SPERNER'S LEMMA, BROUWER'S FIXED-POINT THEOREM, AND THE SUBDIVISION OF SQUARES INTO TRIANGLES

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ABSTRACT. These are notes from a talk I gave for high-schoolers at the Harvard-MIT Mathematics Tournament on two applications of Sperner's lemma: to the Brouwer fixed-point theorem and to a clever plane-geometry result of Monsky. I thank Omar Antolin for suggesting that I look at *Proofs from the Book* for ideas when I realized that I had procrastinated way too much. I also thank Serge Tabachnikov, who first introduced me to this idea during a talk I went to.

Theorem 1 (Monsky [2]). *A square cannot be subdivided into an odd number of triangles of equal area.*

There are two components to this proof:

(1) Combinatorial — Sperner's lemma
(2) Number-theoretical — the 2-adic valuation and its extension to $\mathbb{R}$.

The proof has surprisingly little geometry, but a fair bit of algebra.

1. Sperner's lemma

Let P be a plane polygon. Let $\mathcal{T}$ be a triangulation of P, i.e. a subdivision of the interior of P into triangles. Let the vertices of this triangulation be $\mathcal{V}$, and let $c : \mathcal{V} \to \{0, 1, 2\}$ be a **3-coloring** of the vertices. For simplicity, and without much conceren for creativity, we call the **colors** $0, 1$, and 2.

Given this, we can assign a set of colors to any triangle or edge between vertices in $\mathcal{T}$ as well. For instance, we can call a triangle in $\mathcal{T}$ **full** if all the colors are taken on its vertices.

We will call an edge a **01 edge** if the colors taken on the vertices of the edge are 0 and 1. In fact, this makes sense for the edges on the boundary ∂P, even though these may not strictly lie in the triangulation ∂P.

Lemma 1 (Sperner). *Suppose the number of 01 edges in the triangulation on the boundary ∂P is odd. There is at least one full triangle in P.*

This is really interesting, because we have assumed something only about the boundary ∂P, not the triangulation in the interior. (Draw a picture.) Note that the statement and the conclusion are talking about different things. The statement is talking about edges of the *polygon*. The conclusion is talking about triangles in the *triangulation* (which may be very small).

Example 1. Suppose given a triangulation of a triangle, and a coloring on the vertices, such that the *outer* three vertices (which may not be a triangle in the

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triangulation) take the three different colors $0, 1, 2$ such that v_0 is mapped to 0, v_1 is mapped to 1, and v_2 mapped to 2. Suppose furthermore that for vertices on the edges $[v_i, v_j]$ of the triangle, only the colors i, j are taken.

Then there is a (*small*) triangle in the triangulation which takes the three different colors.

Proof. We will show that the *number* of full triangles is *odd*. Since 0 is not odd, this is it.

Let Δ be a triangle in the triangulation $\mathcal{T}$. Define $\tau(\Delta)$ to be the number of 01 edges contained in Δ.

The claim is:

$\tau(\Delta)$ is odd precisely when Δ is a full triangle.

This is easy to check. If Δ is full, then $\tau(\Delta) = 1$ evidently. Otherwise $\tau(\Delta)$ will be odd. For instance, if Δ is colored $0, 1, 1$, then $\tau(\Delta) = 2$.

In particular, if we show that $\sum_{\Delta} \tau(\Delta)$ is odd, then we will be done.

But this sum counts the 01 edges in $\mathcal{T}$ with certain multiplicities. Edges in the interior are counted twice, because they occur in two triangles $\Delta \in \mathcal{T}$; edges on the boundary are counted once. Mod 2, the hypotheses imply that this sums to 1. ☐

This proof highlights the virtues of sometimes disregarding parts of mathematical structure. Working mod 2 often has many advantages like being able to focus on parity. Another is that signs become gloriously irrelevant. If, as you will see when you study topology, you are working with something particularly sign-heavy, and it becomes a serious concern to make only even numbers of sign mistakes, it is often a welcome relief to quotient by 2.

2. The Brouwer fixed-point theorem

This is one of those hyper-famous results that everyone doing mathematics has to hear—and, for the first time, it must be introduced with wondrous fanfare.

Theorem 2. *Let B^n be the n-dimensional ball in euclidean space $\mathbb{R}^n$ (i.e. the set $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$; the generalization of the unit disk). Then any continuous function $f : B^n \to B^n$ has a fixed point.*

Recall that a **fixed point** of a function f is a *point* x which is *fixed* under f, i.e. $f(x) = x$.

Much of mathematics has concerned itself with the finding of fixed points of functions, or at least the discovery of sufficient conditions for their existence.

Example 2. In the case $n = 1$, this is an exercise based on the intermediate value theorem of elementary calculus.

In higher dimensions, the Brouwer theorem is considerably less trivial. The standard proof, which you will (surely) see when you take a topology course, has it follow as a not-too-difficult consequence of a fairly complicated invariant called *homology.* We will make it follow as a not-too-difficult consequence of a much simpler result, namely Sperner's lemma above, following [1].

Proof in case $n = 2$. A ball is basically a triangle. A mathematician would say that these are *homeomorphic*, i.e. that there is a bijective map between them which is continuous with its inverse. The point is, we shall just prove the result for triangles, because then there is some nice notion of triangulation.

Consider the triangle given by[1]

$$\Delta = \left\{(a_0, a_1, a_2) \in [0,1]^3 : a_0 + a_1 + a_2 = 1\right\}.$$

This is a triangle, whose vertices are the unit vectors $e_0 = (1,0,0), e_1 = (0,1,0), e_2(0,0,1)$. Let $f : \Delta \to \Delta$ be a continuous map. Suppose f has no fixed point. From this, we have to deduce a wacky statement so outrageous that our initial statement must have been false.

Suppose $\mathcal{T}$ is any triangulation of Δ. Given a vertex $v \in \mathcal{T}$, we define the coloring of v to be the minimum i such that $f(v) - v$ has negative ith coordinate. We know that the *sum* of the coordinates of $v, f(v)$ are the same. So if $v \neq f(v)$, one of those differences at least must be negative. In particular, we have not broken any mathematical rules yet.

But I claim that, with this coloring, the vertices e_0, e_1, e_2 are colored differently. This is because e_0 maximizes the zeroth coordinate in Δ, so $f(e_0) - e_0$ must have the first coordinate zero or negative. (It can't be zero, or $f(e_0) = e_0$.) Similarly for the others. More strongly, if we have a vertex $v \in \mathcal{T}$ on the edge $[e_0, e_1]$, then the second (i.e. last, e_2) coordinate is zero. So $f(v) - v$ has nonnegative last coordinate and must be colored zero or one.

Thus we are in the situation of the example.

So we find that there is a small triangle Δ in the triangulation $\mathcal{T}$ such that the three vertices v_0, v_1, v_2 are colored $0, 1, 2$. Do this repeatedly for a sequence of triangulations $\mathcal{T}_1, \mathcal{T}_2, \dots$ of diameters tending to zero. In each of these triangulations $\mathcal{T}_n$, we can find a *small* full triangle Δ_n.

By a little analysis (the Bolzano-Weierstrass theorem), we can find a point $x \in \Delta$ such that there is a descending sequence of triangles $\Delta_{n_1}, \Delta_{n_2}, \dots$ converging to x. We know for each i, there are vertices $v_{i,0}, v_{i,1}, v_{i,2}$ really close to x such that $f(v_{i_0})$ has its zeroth coordinate coordinate less than that of $v_{i,0}$. By approximation, $f(x)$ has its zeroth coordinate at most that of x. Doing the same for the other coordinates, we find that all the coordinates of $f(x)$ are at most that of x. This means that

$$f(x) = x.$$

□

3. The 2-adic valuation

Suppose we had an equal-area triangulation of a square Π. We are going to use a suitable number-theoretic way of partitioning the vertices to show that the number of triangles is even.

Let $m \in \mathbb{Z}$. We define:

Definition 1. The **2-adic valuation** of m, denoted $|m|_2$, is 2^{-d} where d is the number of factors of 2 in m.

If $m = 0$, then $|m|_0 = 0$.

More generally, if $m/n \in \mathbb{Q}$ is a reduced fraction, then we define the **2-adic valuation** of this to be $|m|_2 / |n|_2$.

The 2-adic valuation is simply a measure of how 2-divisible an integer is. The more factors of 2 in a number, the smaller it is.

[1] Like computer programmers, we start our indices at zero.

Example 3. An integer has its 2-adic valuation less than one iff it is even. Otherwise, the 2-adic valuation is one.

It turns out that the 2-adic valuation satisfies properties analogous to the standard absolute value.

Proposition 1. (1) $|mn|_2 = |m|_2\,|n|_2$.
(2) $|m+n|_2 \leq |m|_2 + |n|_2$.

Proof. This is essentially unique factorization. □

Let's say you have number divisible three times by 2. If you add a number divisible four times by 2, then the resulting sum is divisible precisely three times by 2.

This leads to:

Proposition 2. Suppose $x, y \in \mathbb{Q}$ and $|x|_2 < |y|_2$. Then

$$|x+y|_2 = |y|_2\,.$$

Although we shall not need this, it is similarly possible to construct ***p*-adic valuations** for any prime $p \in \mathbb{Z}$.

4. Monsky's proof

The first step is to connect this whole business of valuations with the areas of triangles.

Fix $(x, y) \in \mathbb{Q}^2$. We shall assign a number or "color" to x. This will be used to define a coloring on vertices.

(0) 0 if $|x|_2\,, |y|_2 < 1$. These are the "good" points.
(1) 1 if $|x|_2 \geq 1, |x|_2 \geq |y|_2$.
(2) 2 otherwise. That is, $|y|_2 > |x|_2\,, 1$.

Note that classes 1 and 2 are invariant under translation by points in 0.

Lemma 2. *The 2-adic valuation of a full triangle with vertices in* $\mathbb{Q}^2$ *is* > 1.

Proof. Let $[v_0, v_1, v_2]$ be a full triangle. By translation-invariance, the translate $[0, v_1 - v_0, v_2 - v_0]$ is full as well. So we may reduce to this case of $v_0 = 0$.

So let $v_1 = (x_1, y_1), v_2 = (x_2, y_2)$ be colored 1, 2. The area of the triangle with vertices at v_1, v_2 is given by

$$\frac{1}{2}\,|x_1 y_2 - x_2 y_1|\,,$$

by elementary geometry. But (x_1, y_1) is colored 1. This means $|x_1|_2 \geq |y_1|_2\,, 1$. Similarly we have that $|y_2| > |x_2| \geq 1$. This means that $x_1 y_2$ is the "big" term—it has valuation strictly more than $x_2 y_1$, and more than one. So the difference has valuation more than one, and dividing by two can only push this up further. □

Lemma 3. *Any line can only contain points of at most two types.*

Proof. Suppose not. Again, we may suppose that the line passes through the origin. Let the slope be λ. Suppose that $|\lambda|_2 < 1$; the other case is treated similarly. It is then easy to see that points of type two cannot occur. □

We can now prove:

Theorem 3 (Weak Monsky theorem)**.** *Suppose the unit square S can be subdivided into m triangles of equal area with vertices in $\mathbb{Q}^2$. Then m is even.*

Proof. Suppose given such a subdivision. Then we have a *triangulation* $\mathcal{T}$ of S. Moreover, using the above mapping

$$\mathbb{Q}^2 \to \{0, 1, 2\},$$

we can color the vertices of $\mathcal{T}$. On the boundary ∂S, we have precisely one edge of type 01, namely the horizontal edge between $(0, 0), (1, 0)$. In the triangulation, we have

Thus there is one triangle in $\mathcal{T}$ of full type. So its area must have a 2-adic valuation of more than one, say ξ. But the area of S is $m\xi$. So the 2-adic valuation of m is less than one. This is the same as saying that m is even. ☐

We close by explaining how the weak Monsky theorem, which appears purely number-theoretical in nature, actually extends to the strong one. The point of the argument is that the only reason triangles with $\mathbb{Q}$-rational coordinates were considered is that there is a nice 2-adic valuation on $\mathbb{Q}$. But in fact:

Theorem 4. *There is a function $|\cdot|_2' : \mathbb{R} \to \mathbb{Q}$ such that*

(1) $|\cdot|_2'$ *extends the 2-adic valuation on* $\mathbb{Q}$.
(2) $|xy|_2' = |x|_2' \, |y|_2'$.
(3) $|x + y|_2' \leq \max |x|_2' \, , |y|_2'$.

This theorem is quite sophisticated, and we cannot prove it here. The point is, though, that with this theorem, the previously given proof of the weak Monsky theorem extends verbatim without the restriction that the points have coordinates in $\mathbb{Q}$.

References

1. Martin Aigner and Günter Ziegler, *Proofs from the book*, Springer, 1998.
2. Paul Monsky, *On dividing a square into triangles*, American Math. Monthly (1970).

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