SIMPLICIAL COMMUTATIVE RINGS, I

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1. Introduction

Classical algebraic geometry begins with the category $\mathrm{Alg}_{k/,\mathrm{red}}$ of finitely generated, reduced algebras over an algebraically closed field k. The opposite category is the category of *affine algebraic varieties,* and general algebraic varieties are obtained by gluing affine ones in a similar way as manifolds are obtained from charts. After Grothendieck, algebraic geometry begins with the category of *affine schemes*: the opposite category of the category **CR** of *all* commutative rings. In particular, one allows nilpotents, and one allows rings which are very large. For the purposes of DAG, we want to make things a bit more homotopyish, and work with slightly fancier rings.

A natural "homotopical" replacement is provided by the category of topological commutative rings. To avoid point-set technicalities, it is convenient to work in *simplicial* commutative rings. Simplicial sets are a nicer category than topological spaces (they are *presentable* and are specifiable by first-order data).

An alternate motivation for the use of simplicial methods is provided by the following. One of the examples in which one makes a category more homotopyish comes from homological algebra: given an abelian category $\mathcal{A}$, one can form the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$. We might try to generalize this procedure to the category of commutative rings. Commutative rings do not form an abelian category, but we might recall that the Dold-Kan correspondence furnishes an equivalence

$$\mathrm{DK} : \mathrm{Ch}_{\geq 0}(\mathcal{A}) \simeq \mathrm{Fun}(\Delta^{op}, \mathcal{A})$$

between nonnegatively graded chain complexes in $\mathcal{A}$ and simplicial objects in $\mathcal{A}$. This suggests that the use of simplicial methods might allow a way of making a category more homotopyish.

Definition 1. A *simplicial commutative ring* $A_\bullet$ is a simplicial object in the category of commutative rings.

What does this mean? Essentially, we have a simplicial *set* $A_\bullet$, but we also have the structure of a commutative ring on each $A_n, n \in \mathbb{Z}_{\geq 0}$, such that the simplicial operators are ring-homomorphisms with respect to this. A *morphism* of simplicial commutative rings is just a morphism of simplicial sets which in each degree is a ring-homomorphism. This yields a category **SCR** of simplicial commutative rings.

Example. A rather uninteresting example of a simplicial commutative ring is the constant simplicial commutative ring at a given ring R. This gives a fully faithful imbedding

$$\mathbf{CR} \hookrightarrow \mathbf{SCR}.$$

We would expect ordinary rings to imbed inside "brave new" rings, so this is not too surprising.

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Example. Given any simplicial set $X_\bullet$ and any ring R, we can form the simplicial commutative ring $R[X_\bullet]$, which in each degree is the polynomial algebra $R[\{X_n\}]$.

Observe that the singular complex of a topological commutative ring is a simplicial commutative ring, and the geometric realization of a simplicial commutative ring is a topological commutative ring.

2. Homotopy groups

Every simplicial commutative ring is a simplicial group, so it is in particular a Kan complex. Let $R_\bullet \in \mathbf{SCR}$. Then, because $R_\bullet$ is a Kan complex, we can talk about its *homotopy groups* relative to a basepoint $* \in R_0$, which we take to be the zero element. A priori, $\pi_0 R_\bullet$ is only a set, but the fact that $R_\bullet$ is a simplicial abelian group identifies $\pi_* R_\bullet$ with the homology of the associated normalized chain complex (by the Dold-Kan correspondence). Consequently, all the homotopy groups are abelian groups. However, there is additional structure:

Proposition 1. *The homotopy groups $\pi_* R_\bullet$ form a graded commutative ring.*

We consequently get a functor

$$\pi_* : \mathbf{SCR} \to \mathbf{GrCommAlg}_{\geq 0},$$

where $\mathbf{GrCommAlg}_{\geq 0}$ is the category of nonnegatively graded commutative rings.

Proof. Let S^n be the simplicial sphere. Given based maps $f, g : S^n \to R_\bullet$ representing classes in $\pi_n R_\bullet$, the sum $f + g \in \pi_n R_\bullet$ can be be represented by the pointwise sum $f + g : S^n \to R_\bullet \times R_\bullet \stackrel{+}{\to} R_\bullet$, by the Eckmann-Hilton argument. Consequently, all the homotopy groups are abelian groups. Given $f : S^n \to R_\bullet, g : S^m \to R_\bullet$, we define their product in $\pi_{n+m} R_\bullet$ to be

$$f \wedge g : S^m \wedge S^n \to R_\bullet \wedge R_\bullet \to R_\bullet,$$

where the product operation factors through the smash product $R_\bullet \wedge R_\bullet$. This is well-defined, and is easily seen to satisfy the relevant conditions. Graded-commutativity follows from the fact that the involution $S^m \wedge S^n \to S^n \wedge S^m$ creates a sign $(-1)^{nm}$. ☐

For instance, $\pi_0 R_\bullet$ can be obtained as the ring

$$R_0/(d_1 - d_0)R_1.$$

A quick way to see that $(d_1 - d_0)R_1$ is an ideal is to observe that the degeneracy map $s_0 : R_0 \to R_1$ is a section of both d_1 and d_0, and all these maps are ring-homomorphisms.

We want to think of the relationship between simplicial commutative rings and ordinary rings as being something like the relationship between ordinary rings and ordinary *reduced* rings. Just as Grothendieck-style algebraic geometry allows one to add in nilpotents, derived algebraic geometry offers a fancier version of nilpotents: higher homotopy groups. The analog of the canonical map $R \to R_{\mathrm{red}}$ valid for any ring R is the map

$$R_\bullet \to \pi_0 R$$

for any simplicial ring. Here $\pi_0 R$ is regarded as a constant simplicial ring. In fact, this map is just a special case of the map $X_\bullet \to \pi_0 X$ valid for any simplicial set $X_\bullet$ (π_0 is the right adjoint for the inclusion of sets in simplicial sets).

Example. We might think of the relationship between $R_\bullet$ and $\pi_0 R_\bullet$ as a homotopyish version of killing nilpotents; the later talks will make this precise.

Example. Simplicial fields are constant. In fact, the map $R_\bullet \to \pi_0 R$ is surjective in each degree, so if the R_n are fields, then these maps are isomorphisms.[1]

There is an analogous theory of *simplicial modules* $M_\bullet$ over a simplicial ring $R_\bullet$. Such an $M_\bullet$ is equipped with a simplicial map

$$R_\bullet \times M_\bullet \to M_\bullet$$

satisfying all the usual axioms degreewise. Using similar reasoning, we find that there is a natural map

$$\pi_* R \times \pi_* M \to \pi_* M,$$

which makes $\pi_* M$ a *graded module* over the graded ring $\pi_* M$. In particular, $\pi_0 M$ is a $\pi_0 R$-module.

3. The homotopy theory

We said we wanted to do homotopy theory. In fact, we have a good notion of simplicial homotopy in **SCR** (unlike **CR**): the category **SCR** comes with an *enrichment* over the category of simplicial sets. In other words, **SCR** is a *simplicial category*: between any two objects there is a *simplicial set* (rather than simply a set) of maps.

In any category $\mathcal{C}$ with coproducts, the category $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ comes with an enrichment over the category of simplicial sets: one defines the tensor $K_\bullet \otimes X_\bullet$ for $K \in \mathbf{SSet}$ and $X_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ via

$$(K \otimes X)_n = \bigsqcup_{K_n} X_n.$$

The simplicial structure on **SCR** is a special case of this. So, given a simplicial set $K_\bullet$ and a simplicial commutative ring $R_\bullet$, we define

$$(K_\bullet \otimes R_\bullet)_n = \bigotimes_{K_n} R_n.$$

Given $X_\bullet, Y_\bullet \in \mathbf{SCR}$, we define the simplicial set $\underline{\hom}(X_\bullet, Y_\bullet) \in \mathbf{SSet}$ via

$$\underline{\hom}(X_\bullet, Y_\bullet)_n = \hom_{\mathbf{SCR}}(K_\bullet \otimes X_\bullet, Y_\bullet).$$

In a similar way, we can define the structure of a simplicial category on the category of simplicial modules over $R_\bullet \in \mathbf{SCR}$.

This automatically gives us a notion of simplicial homotopy: a *simplicial homotopy* between two morphisms

$$f, g : X_\bullet \rightrightarrows Y_\bullet$$

as a morphism

$$\Delta^1 \otimes X_\bullet \to Y_\bullet$$

which restricts to f on $\Delta^{\{0\}} \otimes X_\bullet$ and to g on $\Delta^{\{1\}} \otimes X_\bullet$.

In fact, **SCR** comes with a *simplicial model structure* that enables us to say more.

(1) A *fibration* in **SCR** is just a fibration of underlying simplicial sets.
(2) A *weak equivalence* in **SCR** is just a weak equivalence of underlying simplicial sets (that is, an equivalence on π_*).
(3) The cofibrations are determined.

[1] I learned this from Mathoverflow: `http://mathoverflow.net/questions/45273/what-facts-in-commutative-algebra-fail-miserably-for-simplicial-commutative-rings`.

The existence of the model structure is not obvious. It can be established by a process called "transfer" which was already implicit in Quillen's work. The setup is as follows. One has an adjunction:

$$F, U : \mathbf{SSet} \to \mathbf{SCR}$$

where F is the free functor and U is the forgetful functor. We already have a nice model structure ("homotopy theory") on **SSet**; the strategy is to "transfer" this to **SCR** in such a way that the above adjunction is actually a Quillen adjunction. Since U has to preserve fibrations and weak equivalences, this gives us a natural definition of those in **SCR**. This process applies more generally to algebraic structures in a model category.

In any event, one has:

Theorem 1 (Quillen)**.** *There exists a cofibrantly generated simplicial model structure on the category* **SCR** *of simplicial commutative rings with the above weak equivalences and fibrations. One obtains a Quillen adjunction between* **SSet** *and* **SCR** *from the free-forgetful adjunction.*

In fact, **SCR** *is a monoidal model category under the tensor product, and it is proper.*

The proof of Quillen's theorem requires some work, and it will not be given. Nonetheless, we will indicate a basic outline. The theorem can be deduced from the following more general result.

Theorem 2. *Let $\mathcal{A}, \mathcal{B}$ be complete, cocomplete categories together with an adjunction $F, G : \mathcal{A} \to \mathcal{B}$. Suppose that:*

(1) *$\mathcal{A}$ is a cofibrantly generated model category.*
(2) *$\mathcal{B}$ admits a path object factorization and a fibrant replacement functor.*
(3) *$\mathcal{A}$ and $\mathcal{B}$ are presentable categories (or satisfy some weaker cardinality hypotheses), and G is an accessible functor. (This is to be able to run the small object argument.)*

Then there is a model structure on $\mathcal{B}$ such that a morphism f in $\mathcal{B}$ is a fibration or weak equivalence if and only if Gf is so.

This result is a machine for transferring model structures along right adjoint functors, and can be used to put model structures on simplicial groups, simplicial associative rings, simplicial Lie algebras, and so forth, by transferring the model structure from simplicial sets along a free-forgetful adjunction.

Outline of proof. Define a map f to be a fibration or weak equivalence in $\mathcal{B}$ if and only if Gf is one in $\mathcal{A}$. Define a map f in $\mathcal{B}$ to be a cofibration precisely if it has the left lifting property with respect to the fibrations which are also weak equivalences.

Choose generating sets I, J of cofibrations and trivial cofibrations in $\mathcal{A}$; then FI and FJ are chosen as generating cofibrations and trivial cofibrations in $\mathcal{B}$. It is "formal" to see the lifting axioms are satisfied (by adjointness). The factorization axioms require more work, and it's also not obvious that FJ actually represents a set of generating trivial cofibrations (or even that the FJ are weak equivalences).

The key strategy to make this work is to show that F preserves trivial cofibrations, by arguing that $F(j)$ for a trivial cofibration j has the left lifting property with respect to all fibrations. One can show, using the second axiom, that any morphism in $\mathcal{B}$ with the left lifting property against all fibrations is a weak equivalence. ☐

Repeating the same arguments, for any ring R, we can form a model category $\mathbf{SCR}_{R/}$ of simplicial R-algebras. This is simply the model structure obtained from the model category **SCR** on the undercategory.

Example. Let $X_\bullet$ be a simplicial set and consider the free simplicial commutative ring $\mathbb{Z}[X_\bullet]$. We'd like to compute the homotopy groups of this simplicial ring. Observe that the homotopy type of this (by the discussion of the model structure, since $\mathbb{Z}[]$ is a left Quillen functor) only depends on the homotopy type of $X_\bullet$ and that the image of this functor lands in the cofibrant-fibrant objects.

We observe that $\mathbb{Z}[X_\bullet]$ is the *symmetric algebra* (dimensionwise) of the free simplicial abelian group $\mathbb{Z}X_\bullet$ As a simplicial abelian group, there is a weak equivalence

$$\mathbb{Z}X_\bullet \simeq \bigoplus K(H_n(X_\bullet), n),$$

where $K(H_n(X_\bullet), n)$ is a simplicial abelian group with one homotopy group in dimension n, equal to $H_n(X_\bullet)$ (i.e., an Eilenberg-MacLane space). In particular, if we assume that the symmetric algebra is independent of the homotopy type, we find

$$\mathbb{Z}[X_\bullet] = \mathrm{Sym}^\bullet \mathbb{Z}X_\bullet \simeq \bigotimes \mathrm{Sym}^\bullet K(H_n(X_\bullet), n).$$

We should be careful, though: here we have to make sure that $K(H_n(X_\bullet), n)$ is taken to be a *cofibrant* simplicial abelian group, for each n, because $\mathbb{Z}X_\bullet$. In other words, we are really taking the *derived* symmetric algebra functor. To be more precise, then we have a weak equivalence in **SCR**

$$\mathbb{Z}[X_\bullet] \simeq \bigotimes \mathbb{L}\mathrm{Sym}^\bullet K(H_n(X_\bullet), n).$$

In principle, we can completely work out the homotopy groups of $\mathbb{Z}[X_\bullet]$ once we know $\mathbb{L}\mathrm{Sym}^k$ and the Künneth formula. Unfortunately, the former is somewhat complicated: see the discussion at `http://mathoverflow.net/questions/97035/derived-functors-of-symmetric-powers/97225#97225`. In [4], a small portion of $\mathbb{L}\mathrm{Sym}^k$ is computed.

The existence of the model structure enables us to define a notion of *simplicial resolution* analogous to the classical notion of a projective resolution in homological algebra.

Definition 2. Let S be an R-algebra. Then a **simplicial resolution** for S is a cofibrant replacement for S in the category of simplicial R-algebras. In other words, it is the data of a trivial fibration of simplicial algebras

$$X_\bullet \to S$$

such that $X_\bullet$ is a cofibrant simplicial R-algebra.

The existence of the model structure enables us to see very efficiently that $X_\bullet$ is unique up to homotopy.

Example. In view of the model structure, we can define the *derived functor* of a left Quillen functor $F : \mathbf{SCR} \to \mathcal{A}$ for a model category $\mathcal{A}$. Namely, we set $\mathbb{L}F(X_\bullet) = F(\widetilde{X}_\bullet)$ for $\widetilde{X}_\bullet$ a cofibrant replacement of $X_\bullet$. This is well-defined up to homotopy.

In a similar vein, we can use the monoidal model structure on **SCR** to define a (symmetric) monoidal structure on the homotopy category given by the *derived tensor product.* Given $A_\bullet \in \mathbf{SCR}$ and $A_\bullet \to B_\bullet, A_\bullet \to C_\bullet$, we define the **derived tensor product** as

$$B_\bullet \overset{\mathbb{L}}{\otimes}_{A_\bullet} C_\bullet = \widetilde{B}_\bullet \otimes_{A_\bullet} \widetilde{C}_\bullet$$

where $\widetilde{B}_\bullet, \widetilde{C}_\bullet$ are cofibrant replacements for $A_\bullet \to B_\bullet, A_\bullet \to C_\bullet$.

Let us try to elucidate what the condition "cofibrant" means. Let $A_\bullet A'_\bullet$ be a cofibration of simplicial sets (that is, a monomorphism). Then $\mathbb{Z}[A_\bullet] \to \mathbb{Z}[A'_\bullet]$ is a cofibration of simplicial commutative rings, since the free functor is a left Quillen functor $\mathbf{SSet} \to \mathbf{SCR}$. If we were working in $\mathbf{SCR}_{R/}$, we would take $R[A_\bullet] \to R[A'_\bullet]$ instead. Moreover, as $A_\bullet \to A'_\bullet$ ranges over a set of generating cofibrations for simplicial sets (for instance, $\partial\Delta[n]_\bullet \to \Delta[n]_\bullet$), we get a set of generating cofibrations for $\mathbf{SCR}_{R/}$, by the argument sketched above. It follows that if we have a map $R \to X_\bullet$ of simplicial commutative rings, then to say that it is a cofibration implies that each X_n is a retract of a polynomial algebra on R—in particular, each X_n is a (formally) *smooth* R-algebra. This smoothness condition suggests that a cofibrant replacement of S might be the appropriate place to apply the relative Kähler differential functor (if one makes the analogy between "smooth" and "projective"), and in a later talk this will be discussed further.

Example. Let R be a ring, and let $S_\bullet$ be a simplicial R-algebra. We say that $S_\bullet$ is **free** if there exist subsets $C_n \subset S_n$ such that:

(1) S_n is R-free (as an algebra) on the set C_n.
(2) For any $\phi : [m] \twoheadrightarrow [n]$, we have $\phi^* C_n \subset C_m$.

In this case, the inclusion $R \to S_\bullet$ is a cofibration in $\mathbf{SCR}$. In fact, $S_\bullet$ can be obtained by "attaching cells" from R for each of the generators in the C_n (well, at least the nondegenerate ones), upwards on n.

Let $R_\bullet \in \mathbf{SCR}$. Then the category $\mathrm{Mod}_{R_\bullet}$ of simplicial $R_\bullet$-modules has been defined. Then $\mathrm{Mod}_{R_\bullet}$ comes with a simplicial enrichment: given a module $M_\bullet$ and a simplicial set $K_\bullet$, one defines the tensor $K_\bullet \otimes M_\bullet = \mathbb{Z}[K_\bullet] \otimes_{\mathbb{Z}} M_\bullet$. It is, in a similar manner, a *simplicial model category*:

(1) The fibrations are the underlying fibrations of simplicial sets (in particular, every simplicial module is fibrant).
(2) The weak equivalences are the weak equivalences of underlying simplicial sets (equivalently, the quasi-isomorphisms of chain complexes).
(3) The cofibrations are determined.

We can obtain this model structure from transfer along the free-forgetful adjunction from simplicial abelian groups, for instance, or from simplicial sets.

4. Examples of simplicial resolutions

The model structure on $\mathbf{SCR}$ (or on $\mathbf{SCR}_{R/}$) enables us to see that we can always find a simplicial resolution for any ring (or R-algebra). Nonetheless, for the purposes of the cotangent complex (or otherwise), we might want to explicitly be able to compute one. Here are two inefficient ways of doing this, one efficient one, and a general fact.

We start with a general finiteness result, which implies, for instance, finiteness properties of the cotangent complex.

Proposition 2. *Let R be a noetherian ring, S a finitely generated R-algebra. Then there exists a simplicial resolution $X_\bullet \to S$ where $X_\bullet$ is a cofibrant (even free) simplicial R-algebra such that X_n is finitely generated over R for each n.*

Proof. We will build up the resolution in stages. Namely, we will find approximations $X_\bullet^{(1)}, X_\bullet^{(2)}, \dots$ in $\mathbf{SCR}_{R//S}$ such that each $X_\bullet^{(i)} \to X_\bullet^{(i+1)}$ is a cofibration and such that $X^{(i)}$ has a homotopy type closer to that of S as i increases.

Choose a surjection $R[x_1, \dots, x_n] \twoheadrightarrow S$. Let $X_\bullet^{(0)}$ be the simplicial R-algebra which is constant at $R[x_1, \dots, x_n]$; then $X_\bullet^{(0)}$ is cofibrant, and one has a map

$$X_\bullet^{(0)} \to S$$

which is a surjection in degree zero. Let $I \subset X_0^{(0)}$ be the kernel; it is a finitely generated ideal by Hilbert's basis theorem. Choose generators $x_1, \dots, x_m$ and for each $1 \leq j \leq m$, a based simplicial map $\phi_j : \partial\Delta[1]_\bullet \to X_\bullet^{(0)}$ sending the non-basepoint to x_j. These defines maps of simplcial R-algebras $R[\partial\Delta[1]_\bullet] \to X_\bullet^{(0)}$.

Let $X^{(1)}$ be the pushout

$$X_\bullet^{(0)} \otimes_{\bigotimes_{j=1}^m R[\partial\Delta[1]]_\bullet} \otimes \bigotimes_{j=1}^m R[\Delta[1]]_\bullet.$$

In other words, we have attached 1-cells to annihilate I. It is easy to see that $\pi_0 X_\bullet^{(1)} = S$, and that $X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)}$ is a cofibration.

Inductively, suppose that i is given, and that we have a sequence

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \dots \hookrightarrow X_\bullet^{(i)} \to S$$

of cofibrations in $\mathbf{SCR}_{R/}$, such that $\pi_j X^{(i)} \to S$ is an isomorphism for $j \leq i$, and such that everything is finitely generated. We want to extend this chain a step further (to make a better approximation). Let $I \subset \pi_{i+1} X^{(i)}$ be the kernel of $\pi_{i+1} X^{(i)} \to S$; choose generators $x_1, \dots, x_m$ represented by maps $\partial\Delta[i+1]_\bullet \to X^{(i)}$. As before, form a pushout

$$X^{(i+1)} = X^{(i)} \otimes_{\bigotimes_{j=1}^m R[\partial\Delta[i+1]_\bullet]} \bigotimes_{j=1}^m R[\Delta[i+1]_\bullet].$$

If we iterate this, we find (since homotopy groups are compatible with filtered colimits) that there is a *cofibrant* simplicial R-algebra $X_\bullet$ with a surjective map to S which is an isomorphism on all homotopy groups. Note that any surjection of simplicial abelian groups is a Kan fibration; consequently, $X_\bullet$ is the requisite simplicial resolution. As $X_\bullet$ was built by attaching a finite number of cells in each dimension, we get the finiteness condition as well. □

Next, we will write down a few examples of simplicial resolutions.

Example (The bar construction)**.** We begin with a general bit of category theory. Let $\mathcal{C}$ be any category, and $T : \mathcal{C} \to \mathcal{C}$ a monad. Let $X \in \mathcal{C}$ be a T-algebra. Then we can form a simplicial object $B(T, X)_\bullet \in \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ (called the **bar construction**). We have

$$B(T, X)_n = T^{n+1} X$$

and the simplicial operators come from the action of T on itself as well as the action of T on X. If one thinks of T as a monoid object in the functor category, this becomes more transparent: it is analogous to the construction of the (simplicial) universal space $EG_\bullet$ of a group G. There is a map of simplicial T-algebras

$$B(T, X)_\bullet \to X$$

(where X is the constant simplicial object). This map comes from applying the action of T to X repeatedly. By "formal" arguments (direct combinatorics), this map is a simplicial homotopy equivalence in $\mathrm{Fun}(\Delta^{op}, \mathcal{C})$ (and thus, often a *weak* equivalence in $\mathrm{Fun}(\Delta^{op}, T-\mathrm{alg})$). In particular, since T is often a "free" functor in some sense, the bar construction provides a way to obtain a simplicial resolution of an object.

For instance, let $\mathcal{C}$ be the category of sets, and let T be the functor which sends a set S to the underlying set of the ring $R[S]$. The algebras over this monad are precisely the R-algebras. Given an

R-algebra R', one gets a simplicial R-algebra $B(T, R')_\bullet$ together with a weak equivalence of simplicial algebras

$$B(T, R')_\bullet \to R'.$$

In fact, $B(T, R')_\bullet$ is *cofibrant* as a simplicial R-algebra, so we have obtained an explicit (if extremely inefficient) choice of simplicial resolution of R'. To see that it is cofibrant, observe that it is free as a simplicial R-algebra in the previous sense. In each dimension, $B(T, R')_n$ is defined as a free algebra on a (large) set of generators; the simplicial operators that raise degree send generators to generators.

We give a (very) slightly more economical illustration of the bar construction.

Example. We have a monad F on the category of R-modules which sends any R-module M to the free R-algebra $FM = \mathrm{Sym}^\bullet M$ on R' (considered as an R-module). This monad comes from the adjunction between R-modules and R-algebras. Note that F takes values in the category of R-algebras, and an algebra over the monad F is just an R-algebra. As in the previous example, we can associate to any R-algebra S a simplicial R-algebra $B(F, S)_\bullet$ together with a morphism of simplicial R-algebras

$$B(F, S)_\bullet \to S$$

which is a weak equivalence of simplicial R-algebras (in fact, a homotopy equivalence of simplicial R-modules). If S is projective as an R-module, then $B(F, S)_\bullet$ is free and hence cofibrant.

Example (Killing a polynomial generator)**.** The real reason we wanted to introduce the bar construction was for this example, from Iyengar's paper in [1]. Consider a ring R and the map $R[y] \to R$ sending $y \mapsto 0$. We can use the bar construction to obtain a simplicial resolution of R as an $R[y]$-algebra.

The strategy is to consider the monad $T = R[y] \otimes -$ in the category of R-algebras, and observe that an algebra over T is an $R[y]$-algebra. We can thus form a simplicial $R[y]$-algebra $B(T, R)_\bullet$ and a map

$$B(T, R)_\bullet \to R$$

which is a weak equivalence. As a simplicial object, we have $B(T, R)_n = R[y]^{\otimes(n+1)}$, and the face simplicial operators come either from the multiplication $R[y]^{\otimes 2} \to R[y]$ or the map $R[y] \to R, y \mapsto 0$. This is cofibrant as a simplicial $R[y]$-algebra (indeed, free) and we have a simplicial resolution.

Very concretely, we see that elements of $B(T, R)_n$ can be represented as sums of R-linear formal symbols

$$g[f_1|f_2|\dots|f_n], \quad g, f_1, \dots, f_n \in R[y].$$

(The g is the $R[y]$-algebra structure on $B(T, R)_n = T^{n+1}R$: it comes from the action of T on the very left.) We can express the face and degeneracy operators as follows:

(1) For $0 \leq i \leq n$, we have

$$d_i(gf_1[f_2|\dots|f_n]) = \begin{cases} gf_1[f_2|\dots|f_n] & \text{if } i = 0 \\ g[f_1|f_2|\dots|f_i f_{i+1}|\dots|f_n] & \text{if } 0 < i < n \\ g\phi(f_n)[f_1|f_2\dots|f_{n-1}] & \text{if } i = n \end{cases}$$

where $\phi : R[y] \to R$ is the homomorphism, $y \mapsto 0$.

(2) For $0 \leq i \leq n$, we have

$$s_i(g[f_1|f_2|\dots|f_n]) = g[f_1|f_2|\dots|f_i|1|f_{i+1}|\dots f_n].$$

We have a map $B(T,R)_\bullet \to R$; this sends a bar $g[f_1|f_2|\dots|f_n]$ to $\phi(g)\phi(f_1)\dots\phi(f_n) \in R$.

As an application of this factorization $R[y] \hookrightarrow B(T,R)_\bullet \twoheadrightarrow R$, we can compute the homotopy groups of the derived tensor product

$$R \overset{\mathbb{L}}{\otimes}_{R[y]} R.$$

This is represented by the simplicial commutative ring $X_\bullet = B(T,R)_\bullet \otimes_{R[y]} R$. Elements of X_n are represented by "bars:" they are sums of R-linear symbols $g[f_1|\dots|f_n]$ where each $f_i \in R[y]$ and $g \in R$; the simplicial operators are given by the same formulas as before.

Let us compute the homotopy groups of $X_\bullet$, or the homology of the Moore complex. Here $X_0 = R$, $X_1 = R[y]$, and the boundary map (for the Moore complex) is given by $\partial([f]) = 0$. So $\pi_0 X_\bullet = R$ (which is not surprising). Here X_2 consists of R-linear combinations of symbols $[f_1|f_2], f_1, f_2 \in R[y]$ and the differential is given by

$$\partial[f_1|f_2] = \phi(f_1)[f_2] - [f_1 f_2] + \phi(f_2)[f_1].$$

So in particular

$$\partial[y^n|y^m] = \begin{cases} -[y^{n+m}] & \text{if } n,m > 0 \\ 0 & \text{if one of } n,m \text{ is } 0 \\ 1 & \text{if } n = m = 0 \end{cases}.$$

In particular, the cokernel of ∂ is precisely a free module of rank one (corresponding to $[y]$). So $\pi_1(X_\bullet) = \pi_0(X_\bullet) = R$.

It gets a little messy after that, but the higher homotopy groups are all zero. This isn't, of course, too surprising: the homotopy groups $\pi_*(R \overset{\mathbb{L}}{\otimes}_{R[y]} R)$ are just the groups $\mathrm{Tor}_i^{R[y]}(R,R)$, which can be computed in terms of the resolution $0 \to R[y] \xrightarrow{y} R[y] \to 0$ of the $R[y]$-module R.

In general, given a simplicial commutative ring $R_\bullet$, and maps $R_\bullet \to S_\bullet, R_\bullet \to S'_\bullet$, there is a spectral sequence

$$\mathrm{Tor}_{p,q}^{\pi_* R_\bullet}(\pi_* S_\bullet, \pi_* S'_\bullet) \implies \pi_{p+q}(S_\bullet \overset{\mathbb{L}}{\otimes}_{R_\bullet} S'_\bullet).$$

See [5].

References

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