} = \{(1)\}$. Then (8) becomes

$$E^G \to E^{hG} \xrightarrow{\mathrm{def}} (E^{\{1\}})^{hG}. \tag{10}$$

This basic map, from fixed points to homotopy fixed points, has played an important role in algebraic topology.

We also have the map of (9), which becomes

$$E_{hG} \to E^G, \tag{11}$$

called a *transfer* map. If (11) has image containing the unit, then (10) is an equivalence by the above machine.

Remark 2.34. The reader may have noted a difference in our treatment of the derived induction and restriction pictures. In the setting of Green functors M, we considered the map $\bigoplus_{H \in \mathcal{F}} M(G/H) \to M(G/G)$ (given by the sum of the induction maps) and asked whether its image contained the unit. It naturally factored through a colimit $\varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} M(G/H) \to M(G/G)$, which the above direct sum surjected onto. In the setting of derived induction theory, the homotopy colimit $\mathrm{hocolim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} E^H$ can have classes in homotopy that do not show up in the direct sum $\bigoplus_{H \in \mathcal{F}} E^H$. For example, if X is a spectrum with a G-action, X_{hG} can have larger homotopy groups than X. This is in fact what will happen in our applications.

In the next lecture, we will discuss more the consequences of derived induction and restriction theory and describe a powerful tool for verifying that the above hypotheses are satisfied.

3. $\mathcal{F}$-NILPOTENT G-SPECTRA

We will restart the discussion of derived induction and restriction theory in this lecture from a different point of view.

3.1. **Thick subcategories and nilpotence.** Let $\mathcal{C}$ be a triangulated category which is idempotent-complete. The latter condition means that if $X \in \mathcal{C}$ and if $e : X \to X$ is an idempotent endomorphism, then there is a splitting in $\mathcal{C}$ given by $X \simeq Xe \oplus X(1-e)$.

Definition 3.1. A *thick subcategory* of $\mathcal{C}$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ such that $\mathcal{D}$ is triangulated and idempotent-complete. This means:

(1) $0 \in \mathcal{D}$.
(2) Given an exact triangle $X' \to X \to X''$ in $\mathcal{C}$, if two out of the three objects belong to $\mathcal{D}$, then the third does as well.
(3) If $X \in \mathcal{D}$ and admits a decomposition in $\mathcal{C}$, $X \simeq Y' \oplus Y''$, then $Y', Y'' \in \mathcal{D}$ as well.

The classification and analysis of thick subcategories of triangulated categories is an active field, starting with the classification for p-local finite spectra by Hopkins-Smith [17].

Let $\mathcal{C}$ be a $\otimes$*-triangulated category.* That is, we assume that:

(1) $\mathcal{C}$ is a symmetric monoidal category with tensor product $\otimes$ and unit $\mathbf{1}$.
(2) $\mathcal{C}$ is a triangulated category.
(3) The functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ is exact in each variable.

Definition 3.2. A thick subcategory $\mathcal{I} \subset \mathcal{C}$ is called a **thick $\otimes$-ideal** if for all $X \in \mathcal{C}, Y \in \mathcal{I}$, we have $X \otimes Y \in \mathcal{I}$.

Thick $\otimes$-ideals in particular have been studied extensively, e.g., in the work of Balmer [4] on the *spectrum* of a $\otimes$-triangulated category.

Definition 3.3 (Bousfield)**.** Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$ and let A be an associative algebra object in $\mathcal{C}$. Then we say that an object $X \in \mathcal{C}$ is **A-nilpotent** if it belongs to the thick $\otimes$-ideal generated by A.

3.2. **$\mathcal{F}$-nilpotence.** Let G be a finite group. The homotopy category $\mathrm{Ho}(\mathrm{Sp}_G)$ is a $\otimes$-triangulated category and we will work in it.

Construction 3.4. Given $H \leq G$, we define a commutative algebra object $A_H = F(G/H_+, S^0_G) \in \mathrm{Sp}_G$ as the Spanier-Whitehead dual to the G-space G/H. The commutative algebra object A_H has the property that

$$\mathrm{Mod}_{\mathrm{Sp}_G}(A_H) \simeq \mathrm{Sp}_H,$$

cf. [6] and [21, Sec. 5.3].

Definition 3.5. Let G be a finite group and let $\mathcal{F}$ be a family of subgroups. We let $\mathcal{C} = \mathrm{Sp}_G$ and take $A_{\mathcal{F}} = \prod_{H \in \mathcal{F}} A_H$. We say that a G-spectrum is **$\mathcal{F}$-nilpotent** if it is $A_{\mathcal{F}}$-nilpotent.

Remark 3.6. Given a subgroup $H \subset G$, there is a restriction functor $\mathrm{Res}^G_H : \mathrm{Sp}_G \to \mathrm{Sp}_H$ from G-spectra, which has a biadjoint functor $\mathrm{Ind}^G_H : \mathrm{Sp}_H \to \mathrm{Sp}_G$. A G-spectrum is $\mathcal{F}$-nilpotent if and only if it belongs to the smallest thick subcategory of $\mathrm{Ho}(\mathrm{Sp}_G)$ generated by the G-spectra which are *induced* from a subgroup in $\mathcal{F}$.

The notion of $\mathcal{F}$-nilpotence is very closely connected to the induction and restriction theory discussed in the previous lecture.

Proposition 3.7. *Given an $\mathcal{F}$-nilpotent G-spectrum E, the comparison maps* (8) *and* (9) *are equivalences.*

Proof. When the G-spectrum E is itself induced from an H-spectrum for some $H \in \mathcal{F}$, a formal argument (amounting to a certain augmented cosimplicial object being *split*) shows that the comparison maps are equivalences. Since (8) and (9) are morphisms between exact functors, the collection of G-spectra E for which they are equivalences forms a thick subcategory, which therefore contains all $\mathcal{F}$-nilpotent G-spectra. □

For a G-ring spectrum, $\mathcal{F}$-nilpotence is equivalent to the derived induction map (9) being an equivalence, or to being induced from $\mathcal{F}$.

Proposition 3.8. *Let R be a G-ring spectrum. Then R is $\mathcal{F}$-nilpotent if and only if R is induced from $\mathcal{F}$. This happens if and only if the G-ring spectrum $R \wedge \widetilde{E\mathcal{F}}$ is contractible.*

For a general G-spectrum, one can reduce to the case of a ring spectrum by considering the endomorphism ring. This follows from the following result.

Proposition 3.9. *A G-spectrum E is $\mathcal{F}$-nilpotent if and only if the endomorphism ring object $F(E, E) \in \mathrm{Sp}_G$ is $\mathcal{F}$-induced.*

3.3. **Consequences of $\mathcal{F}$-nilpotence.** Let R be a G-ring spectrum. Consider the Green functor $\pi_*^- R$. If R is $\mathcal{F}$-nilpotent, we can ask whether $\pi_*^- R$ is induced from the family of subgroups $\mathcal{F}$ in the sense of Definition 2.20. We can also ask whether $\pi_*^G R$ can be recovered from $\pi_*^H R$ for $H \in \mathcal{F}$, i.e., whether (4) is an isomorphism. The answer to the both questions is no in general.

Since R is $\mathcal{F}$-nilpotent, we have

$$R^G \simeq \mathrm{hocolim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} R^H$$

under the induction maps. This gives a homotopy colimit *spectral sequence* converging to $\pi_* R^G$ where the E_2 term is given

$$\varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)}{}^{s} \pi_t(R^H),$$

where $\varinjlim^s{}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)}$ denotes the left derived functors of the colimit functor. The $E_2^{0,t}$ line is given by $\varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} \pi_t(R^H)$. We have a natural map

$$\varinjlim_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} \pi_t(R^H) \to \pi_t R^G,$$

and this is surjective for all t if and only if $\pi_0 R^G$ is induced in the sense of Definition 2.20. However, this map is simply the edge homomorphism in a spectral sequence. In particular, $\mathcal{F}$-nilpotence *does not* imply that the homotopy groups are induced. However, it implies something close.

Proposition 3.10 ([20])**.** *Suppose the G-ring spectrum R is $\mathcal{F}$-nilpotent. Then:*

(1) *The Green functor $\pi_*^- R \otimes_{\mathbb{Z}} \mathbb{Z}[1/|G|]$ is induced from $\mathcal{F}$. In particular, the comparison map*

$$\pi_*^G R \to \varprojlim_{\mathcal{O}_{\mathcal{F}}(G)^{op}} \pi_*^H R \tag{12}$$

becomes an isomorphism after inverting $|G|$.

(2) *Moreover, the map* (12) *is an* $\mathcal{N}$*-isomorphism: that is,*
- *Any element in the kernel of this map is nilpotent.*
- *Given* $x \in \varprojlim_{\mathcal{O}_{\mathcal{F}}(G)^{op}} \pi_*^H R$*, there exists* N *such that* x^N *belongs to the image. If* R *is* p*-local, then we can take* N *to be a power of* p*.*

Remark 3.11. The class of $\mathcal{F}$-nilpotent G-spectra is closed under cotensors: if X is a G-spectrum and E is $\mathcal{F}$-nilpotent, then the function spectrum $F(X, E)$ is $\mathcal{F}$-nilpotent. In particular, given an $\mathcal{F}$-nilpotent G-ring spectrum E and a G-space Y, then the G-ring spectrum $F(Y_+, E)$ is $\mathcal{F}$-nilpotent. Therefore, one obtains analogous results for the (Mackey-functor valued) E-cohomology of Y.

3.4. **Some examples of derived defect bases.** Given any G-spectrum M, there is a unique smallest family $\mathcal{F}$ such that M is $\mathcal{F}$-nilpotent.

Definition 3.12. The minimal family $\mathcal{F}$ such that M is $\mathcal{F}$-nilpotent is called the *derived defect base* of M.

We now describe the derived defect bases of the two key examples of equivariant spectra discussed earlier.

Theorem 3.13 (Quillen [28] and Carlson [11]; cf. also Balmer [5])**.** *The* G*-spectrum* $\underline{Hk}$ *representing Borel-equivariant cohomology with coefficients in a field* k *of characteristic* p *is* $\mathcal{F}$*-nilpotent for* $\mathcal{F}$ *the family of* elementary abelian p-subgroups, *i.e., subgroups isomorphic to* $(C_p)^n$ *for some* n*.*

We can apply Proposition 3.10 to the example of $\underline{Hk}$. After inverting $|G|$, there is no interesting information as the higher cohomology is all annihilated. However, the second statement leads to an interesting consequence. In particular, one has the following celebrated result.

Theorem 3.14 (Quillen [28])**.** *Let* G *be a finite group and let* $\mathcal{F}$ *denote the family of elementary abelian* p*-subgroups. Then the comparison map*

$$H^*(G; k) \to \varprojlim_{G/A \in \mathcal{O}_{\mathcal{F}}(G)^{op}} H^*(A; k)$$

is a uniform F*-isomorphism: that is, there exists* N *such that*

(1) *Every element of the kernel is nilpotent of exponent* N*.*
(2) *Given any* $x \in \varprojlim_{G/A \in \mathcal{O}_{\mathcal{F}}(G)^{op}} H^*(A; k)$*,* x^{p^N} *belongs to the image of the comparison map.*

Remark 3.15. Note that the group cohomology is generally difficult to compute for a general group, but there are *certain* groups for which one knows the answer. For instance, one knows the answer for an abelian group. If $G = C_{p^n}$, then we have $H^*(C_p; k) \simeq \Lambda(\alpha_{-1}) \otimes_k P(\beta_{-2})$, and for any abelian p-group, the cohomology is determined by the Künneth theorem as a tensor product of an exterior and a polynomial algebra.

We can also determine the answer for equivariant K-theory.

Theorem 3.16. *The derived defect base of equivariant* K*-theory* KU_G *is given by the family of cyclic subgroups of* G*.*

Remark 3.17. After inverting the order of G, the above result combined with Equation (12) gives that the representation ring Green functor becomes induced from the family of cyclic subgroups after inverting $|G|$. This is, of course, the statement of Artin's induction theorem.

We list some examples for geometrically natural G-spectra.

G-spectrum	Derived defect base
$\underline{H\mathbb{F}_p}$	Elementary abelian p-subgroups (Quillen, Carlson)
$\underline{H\mathbb{Z}}$	Elementary abelian subgroups (at any prime)
$\underline{MU}$	Abelian l-subgroups for any l (Quillen's "complex-oriented descent")
$\underline{E_n}$	Abelian p-subgroups of rank $\leq n$ (Hopkins-Kuhn-Ravenel)
KU_G, KO_G	Cyclic subgroups (Artin, Bojanowska-Jackowski)

3.5. **The main tool.** The main tool in our descent results is a method for transferring rational descent information (which is usually much easier to come by) to chromatic information.

Fix a G-ring spectrum R. Our goal is to consider comparison maps (8) and to prove that, after modifying them by smashing with a telescope, they become an equivalence.

Theorem 3.18. *Let R be an $\mathbb{E}_\infty$-algebra in G-spectra. Suppose $R_{\mathbb{Q}}$ is $\mathcal{F}$-nilpotent. Let T be any telescope. Then $R \wedge T$ is also $\mathcal{F}$-nilpotent. In particular, the maps*

$$T \wedge R^G \to T \wedge \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} R^H \to \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} (T \wedge R^H) \tag{13}$$

is an equivalence.

Proof. We can choose T to be a ring spectrum itself. To show that $R \wedge T$ is $\mathcal{F}$-nilpotent, it suffices to show that the G-ring spectrum $R \wedge T \wedge \widetilde{E}\mathcal{F}$, or equivalently its G-fixed points $(R \wedge T \wedge \widetilde{E}\mathcal{F})^G$, is contractible. Since T is arising as a nonequivariant spectrum here, we have

$$(R \wedge T \wedge \widetilde{E}\mathcal{F})^G = T \wedge (R \wedge \widetilde{E}\mathcal{F}).$$

The condition that $R_{\mathbb{Q}}$ is $\mathcal{F}$-nilpotent is equivalent to the condition that $R \wedge \widetilde{E\mathcal{F}}$ is rationally trivial, where $\widetilde{E\mathcal{F}}$ is an $\mathbb{E}_\infty$-algebra in Sp_G.[9] Equivalently, one has that the non-equivariant $\mathbb{E}_\infty$-ring $(R \wedge \widetilde{E\mathcal{F}})^G$ is rationally contractible. By the next result below, this forces that $T \wedge (R \wedge \widetilde{E\mathcal{F}})^G$ is contractible.

Since $T \wedge R$ is $\mathcal{F}$-nilpotent, it follows that the map

$$T \wedge R^G \to \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} (T \wedge R^H)$$

is an equivalence. To see that the map from the second to the third term in (13) is an equivalence, we observe that if $R' = F(E\mathcal{F}_+, R)$, then $T \wedge R'$ is also $\mathcal{F}$-nilpotent as a ring spectrum receiving a map from $T \wedge R$. Taking G-fixed points and using that (8) is an equivalence, we find that the second and third terms in (13) are equivalent. ☐

Remark 3.19. The condition that $R_{\mathbb{Q}}$ is $\mathcal{F}$-nilpotent is purely algebraic. It is equivalent to assuming that the Green functor $\pi_0^- R_{\mathbb{Q}}$ is induced from the family $\mathcal{F}$.

3.6. **The May nilpotence conjecture.**

Theorem 3.20 ([22])**.** *Let R be an $\mathbb{E}_\infty$-ring spectrum. Suppose $R_{\mathbb{Q}} = 0$. Then $R \wedge T = 0$ for any telescope T.*

Proof. This is a consequence of the nilpotence theorem of [13] together with some analysis with power operations, which is carried out in [22]. ☐

Example 3.21. Let us show that KU_G is nilpotent for the family of cyclic subgroups. When we rationalize, we have $\pi_0 KU_G \otimes \mathbb{Q} \simeq R(G) \otimes \mathbb{Q}$, and this is induced from the cyclic subgroups by Artin's theorem. Therefore, $KU_G \otimes \mathbb{Q}$ is nilpotent for the family of cyclic subgroups.

[9] Though, notably, not a genuine commutative ring spectrum in the sense of Hill-Hopkins-Ravenel.

4. Applications to algebraic K-theory

In this final lecture, we will outline the proofs of our main descent results in algebraic K-theory.

4.1. **The general setup.** We first discuss a generalization of the Galois descent problem in algebraic K-theory. These ideas appear in Thomason [34] and a detailed exposition of them appears in the early sections of the thesis of Merling [25].

Let $\mathcal{C}$ be a category (or ∞-category) that can be the input to algebraic K-theory. For example, $\mathcal{C}$ could be:

(1) An exact category (as in Quillen [29]).
(2) A Waldhausen category (as in Waldhausen [37] or Barwick [8]).
(3) A stable ∞-category (as in [10]).

Suppose $\mathcal{C}$ is acted on by a finite group G. This is not necessarily a strict action, but rather an action up to coherent homotopy.

Example 4.1. Suppose $\mathcal{C}$ is an ordinary category with an action of a finite group G. Let us spell out what this means concretely:

- We have functors $F_g : \mathcal{C} \simeq \mathcal{C}$ for each $g \in G$. We assume $F_e = \mathrm{id}$.
- We have an isomorphisms $c_{g_1 g_2} : F_{g_1} \circ F_{g_2} \simeq F_{g_1 g_2}$.
- For a triple g_1, g_2, g_3, we have an equality of the two natural isomorphisms $F_{g_1} \circ F_{g_2} \circ F_{g_3} \simeq F_{g_1 g_2 g_3}$ that one can write down. (This is the analog of a cocycle condition.)

Fix a G-action on $\mathcal{C}$.

Construction 4.2. We can form the *homotopy fixed points* $\mathcal{C}^{hG}$, which will be of the same form – an exact, Waldhausen, or stable ∞-category. To give an object $X \in \mathcal{C}^{hG}$ amounts to specifying an object $X_0 \in \mathcal{C}$ together with a *coherent choice* of isomorphisms $X_0 \simeq F_g X_0$ for $g \in G$ (satisfying a cocycle condition as above if $\mathcal{C}$ is an ordinary category, and higher analogs if $\mathcal{C}$ is an ∞-category).

Example 4.3. Suppose G acts trivially on $\mathcal{C}$. Then $\mathcal{C}^{hG} = \mathrm{Fun}(BG, \mathcal{C})$ is the ∞-category of objects in $\mathcal{C}$ equipped with a G-action and G-equivariant maps between them.

Example 4.4. Let G be a finite group and let $\gamma \in H^2(G; \mathbb{C}^\times)$. Choose a cocycle $c : G \times G \to \mathbb{C}^\times$ representing α. Then there is a G-action on the category $\mathrm{Vect}_{\mathbb{C}}$ of complex vector spaces such that, for each $g \in G$, the functor F_g is the identity. However, for $g_1, g_2 \in G$, the isomorphism $c_{g_1 g_2}$ is given by multiplication by $c(g_1, g_2)$. Unwinding the definitions, it follows that $\mathrm{Vect}_{\mathbb{C}}^{hG}$ is given by a category of c-twisted representations of G.

In general, algebraic K-theory does not commute with inverse limits of categories. Thus, one may ask:

Question 4.5. How close is the map $K(\mathcal{C}^{hG}) \to K(\mathcal{C})^{hG}$ to being an equivalence?

4.2. **Galois descent.** Our claim is that the above question is a (vast) generalization of the K-theory Galois descent question, through the theory of Galois descent for modules. The reason that this construction is relevant to our problem is the following.

Example 4.6 (Classical Galois descent)**.** Let $E \subset F$ be a G-Galois extension of fields. Then there is a G-action on the category Vect_F of F-vector spaces. Given an F-vector space V, an element of the group $g \in G$ takes V to F-vector space V^g, which has the same underlying set, though the

F-action is twisted by g. The homotopy fixed points are given by the category of E-vector spaces, i.e.,

$$\mathrm{Vect}_F^{hG} \simeq \mathrm{Vect}_E.$$

We describe this explicitly for $\mathbb{R} \subset \mathbb{C}$.

Example 4.7. There is a C_2-action on the category of complex vector spaces $\mathrm{Vect}_{\mathbb{C}}$ which sends a $\mathbb{C}$-vector space V to its complex conjugate $\overline{V}$. Here $\overline{V}$ has the same underlying set as V, but the $\mathbb{C}$-action is twisted by complex conjugation. The homotopy fixed points of the C_2-action are given by the category of pairs (V, ι) where $V \in \mathrm{Vect}_{\mathbb{C}}$ and $\iota : V \simeq V$ is a $\mathbb{C}$-*antilinear* involution. Under the functor $(V, \iota) \mapsto \ker(\iota - 1)$, this category is equivalent to the category of $\mathbb{R}$-vector spaces.

It is possible to formulate a derived version of Galois descent in the setting of faithful Galois extensions of $\mathbb{E}_\infty$-rings. The following result seems to have been noticed independently by several authors.

Theorem 4.8 (Gepner-Lawson, Meier [24], Banerjee)**.** *Let $A \to B$ be a faithful G-Galois extension of $\mathbb{E}_\infty$-ring spectra in the sense of Rognes. Then there is an equivalence of symmetric monoidal ∞-categories* $\mathrm{Mod}(A) \simeq \mathrm{Mod}(B)^{hG}$.

Given an $\mathbb{E}_\infty$-ring A, the algebraic K-theory $K(A)$ of A is defined to be the algebraic K-theory of the stable ∞-category $\mathrm{Perf}(A)$ of *perfect* (or dualizable) A-modules. Taking dualizable objects in the above equivalence, one obtains a Galois descent statement

$$\mathrm{Perf}(A) \simeq \mathrm{Perf}(B)^{hG}.$$

Therefore, Question 4.5 for the G-action on $\mathrm{Perf}(B)$ is precisely the question of how close $K(A) \to K(B)^{hG}$ is to being an equivalence.

4.3. **Equivariant algebraic K-theory.** The idea of Carlsson is that the above map $K(\mathcal{C}^{hG}) \to K(\mathcal{C})^{hG}$ is analogous to the map from fixed points to homotopy fixed points for a G-spectrum. To make that into more than an analogy, one uses the following construction of an *equivariant algebraic K-theory* spectrum which we outline.

Construction 4.9. Let $\mathcal{C}$ be an ∞-category with G-action. Suppose that $\mathcal{C}$ admits finite coproducts.

Let Cat_∞ denote the ∞-category of ∞-categories. There is a functor $F_{\mathcal{C},G} : \mathbf{Burn}_G^{op} \to \mathrm{Cat}_\infty$ as follows:

(1) We have $G/H \mapsto \mathcal{C}^{hH}$.
(2) The restriction morphisms are given as follows. Given a morphism $f : G/H \to G/H'$ in $\mathcal{O}(G)$, we have the obvious functor $f^* : \mathcal{C}^{hH'} \to \mathcal{C}^{hH}$.
(3) The dual morphisms are given as follows. Given a morphism $f : G/H \to G/H'$ in $\mathcal{O}(G)$, we have the functor $f_* : \mathcal{C}^{hH} \to \mathcal{C}^{hH'}$ which is left adjoint to f^*.

Definition 4.10 (Compare Merling [25], Barwick [7], Barwick-Glasman-Shah [9])**.** Let $\mathcal{C}$ be a small stable ∞-category equipped with a G-action.

Compose the functor $F_{\mathcal{C},G}$ with algebraic K-theory (which, as in [10], defines a functor from stable ∞-categories to spectra). We obtain a spectral Mackey functor, or equivalently a G-spectrum, $K_G(\mathcal{C})$ such that, for every $H \leq G$.

$$K_G(\mathcal{C})^H = K(\mathcal{C}^{hH}).$$

We can also ask a generalization of Question 4.5.

Question 4.11. Let $\mathcal{C}$ be a category with a G-action. Given a family $\mathcal{F}$, how close is the map

$$K(\mathcal{C}^{hG}) \to \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} K(\mathcal{C}^{hH})$$

an equivalence? This map is (8) applied to the G-spectrum $K_G(\mathcal{C})$.

4.4. **General descent results.** Suppose $\mathcal{C}$ is a symmetric monoidal, stable ∞-category equipped with a G-action. For each $H \leq G$, we have a natural functor $\mathcal{C}^{hG} \to \mathcal{C}^{hH}$. We let Ind_H^G denote its biadjoint.

Theorem 4.12. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category with a G-action. Fix a family $\mathcal{F}$. Suppose there exist objects $x_H \in \mathcal{C}^{hH}$ for each subgroup $H \in \mathcal{F}$ such that*

$$\sum_{H \in \mathcal{F}} \left[\mathrm{Ind}_H^G(x_H) \right] \in K_0(\mathcal{C}^{hG})$$

becomes a unit in $K_0(\mathcal{C}^{hG}) \otimes \mathbb{Q}$. Then for any telescope T, the map

$$T \wedge K(\mathcal{C}^{hG}) \to T \wedge \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)^{op}} K(\mathcal{C}^{hH})$$

is an equivalence.

Proof. The hypotheses imply that $K_G(\mathcal{C})_{\mathbb{Q}}$ is $\mathcal{F}$-nilpotent. Now $K_G(\mathcal{C})$ is an $\mathbb{E}_\infty$-algebra in Sp_G by Theorem 3.18. ☐

4.5. **Example: Galois descent in algebraic K-theory.**

Theorem 4.13. *Let $A \to B$ be a faithful G-Galois extension and suppose that the map $K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ is surjective. Then the map $T \wedge K(A) \to T \wedge K(B)^{hG}$ is an equivalence for any telescope T.*

Proof. This follows from the above result. By definition, the algebraic K-theory of a ring spectrum R is the algebraic K-theory of the stable ∞-category of perfect R-modules. We take $\mathcal{C} = \mathrm{Perf}(B)$ with the G-action induced from the G-action on B. By Galois descent, we have $\mathcal{C}^{hG} = \mathrm{Perf}(A)$. The induction functor $\mathcal{C}^{hG} \to \mathcal{C}$ corresponds in this picture to the forgetful functor $\mathrm{Perf}(B) \to \mathrm{Perf}(A)$. Thus, this result is a special case of the above. ☐

Proposition 4.14. *Suppose A is an $\mathbb{E}_\infty$-ring spectrum and the following holds:*

1. *$\pi_0 A$ has no nontrivial idempotents.*
2. *$K_0(A)$ and has no nontrivial idempotents.*

Then any faithful G-Galois extension $A \to B$ has the property that $K_0(B) \otimes \mathbb{Q} \to K_0(A) \otimes \mathbb{Q}$ is surjective, so that the above Galois descent result applies.

4.6. **Representation theory.** We consider now a polar opposite to the situation of Galois descent.

Let A be an $\mathbb{E}_\infty$-ring and fix a finite group G. Consider $\mathcal{C} = \mathrm{Perf}(A)$ with trivial G-action. Then for any H,

$$K_G(\mathcal{C})^H = K(\mathcal{C}^{hH}) = K(\mathrm{Fun}(BH, \mathrm{Perf}(A)))$$

We can think of this as a sort of "representation category" over A.

Definition 4.15. For a finite group G, we define $R(G; A) = K_0(\mathrm{Fun}(BG, \mathrm{Perf}(A)))$. With G fixed, the collection $\{R(H, A)\}_{H \leq G}$ naturally assemble into a Green functor on G. In particular, if $H \subset G$, we have induction and restriction morphisms $\mathrm{Ind}_H^G : R(H; A) \to R(G; A)$ and $\mathrm{Res}_H^G : R(G; A) \to R(H; A)$.

Definition 4.16. Fix a finite group G and a family $\mathcal{F}$ of subgroups of G. We will say that *A-based Artin induction holds* with respect to the family $\mathcal{F}$ if there exist classes $x_H \in R(H; A) \otimes \mathbb{Q}$ for each $H \in \mathcal{F}$ such that $1 = \sum_{H \in \mathcal{F}} \mathrm{Ind}_H^G(x_H)$.

Example 4.17. The classical Artin induction theorem states that if $A = \mathbb{C}$, then A-based Artin induction holds for any finite group for the family of cyclic subgroups.

4.7. We can now state the basic connection between Artin induction and descent in algebraic K-theory.

Theorem 4.18. *Fix a finite group G and a family $\mathcal{F}$ of subgroups of G. Let A be an $\mathbb{E}_\infty$-ring. Suppose A-based Artin induction holds with respect to $\mathcal{F}$. Then for any R-linear ∞-category $\mathcal{C}$ with G-action, the G-spectrum $L_n K_G(\mathcal{C})$ is $\mathcal{F}$-nilpotent. In particular,* (**??**) *is an equivalence.*

Proof sketch. Since a module over an $\mathcal{F}$-nilpotent ring spectrum is $\mathcal{F}$-nilpotent, it suffices to consider the case $\mathcal{C} = \mathrm{Perf}(R)$ with trivial G-action. It is now easy to see that $\mathcal{F}$-nilpotence for the rationalization is equivalent to the A-based Artin induction theorem.

□

Example 4.19. Suppose A-based Artin induction holds with respect to the family of subgroups $\mathcal{F}$ of the group G. In this case, the *assembly* map

$$\mathrm{hocolim}_{G/H \in \mathcal{O}_{\mathcal{F}}(G)} K(A[H]) \to K(A[G])$$

becomes an equivalence after L_n-localization.

4.8. **Some examples of Artin induction.** We now give some examples where A-based Artin induction holds for interesting families of subgroups.

Example 4.20. Let $A = \mathbb{Z}$. Then for any finite group G, A-based Artin induction holds for the family of cyclic subgroups by a theorem of Swan [33]. It is easy to see that $R(G; \mathbb{Z})$ is the Grothendieck group of the (exact) category of finitely generated free abelian groups with G-action (which Swan considers).

Example 4.21. Let A be a complex orientable $\mathbb{E}_\infty$-ring (we do not assume the map $MU \to A$ is $\mathbb{E}_\infty$ though). Then A-based Artin induction holds for the family of *abelian* subgroups.

To see this, let G be a finite group. Choose a faithful complex representation V of G and consider the flag variety $F(V)_+$; it admits a finite cell decomposition with abelian stabilizers. In particular, the class that $F(V)_+ \wedge A \in \mathrm{Fun}(BG, \mathrm{Perf}(A))$ defines in $R(G; A)$ is a linear combination of the classes of $G/H_+ \wedge A$ for $H \subset G$ abelian. However, complex orientability also implies that $F(V)_+ \wedge A \in \mathrm{Fun}(BG, \mathrm{Perf}(A))$ is a direct sum of even shifts of the unit. This gives an Artin induction theorem.

Example 4.22. Let $A = KU$. Let G be a finite group such that $|G|$ is only divisible by $2, 3, 5$. Then A-based Artin induction holds for the family of *rank ≤ 2 abelian subgroups.*

This result reduces to the case where $G = (\mathbb{Z}/p)^3$ for $p = 2, 3, 5$. For such p, there exists a nontoral imbedding $G \hookrightarrow \mathcal{G}$ for $\mathcal{G}$ a compact, simply connected Lie group [**?**]. One can then imitate the argument above by considering the G-action on the flag variety of $\mathcal{G}$; it is crucial that the $\mathcal{G}$-action on the flag variety can be trivialized after smashing with complex K-theory.

Conjecture. *For any finite group G, KU-based Artin induction holds for the family of rank ≤ 2 abelian subgroups.*

Appendix A. A primer on nilpotence and descent

A.1. **Thick and localizing subcategories.** Let $\mathcal{C}$ be a stable ∞-category (such as the derived category of a ring, or the homotopy category of spectra) which is idempotent-complete. The latter condition means that if $X \in \mathcal{C}$ and if $e : X \to X$ is an idempotent endomorphism, then there is a splitting in $\mathcal{C}$ given by $X \simeq Xe \oplus X(1-e)$.

Definition A.1. A **thick subcategory** of $\mathcal{C}$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ such that $\mathcal{D}$ is triangulated and idempotent-complete. This means:

(1) $0 \in \mathcal{D}$.
(2) Given an exact triangle $X' \to X \to X''$ in $\mathcal{C}$, if two out of the three objects belong to $\mathcal{D}$, then the third does as well.
(3) If $X \in \mathcal{D}$ and admits a decomposition in $\mathcal{C}$, $X \simeq Y' \oplus Y''$, then $Y', Y'' \in \mathcal{D}$ as well.

Suppose that $\mathcal{C}$ has all colimits, or equivalently all direct sums. Then we say that a full subcategory $\mathcal{D} \subset \mathcal{C}$ is **localizing** if $\mathcal{D}$ is thick and is also closed under colimits (equivalently, all direct sums).

Definition A.2. Given any set S of objects of $\mathcal{C}$, there is a smallest thick subcategory of $\mathcal{C}$ that S generates, denoted $\langle S \rangle$. There is similarly a smallest localizing subcategory $\langle S \rangle_{\mathrm{loc}}$ generated by S.

We now give two basic and important examples of localizing and thick subcategories.

Example A.3. Let $\mathcal{C} = D(\mathbb{Z})$ be the derived ∞-category of $\mathbb{Z}$. Fix a prime number p. Then we have a localizing subcategory $\mathcal{C}_1 \subset \mathcal{C}$ given by those objects $X \in \mathcal{C}$ whose homology is all p-power torsion, or equivalently such that the p^{-1}-localization is contractible, i.e., $X[1/p] \simeq 0$. The fact that $\mathcal{C}_1 \subset \mathcal{C}$ is a localizing subcategory follows easily from the long exact sequence in homology associated to a cofiber sequence of chain complexes and the fact that p^{-1}-localization commutes with direct sums.

Example A.4. We have a thick subcategory $\mathcal{C}_2 \subset \mathcal{C} = D(\mathbb{Z})$ defined as follows. An object $X \in \mathcal{C}$ belongs to $\mathcal{C}_2$ if and only if there exists n such that multiplication by $p^n : X \to X$ is nullhomotopic. A diagram chase shows that if we have a cofiber sequence $X' \to X \to X''$ in $\mathcal{C}$ and if p^a annihilates X' while p^b annihilates X'', then p^{a+b} annihilates X. Using this, one sees that $\mathcal{C}_2$ forms a thick subcategory.

Note the following difference between the above two examples. The subcategory $\mathcal{C}_1$ is not only thick, but *localizing*: it is closed under arbitrary direct sums. However, $\mathcal{C}_2$ is not closed under arbitrary direct sums because $\bigoplus_{i \geq 0} \mathbb{Z}/p^i\mathbb{Z}$ does not belong to $\mathcal{C}_2$. It is the second example that we will primarily be interested in.

The idea is that localizing subcategories are usually specified by "torsion" conditions (such as the condition of being p-power torsion), while thick subcategories can also be specified by analogous *bounded torsion* conditions (such as the condition of being annihilated by a *fixed* power of p). In many cases, when $\mathcal{C}$ is compactly generated, it is easier to classify the thick subcategories of the compact objects $\mathcal{C}^\omega$ than the localizing subcategories of $\mathcal{C}$.

Example A.5. Fix a subset $S \subset \mathrm{Spec}\mathbb{Z}$ of prime ideals in $\mathbb{Z}$. For $s \in S$, we let $k(s)$ denote the residue field at s (so either $\mathbb{F}_p$ or $\mathbb{Q}$). Given a subset $S \subset \mathrm{Spec}\mathbb{Z}$, we have a localizing subcategory $D_S(\mathbb{Z})$ consisting of all objects $X \in D(\mathbb{Z})$ such that $X \otimes_{\mathbb{Z}} k(s) = 0$ for all $s \in S$. Every localizing subcategory of $D(\mathbb{Z})$ is of this form and the localizing subcategories are in bijection with the subsets of $\mathrm{Spec}\mathbb{Z}$. If we take $S = \mathrm{Spec}\mathbb{Z} \setminus \{(p)\}$ for a prime number p, we obtain precisely the localizing subcategory of Example A.3.

Example A.6. Let $\mathcal{C} = \mathrm{Sp}$ be the ∞-category of spectra. In this case, a classification of localizing subcategories of $\mathcal{C}$ is far from known. However, the thick subcategories of the stable ∞-category $\mathcal{C}^\omega$ of finite spectra has been carried out by Hopkins and Smith [17]. Without getting into the details in this lecture, every thick subcategory is cut out by the collection of finite spectra which are acyclic with respect to a certain collection of Morava K-theories.

A.2. **Towers and cosimplicial objects.** Let's give another important example of a thick subcategory.

Example A.7. Fix an arbitrary idempotent-complete stable ∞-category $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$. Let $\mathrm{Tow}(\mathcal{C})$ denote the ∞-category of *towers* in $\mathcal{C}$, i.e., $\mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathcal{C})$. An object is given by a tower

$$\cdots \to X_2 \to X_1 \to X_0$$

and cofiber sequences in $\mathrm{Tow}(\mathcal{C})$ are determined pointwise at each stage. We will say that a tower $\{X_i\}$ is **nilpotent** if there exists N such that each map $X_{i+N} \to X_i$ is nullhomotopic. Note that the nilpotent towers form a thick subcategory $\mathrm{Tow}^{\mathrm{nil}}(\mathcal{C}) \subset \mathrm{Tow}(\mathcal{C})$.

Remark A.8. There is a functor $D(\mathbb{Z}) \to \mathrm{Tow}(D(\mathbb{Z}))$ which sends an object $M \in D(\mathbb{Z})$ to the tower $\cdots \to M \xrightarrow{p} M$. An object M belongs to the subcategory $\mathcal{C}_2$ of Example A.4 if and only if this tower is nilpotent in the above sense.

We will often be dealing with cosimplicial objects rather than towers. Recall [**?**, ??] the ∞-categorical Dold-Kan correspondence between towers and cosimplicial objects in a stable ∞-category which implies that these are essentially the same data.

Example A.9. Let $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{perf}}$. Let $\mathrm{Fun}(\Delta, \mathcal{C})$ denote the ∞-category of cosimplicial objects in $\mathcal{C}$. In this case, one has a totalization $\mathrm{Tot}(X^\bullet)$ which is the inverse limit of the tower $\{\mathrm{Tot}_n(X^\bullet)\}$. We will say that a cosimplicial object $X^\bullet$ is **quickly converging** if it has the property that the quotient tower $\{\mathrm{Tot}_n(X^\bullet)/\mathrm{Tot}(X^\bullet)\}$ is nilpotent.

The notion of a quickly converging cosimplicial object will play an important role for us. Let $X^\bullet \in \mathrm{Fun}(\Delta, \mathcal{C})$. In general, the construction $\mathrm{Tot}(X^\bullet)$ is an infinite homotopy limit (over the category Δ). For example, the construction $X^\bullet \mapsto \mathrm{Tot}(X^\bullet)$ need not commute with a construction like p^{-1}-localization. But when $X^\bullet$ is quickly converging, the totalization $\mathrm{Tot}(X^\bullet)$ behaves much more like a finite homotopy limit.

Proposition A.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be any exact functor. Let $X^\bullet : \Delta \to \mathcal{C}$ be any quickly converging cosimplicial object. Then the cosimplicial object $F(X^\bullet) \in \mathrm{Fun}(\Delta, \mathcal{D})$ is quickly converging and $F(\mathrm{Tot}(X^\bullet)) \simeq \mathrm{Tot}(F(X^\bullet))$.*

Suppose for example $\mathcal{C} = \mathrm{Sp}$. Then we have a spectral sequence

$$E_2^{s,t} = H^s(\pi_t X^\bullet) \implies \pi_{t-s} X^\bullet.$$

The hypothesis of quick convergence has a consequence for this spectral sequence too.

Proposition A.11. *Suppose $X^\bullet \in \mathrm{Fun}(\Delta, \mathcal{C})$ is quickly converging. In this case, the spectral sequence collapses at a finite stage with a horizontal vanishing line.*

A.3. **Approximations by A-modules.** In the rest of this lecture, we will discuss the theory of thick subcategories in the context of approximating objects by modules over an algebra.

Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$ be a symmetric monoidal, stable ∞-category with a biexact tensor product. Let A be an associative (or A_∞) algebra object in $\mathcal{C}$.

Let $X \in \mathcal{C}$. Then we can try to approximate X by objects in $\mathcal{C}$ which are A-modules. For whatever reason, we might expect that A-modules in $\mathcal{C}$ are easier to understand than general objects in $\mathcal{C}$.

Example A.12. If $\mathcal{C} = \mathrm{Sp}$ is the ∞-category of spectra and $A = H\mathbb{F}_p$, then we have an equivalence

$$\mathrm{Mod}_{\mathrm{Sp}}(A) \simeq D(\mathbb{F}_p),$$

and the derived ∞-category $D(\mathbb{F}_p)$ of $\mathbb{F}_p$ is considerably simpler than Sp itself.

Example A.13. Let G be a finite group. Let $\mathcal{C} = \mathrm{Sp}_G$ and let $A = F(G/H_+, S^0)$ be the Spanier-Whitehead dual of the G-space G/H_+ where $H \leq G$. Then $\mathrm{Mod}_{\mathrm{Sp}_G}(A) \simeq \mathrm{Sp}_H$ (cf. [?]). Therefore, approximating a G-spectrum by A-modules amounts to approximating it by G-spectra induced from H.

Assumption A.14. Henceforth, we assume that the unit is compact in $\mathcal{C}$ and A is dualizable in $\mathcal{C}$.

There are essentially two different and dual ways we could try to approximate X by A-modules in $\mathcal{C}$.

(1) Write X as a homotopy limit of A-modules (but where the maps are not necessarily A-module maps).
(2) Write X as a homotopy colimit of A-modules.

There is essentially a tautological way of carrying out each.

Construction A.15. Let $X \in \mathcal{C}$. Then we form a cosimplicial object

$$\mathrm{CB}^\bullet(A, X) = \left\{ A \otimes X \rightrightarrows A \otimes A \otimes X \substack{\to\\ \to\\ \to} \dots \right\}.$$

We have a natural augmentation from X and so we obtain a map $X \to \mathrm{Tot}(\mathrm{CB}^\bullet(A, X))$. The target $\mathrm{Tot}(\mathrm{CB}^\bullet(A, X))$ is called the *A-completion* of X. It is the Bousfield localization of X with respect to A.

Construction A.16. Let $X \in \mathcal{C}$. Then we have a dual construction $\mathbb{D}(\mathrm{CB}^\bullet(A)) \otimes X \in \mathrm{Fun}(\Delta, \mathcal{C})$ which is again augmented over X. The geometric realization of $\mathbb{D}(\mathrm{CB}^\bullet(A)) \otimes X$ is called the **A-cellularization** of X. It is a colimit of A-modules, and is the universal such mapping to X.

Definition A.17. Suppose $\mathcal{C}$ is presentable and the unit is compact. We will say that an object X is **A-complete** if it is equivalent to its A-completion. We will say that an object is **A-torsion** if it is equivalent to its A-cellularization, or equivalently if it belongs to the localizing subcategory of $\mathcal{C}$ generated by the A-modules.

Example A.18. Let $\mathcal{C} = D(\mathbb{Z})$ and let $A = \mathbb{Z}/p\mathbb{Z}$. Then given $X \in D(\mathbb{Z})$, we have that the A-completion of X is given by

$$L_A X \simeq \widehat{X}_p \simeq \varprojlim_n X \otimes_{\mathbb{Z}} \mathbb{Z}/p^n,$$

while the A-cellularization of X is given by

$$\mathrm{Cell}_A(X) \simeq \mathrm{hocolim}_n \Sigma^{-1}(\mathbb{Z}/p^n) \otimes X \simeq \Sigma^{-1}(\mathbb{Z}[p^{-1}]/\mathbb{Z}) \otimes X.$$

An object is A-torsion if and only if it is p-power torsion.

A.4. **Nilpotent objects and the Adams spectral sequence.** In the situation of the previous subsection, we will now single out a class of objects which are well-approximated from both sides by A-modules. These are the *A-nilpotent objects.*

Definition A.19. A thick subcategory $\mathcal{I} \subset \mathcal{C}$ is called a **thick $\otimes$-ideal** if for all $X \in \mathcal{C}, Y \in \mathcal{I}$, we have $X \otimes Y \in \mathcal{I}$. Similarly, we have the definition of a **localizing $\otimes$-ideal.**

Definition A.20 (Bousfield)**.** Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{perf}})$ and let A be an associative algebra object in $\mathcal{C}$. Then we say that an object $X \in \mathcal{C}$ is **A-nilpotent** if it belongs to the thick $\otimes$-ideal generated by A.

Remark A.21. This notion plays an important role in some of the foundational theorems in chromatic homotopy theory.

Proposition A.22. *Suppose $X \in \mathcal{C}$. Then the following are equivalent:*

(1) *X is A-nilpotent.*
(2) *The cobar construction $\mathrm{CB}^\bullet(A, X)$ has totalization given by X (i.e., X is A-complete) and is quickly converging.*

Proof. If X is an A-module, then the cobar construction quickly converges to the appropriate target: in fact, the cobar construction $\mathrm{CB}^\bullet(A) \otimes X$, augmented over X, admits a *splitting* or *extra degeneracy* which gives a cosimplicial homotopy to the constant cosimplicial object X. □

Construction A.23. Suppose that X is A-nilpotent. Then we obtain a spectral sequence

$$H^s(\pi_t(\mathrm{CB}^\bullet(A, X))) \implies \pi_{t-s}X,$$

called the **A-based Adams spectral sequence.**

When X is an A-module, the spectral sequence degenerates at E_2 and everything vanishes in filtration $s > 0$. When X is merely A-nilpotent, the spectral sequence collapses at a finite stage with a horizontal vanishing line.

A.5. **The basic argument.** Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal, stable ∞-category and let A be an algebra object of $\mathcal{C}$. In the previous lecture, we defined an object of $\mathcal{C}$ to be *A-nilpotent* if it belongs to the smallest thick $\otimes$-ideal containing A.

Suppose now that A is dualizable and $\mathbf{1} \in \mathcal{C}$. In that case, we saw (Proposition A.27) that if $R \in \mathcal{C}$ is an *algebra* object, then R is A-nilpotent if and only if R is A-torsion, i.e., belonged to the localizing subcategory of $\mathcal{C}$ generated by the A-modules. The purpose of this lecture is to describe a significant simplification that happens when R has the additional structure of an $\mathbb{E}_\infty$-algebra object.

Assumption A.24. $(\mathcal{C}, \otimes, \mathbf{1})$ is a presentably symmetric monoidal stable ∞-category. A is an algebra object and is dualizable. The unit $\mathbf{1}$ is compact.

We will need to use the following basic construction.

Construction A.25. Fix a prime p and a height n. The assumption that $\mathcal{C}$ is presentable and stable enables us to construct a localization functor $L_n^f : \mathcal{C} \to \mathcal{C}$. This functor annihilates precisely the localizing subcategory generated by objects of the form $F \wedge C$ where F is a type $n+1$ complex and $C \in \mathcal{C}$. Equivalently, L_n^f can be described by smashing with the spectrum $L_n^f S^0$. When $n = 0$, L_0^f is given by rationalization.

The main goal of this lecture is to prove the following result:

Theorem A.26. *Let $\mathcal{C}$ be a presentable, stable ∞-category and suppose the unit is compact. Let $R \in \mathrm{CAlg}(\mathcal{C})$. Suppose $R_{\mathbb{Q}}$ is A-nilpotent. Then $L_n^f R$ is A-nilpotent.*

Theorem A.26 is the key ingredient to proving these K-theory descent results. The rational argument described at the beginning of these lectures (Proposition 1.4) will correspond to the rationalization of an object being A-nilpotent for appropriate A. Theorem A.26 will let us deduce L_n^f-local descent results from rational ones.

We will need the following criterion for nilpotence of *algebra* objects.

Proposition A.27. *If $R \in \mathrm{Alg}(\mathcal{C})$, then R is A-nilpotent if and only if it is A-torsion.*

Proof. The condition that R should be A-torsion is that the natural map

$$|R \otimes \mathbb{D}(\mathrm{CB}^\bullet(A))| \to A$$

is an equivalence. Since homotopy groups commute with filtered colimits, it follows that there exists n such that

$$|\mathrm{sk}_n R \otimes \mathbb{D}(\mathrm{CB}^\bullet(A))| \to R$$

has image in homotopy containing the unit. In particular, we can find a map $R \to |\mathrm{sk}_n R \otimes \mathbb{D}(\mathrm{CB}^\bullet(A))|$ such that the composite

$$R \to |\mathrm{sk}_n R \otimes \mathbb{D}(\mathrm{CB}^\bullet(A))| \to R$$

is the identity. That is, R is a retract of $|\mathrm{sk}_n R \otimes \mathbb{D}(\mathrm{CB}^\bullet(A))|$. Clearly $|\mathrm{sk}_n R \otimes \mathbb{D}(\mathrm{CB}^\bullet(A))|$ is A-nilpotent as a finite colimit of terms each of which is an A-module. Therefore, R is A-nilpotent itself. □

This is an analog of the simple fact that if R_0 is a p-power torsion ring, then in fact R_0 is annihilated by a fixed power of p: specifically, that power which vanishes in R_0. In the next lecture, we will see an additional criterion in the case where R is an $\mathbb{E}_\infty$-algebra object of $\mathcal{C}$.

Our goal is to first reduce to a question about $\mathbb{E}_\infty$-ring spectra.

Reduction to $\mathcal{C} = \mathrm{Sp}$. Suppose $R \in \mathrm{CAlg}(\mathcal{C})$ has the property that $R_{\mathbb{Q}}$ is A-nilpotent. We want to show that $L_n^f R$ is A-nilpotent, or equivalently A-torsion since it is an algebra object. This means that the localization

$$(L_n^f R)[A^{-1}] = 0.$$

Equivalently, form the A^{-1}-localization $T = R[A^{-1}]$, which is an $\mathbb{E}_\infty$-algebra in $\mathcal{C}$. We are given that $T_{\mathbb{Q}} = 0$ and we want to show that $L_n^f T = 0$ as well.

To check that an $\mathbb{E}_\infty$-algebra B in $\mathcal{C}$ vanishes, we can form the construction

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, B) \in \mathrm{CAlg}(\mathrm{Sp})$$

and show that it vanishes. Therefore, if we let $\widetilde{T} = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, T)$, then $\widetilde{T}$ is an $\mathbb{E}_\infty$-ring spectrum. Our hypotheses imply that $\widetilde{T}_{\mathbb{Q}} = 0$, and we need to show that $L_n^f \widetilde{T} = 0$ as well. This is a general statement about $\mathbb{E}_\infty$-ring *spectra* [22]. □

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