algebraic topology

Lectures delivered by Michael Hopkins
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Contents

Introduction

Michael Hopkins taught a course (Math 231a) on algebraic topology at Harvard in Fall 2010. These are my "live-TEXed" notes from the course.

Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date. Some lectures are marked "section," which means that they were taken at a recitation session. The recitation sessions were taught by Eric Larson.

These notes were typeset using LATEX 2.0. I used `vim` to take the notes. I ran the Perl script `latexmk` in the background to keep the PDF output automatically updated throughout class. The `article` class was used for the notes as a whole. The LATEX package `xymatrix` was used to generate diagrams.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe.

Please email corrections to `amathew@college.harvard.edu`.

Lecture 1
9/1

You might just write a song [for the final].

What is algebraic topology? Algebraic topology is studying things in topology (e.g. spaces, things) by means of algebra. In [Professor Hopkins's] first course on it, the teacher said "algebra is easy, topology is hard." The very first example of that is the Euler characteristic.

Forgive me if you know all there is to know about the Euler characteristic.

The Euler characteristic assigns a number to each geometric object. Take a tetrahedron. It's built out of faces, which look like triangles, and edges, and vertices. Call the shape Σ so that

we feel like we are doing math

Euler had the idea of defining

$$\chi(\Sigma) = V - E + F.$$

For the tetrahedron, we get $4 - 6 + 4 = 2$. The point is that however you triangulate something, you always get the same number. Whenever you triangulate a two-dimensional polyhedron (e.g. a cube, tetrahedron), you get two as the Euler characteristic. A classic soccer ball for instance has twelve pentagons, fifteen hexagons, seventy-five edges, and 75 edges, and fifty vertices. The $V - E + F = 2$.

"I'm pretty sure when Euler saw this, he was like "Double rainbow, all the way.""

Euler gave a classical proof of the Greek theorem about the regular platonic solids, i.e. polyhedra where each face has the same shape, and where each vertex has the same number of edges (i.e. the same number of edges meet at each point). An example of a non-platonic solid would be a tetrahedron with another tetrahedron stuck underneath it. This is because the condition on the vertices at each point is not met.

What are they? There is the tetrahedron, the cube, the octahedron, [Hopkins draws pictures],

"This is the point where a lesser professor might quit drawing pictures."

...dodecahedron, icosahedron.

"I was just born naturally with the ability to do these, it's not that I spent a lot of time doing this."

If one writes down vertices, edges, and faces, one always finds that $V - E + F = 2$, of course.

Euler, using this formula, was able to prove that there are at most five platonic solids.

1.1 Theorem (Euler)**.** *There are ≤ 5 platonic solids.*

Proof. We sketch the proof. Suppose F is the number of faces, and suppose each face is an n-gon. Suppose we have k edges coming together at each point.

1. The number of faces is F
2. The number of total edges is $Fn/2$
3. The number of vertices is Fn/k

We find

$$F - \frac{Fn}{2} + \frac{Fn}{k} = 2$$

where $n \geq 3, k \geq 3$, and all these are integers. You can show that there are five solutions in the integers. Just do a bit of casework. ▲

"It might be fun to think it yourself. Actually, it might be more fun to pretend you did."

What is this χ Euler characteristic an invariant of? All these polyhedra are homeomorphic to the sphere. We could try other shapes. For instance, take a torus, and mark it up in some way, to triangulate it. Regardless of how you triangulate it, you get zero. For the torus with g handles, you get $2 - 2g$ for the Euler characteristic. Thus the Euler characteristic distinguishes the surfaces.

So if you lived on a torus, you could find that out by dividing the place into pastures, and then computing the Euler characteristic. Since the Euler characteristic is algebraic and says something about topology, you could think of this about the beginning of algebraic topology.

One of the things we will do in this class is talk about, as we asked, what the Euler characteristic is an invariant of. Note that a convex body with a whisker still has the same Euler characteristic two. Ideally we would make a definition that did not involve triangulations, but just intrinsically associated a number to a space.

In this course, we shall study ways of attaching algebraic invariants to topological spaces, namely abelian groups. We will study the so-called homology groups, which are a fundamental tool for investigating spaces.

What do we mean by a space? We will often restrict to given useful classes of spaces. In the next part, we will talk about this.

We have different models of topological spaces. For instance, spaces that have some kind of combinatorial flavor. More precisely, simplicial complexes.

1.2 Definition. A **simplicial complex** consists of the following data:

1. A set K of **vertices**
2. A collection $\mathcal{S}$ of finite subsets of K, called **simplices**. Loosely speaking, a simplex gets filled in. So it is required that if $S \in \mathcal{S}$ and $S' \subset S$, we have $S' \in \mathcal{S}$. The only other rule is that $\{v\} \in \mathcal{S}$ for each vertex $v \in K$. (Each vertex has to get filled in.)

1.3 Definition. If $S \in \mathcal{S}$ has cardinality n, we call S an n**-simplex**.

So a two-simplex has two vertices, a three-simplex has three vertices, etc. This is useful for many purposes. For instance, a computer can store this kind of data.

Simplicial complexes are thus highly useful, and even enable you to attach a topological space to things which by themselves have no topology.

"Simplicial complexes are triangle-xenophobic." They don't seem to like other shapes. The platonic solids weren't examples of simplicial complexes, e.g. the cube. You could, however, get one by subdividing the square faces of a cube into triangles to get a simplicial complex.

Euler's formula applies, however, to spaces more general than simplicial complexes. This is a notion that we will talk quite a bit more about in a month, much later into the course, but it is a very important notion.

1.4 Definition (J.H.C. Whitehead in the 1950s)**.** A **CW-complex**[1] is defined as follows. First, some notation:

e^n is the n-disk in $\mathbb{R}^n$, the closed one $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$. We call this an n-**cell**. The boundary $\partial e^n = S^{n-1}$ is the sphere, consisting of vectors of length one.

In a CW-complex, you build a space out of cells. More precisely, it is a space X that comes with a **cell decomposition**. There are a bunch of maps $s_\alpha : e^n_\alpha \to X$ (but n varies, α in an indexing set) such that $s_\alpha : Int(e^n_\alpha) \to X$ is a homeomorphism and X is covered bijectively (as a map of sets) by the images of the $Int(e^n_\alpha)$ under s_α. So X is decomposed into the union of the interiors of these cells.

The two conditions we require:

1. The image of the boundary of an n-cell is contained in a finite union of cells of dimension $< n$.
2. X has the weak topology of the s_α. In other words, $f : X \to Y$ is continuous iff $f \circ s_\alpha$ is for each α.

Lecture 2
9/3

CW Complexes

We will have another lecture on the different ways we think about space. Next Wednesday, we will start on the algebraic side of algebraic topology, i.e. homology.

We go back to what we started last time.

§1 A basic construction

We describe how to "attach a cell." Suppose you have a map from an $n-1$ sphere to a topological space X, i.e. $f : S^{n-1} \to X$. Then you can build the quotient

$$X \cup_f D^n$$

which is the disjoint union $X \sqcup D^n$ with $x \in S^{n-1}$ identified with $f(x) \in X$ (i.e. quotient map).

[1]C. for closure finite and W. for weak topology. But it looks suspicious. "I'll probably make all kinds of editorial comments about general topology in the future."

More generally, if $A \subset B$ is a subspace, given $f : A \to X$, we can define

$$X \cup_f B = X \sqcup B/(a \sim f(a)).$$

We shall write $e^n = Int(D^n)$. This is in contrast to the previous notation we used in the previous lecture (i.e., first lecture).

§2 Definition

A **CW-structure** on a space X is a filtration

$$X^0 \subset X^1 \subset X^2 \subset \dots$$

such that

1. $X = \bigcup X^i$.
2. X^0 is a discrete set.
3. X^n is gotten from X^{n-1} by attaching a bunch of n-cells,

   $$X^n = X^{n-1} \cup_{\sqcup f_\alpha} \sqcup D^n_\alpha.$$

   Here the $f_\alpha : S^{n-1} \to X^{n-1}$.
4. X has the weak topology of the $\{X^n\}$, i.e. a map out of X is continuous if and only if the restrictions to X^n are continuous.

X^n is called the n**-skeleton** of X.

There are many other ways of saying this. You can think of this as a decomposition. Let $e^n_\alpha = Int(D^n_\alpha)$. Then as a set X is the disjoint union $X = \sqcup e^n_\alpha$; each of these goes in homeomorphically to the image, the boundaries don't have to.

We're going to meet a bunch of examples of these guys.

You want to think about this as a manufacturing process.

Consider a torus $\mathbb{T}$. We want to define a CW-structure on it. This was described in class; I can't draw it here.

> It's as if you glued two telephones today and made a doughnut for breakfast.

2.1 Theorem. *Any manifold is a CW-complex.*

Proof omitted.

2.2 Example. $\mathbb{RP}^n$ is a CW-complex. The filtration is

$$\mathbb{RP}^0 \subset \mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \dots \subset \mathbb{RP}^n.$$

This is because to get $\mathbb{RP}^n$ from $\mathbb{RP}^{n-1}$, attach the upper hemisphere and glue the boundary to $\mathbb{RP}^{n-1}$. This is a good nontrivial example of a CW complex.

2.3 Example. Riemann spheres are obtained from polygons with sides identified suitably. This, however, is a CW-complex construction. The endpoints of the polygons are the 0-cells; the edges are the 1-cells; the interior is a 2-cell.

§3 Simplicial complexes

We described simplicial complexes last time, but we need to know how to get a space from it.

Let S be a finite set. Let $\Delta^S \in \mathbb{R}^S$ be such that each $x_s, s \in S$ lies in $[0,1]$ and $\sum_{s \in S} x_s = 1$. This is a **simplex**, and for $|S| = 2$ is a line, for instance.

We have done this for an arbitrary set S, without an ordering of the vertices S. Right now, we have no way of assigning an orientation on the faces of this simplex, for instance; this is just a space associated to the set S.

Given a map $S \to S'$, there is a linear map

$$\Delta^S \to \Delta^{S'}.$$

This is defined in a natural manner. So a map of the vertices yields a map of simplices.

Here is a useful notation. We can define, if $x \in \Delta^s$, we can think of x as a sum of its coordinates x_s times standard basis vectors $e_s, s \in S$. Then this map $\Delta^S \to \Delta^{S'}$ sends $e_s \to e_{s'}$ if $s \to s'$.

Given a simplicial complex $(K, \mathcal{S})$ for K a set and $\mathcal{S}$ a collection of finite subsets (called **simplices**), we define:

2.4 Definition. The **geometric realization** of K is the union

$$\bigcup_{S \in \mathcal{S}} \Delta^S \subset \Delta^K.$$

So this was kind of like Baby Bear? I don't want to be baby bear.

2.5 Definition. We write Δ^n for the **standard** n**-simplex**; this is

$$\Delta^{\{0,1,\dots,n\}} \subset \mathbb{R}^{n+1}$$

and has basis vectors $e_0, \dots, e_n$.

There is a difference between this and Δ^S for S a random set. Δ^{n+1} has an **ordering** on the basis vectors because the vertex set $\{0, \dots, n\}$ is ordered. This is an important point.

One of the consequences of this is that the definition of the chain complex of a simplicial complex is more complicated.

Consider the torus. This is the same as a square with opposite edges identified.

Think space invaders.

This is not a simplicial complex. If you cut it on the diagonal edge, it is not a simplicial complex, because you still have only one vertex. You can triangulate the torus, however, and make it into a simplicial complex with a bunch of triangles. The point is that this is inconvenient and complicated; the CW structure was much simpler. However, the CW complex cannot be stored on the computer.

§4 Δ-complex

There is a nice hybrid of the two notions.

2.6 Definition. A **Δ-complex structure** on a space X consists of a collection of maps from standard complexes $S^n_\alpha : \Delta^n \to X$ (which are homeomorphisms when restricted to the interior $Int(\Delta^n)$) with the property that S^n_α restricted to a face is another one of these maps. Finally, we require that a subset $U \subset X$ is open if and only if the inverse images $S^{-1}_\alpha(U) \subset \Delta^n$ is open for all n.

Let us explain what a face is. Suppose I have a set S and $s \in S$. Then the s-face $\delta_s \Delta^S$ of Δ^S is $\Delta^{S-\{s\}} \subset \Delta^S$; if there is a simplex on $\{a, b, c\}$, then the a-face is the line $[b.c]$ opposite a. Note that the ith face $\delta_i \Delta^n = \Delta^{\{1,2,\dots,i-1,i+1,\dots,n\}}$, which has a canonical isomorphism with Δ^{n-1} because of the ordering.

2.7 Example. There is a nice Δ-complex structure on the torus $\mathbb{T}$ when it is viewed as a square with opposite sides identified. There is only one zero-simplex S_0, two 1-simplices (because there are two edges modulo identification), and one 2-simplex. The same works for arbitrary Riemann surface structures via n-gons with sides identified; they, too, have a Δ-complex structure.

2.8 Definition. A **combinatorial Δ-complex**[2] consists of a set $X_n, n \in \mathbb{N}$ (corresponding to the index set for the maps $S^\alpha_n : \Delta^n \to X$ for a Δ-complex) together with ***i*-th face maps**

$$\delta_i : X_n \to X_{n-1}$$

(corresponding to restriction to the ith face). We have one more restriction based on how the face maps commute.

There is a relation we need to work out. For instance, $d_i d_j$ has to be something like $d_j d_i$ or $d_{j-1} d_i$. There is a better and more elegant way to state it, though: for every order-preserving inclusion $[n] \to [m]$ (where $[n] = \{0, 1, \dots, n\}$), there is a map $X_m \to X_n$ such that if you have two of these,

$$[a] \to [b] \to [c]$$

then the composition $X_c \to X_b \to X_a$ is the same as $X_c \to X_a$ from $[a] \to [c]$. In particular, a combinatorial Δ-complex is a contravariant functor on the category of finite ordered sets.

We will also need to spend time in the course with the following important notion:

2.9 Definition. A **simplicial set** is a collection of sets $X_n, n \in \mathbb{N}$ such that for every order-preserving map (not necessarily an inclusion) $[n] \to [m]$, there is a map $X_m \to X_n$ such that the obvious diagram commutes when there is a triple $[l] \to [m] \to [n]$.

We'll do some more with this, especially in the second semester. You can do all of algebraic topology with simplicial sets instead of topological spaces.

[2]This is a notion being tried so early as an experiment.

Lecture 3
9/8

We've spent a while talking about different models for spaces:

1. A topological space
2. A simplicial complex
3. A Δ-complex. These were defined earlier, and include for instance polygons with edges identified to get Riemann surfaces of arbitrary genus. The point of a Δ-complex is that there is potentially collapsing on the boundary. Moreover, there is an ordering on the vertices of each specified simplex.
4. CW-complexes.
5. Simplicial sets.

We would like to attach algebraic things to topological spaces. We will now explain what sorts of algebraic objects we will attach.

3.1 Definition. A **chain complex** is a collection of abelian groups $C_0, C_1, \dots$ with morphisms

$$C_0 \leftarrow^d C_1 \leftarrow^d C_2 \leftarrow^d \dots.$$

The key condition required is

$$\boxed{d^2 = 0}$$

All the morphisms are denoted by d. In fact, there are variants of the definition: for instance, one may define a complex of R-modules. Here the differential would be required to be an R-homomorphism.

One can make another variant where the complex goes in the negative direction as well. Mostly, we will just be considered with this type of chain complex. There are some invariants one extracts from a chain complex known as **homology**.

We will often denote a chain complex by (C_*, d), or even C_*, where d stands for the differential.

3.2 Definition. The **homology** of (C_*, d), written as $H_n(C)$, is

$$\ker(d : C_n \to C_{n+1})/\mathrm{Im}(d : C_{n-1} \to C_n).$$

This makes sense because $d^2 = 0$.

We will end up associating to a space a chain complex, and we will be interested in the homology of this.

We give an unrelated example of a chain complex.

3.3 Example. Let K be a number field. There is a map $K^* \to \mathrm{div}(K)$, where $\mathrm{div}(K)$ is the free abelian group on the set of primes, i.e. the group of nonzero primes. The cokernel is the divisor class group; the kernel is the unit group. This can be viewed as a chain complex with zeros everywhere else.

3.4 Example. Consider $C_n = \mathbb{Z}$ for all n; and $d = 2$ or $d = 0$ alternatively. So $d : C_n \to C_{n+1}$ is multiplication by 2 for n even; 0 for n odd. The homology groups are $\mathbb{Z}$ at $n = 0$ and then, alternatively, $\mathbb{Z}/2$ and 0.

Question. Find a chain complex C_* such that $H_5(C) = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and all the others are zero.

One example would be to place the $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ in dimension five and zeros everywhere else. If we required each of the groups to be free abelian, we could do this as well with $\mathbb{Z}^2$ in dimensions four and five, and zero everywhere else.

Question. Suppose $R = \mathbb{Z}_2[\epsilon]/(\epsilon^2)$. Find a chain complex of free R-modules such that $H_5(C_*) = R/(\epsilon) = \mathbb{Z}_2$, and all the others zero.

> Nobody in the class responded. I was a student once, and I know how much this is torture—and now I just can't resist doing it.

We can put zeros in C_* up until we get to five, and then put $C_5 = R$. We put $C_6 = R$ and $C_6 \to C_5$ be multiplication by ϵ. This achieves the required homology in degree five and below. However, $\ker(C_6 \to C_5)$ is nonzero, so we need something from degree C_7 to come down and kill that kernel. We have $C_7 = R$ and multiplication by ϵ to take up $\ker(C_6 \to C_5)$. We have to keep putting each of them as R with multiplication by ϵ all the way up.

We're going to do much more with chain complexes in the next lecture or so. Now, we will make a chain complex out of a space.

§1 Chain complexes from Δ-complexes

Let X be a space equipped with a structure of a Δ-complex. That's a collection of maps $s^n_\alpha : \Delta^n \to X$ such that various things happen. The most important thing is that the restriction to each face leaves another one in the list.

Here α runs through some index set A, and n through a subset of $\mathbb{Z}$.

3.5 Definition. The **simplicial chain complex** of X is defined as follows. The n-chains are the free-abelian group on the set $\{s^n_\alpha : \Delta^n \to X\}$. So this is free abelian on the n-simplexes in the delta-structure. We write this as $C^\Delta_*(X)$.

We next need to define d. For an n-simplex map s^n_α, we write:

$$ds^n_\alpha = \sum (-1)^i s^n_\alpha|_{\partial_i \Delta^n}.$$

The **simplicial homology groups** $H^\Delta_*(X)$ are the homology groups of this space.

3.6 Example. Consider the torus with the Δ-structure discussed earlier, namely that with a square with opposing sides identified. Let the vertices be $0, 1, 2, 3$. The square is cut into two triangles, which are simplexes in the Δ-complex. There are two 2-simplices $[0, 1, 2], [0, 2, 3]$. The 1-simplices are $[0, 1] = [2, 3], [1, 2] = [0, 3], [0, 2]$. There is one zero-simplex $[0] = [1] = [2] = [3]$. So $C^\Delta_0(X) = \mathbb{Z}$. $C^\Delta_1(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Also, $C^\Delta_2(X) = \mathbb{Z} \oplus \mathbb{Z}$. It is easy to compute the boundary maps explicitly of these guys, though I am having trouble writing it down fast enough.

The map $C_1^\Delta(X) \to C_0^\Delta(X)$ is the zero map since all the vertices are the same. The first homology group is just the cokernel of $C_2^\Delta(X) \to C_1^\Delta(X)$. The differential from C_2 can be written in matrix form.

FIX THIS AS AN EXERCISE

It's not obvious at all, though we will prove this eventually, that this is independent of the particular Δ-structure, though this is true.

Question. What is the simplicial homology of the genus two surface, a polygon with opposite sides identified? You should get $\mathbb{Z}^4$ in dimension one, $\mathbb{Z}$ in dimension two, and $\mathbb{Z}$ in dimension zero.

Let us now jump to the homology of more general spaces, where we will use *all* maps from a simplex into a space; this is daring, since there are a lot of such maps.

3.7 Definition. Let X be a space. The **singular chain complex** $C_*(X)$ is defined so that $C_n(X)$ is the free abelian group on the set of all maps $\Delta^n \to X$. You don't have special ones as in the Δ-complex; you have a huge set.

The differential d sends $s : \Delta^n \to X$ to $\sum(-1)^i s|_{\partial_i \Delta^n}$. The **singular homology** is the homology of the singular chain complex.

This has a lot of miraculous properties, and we will eventually get lots of practice calculating these homology groups. One theorem we will prove, but not quite in the way we're studying it:

3.8 Theorem. *If X is a Δ-complex, the natural inclusion map $C_n^\Delta(X) \to C_n(X)$ (leading to a map of chain complexes) induces an isomorphism on homology.*

In particular, the homology groups are independent of the Δ-structure.

Question. Suppose instead of a Δ-complex, one had a simplicial complex—how do you get a chain complex?

Lecture 4
9-10

Last time we defined the notion of a chain complex and talked about the **homology** of one. We also discussed the **simplicial homology** $H_*^\Delta(X)$ of a Δ-complex as well as the **singular homology** $H_*(X)$ of an arbitrary topological space. Recall that this last thing is the homology of the chain complex $C_*(X)$, where $C_*(X)$ is the complex whose n-th level is free on the maps $\Delta^n \to X$, and the differential is the alternating sum.

Suppose X is the standard n-simplex with the usual Δ-structure. Namely, the k-simplices are just the order-preserving maps $\{0, 1, \dots, k\} \to \{0, \dots, n\}$. This is a convenient combinatorial way of writing the Δ-structure. Let us consider the complex $C_*^\Delta(\Delta^n)$.

4.1 Example. When $n = 0$, we have $\mathbb{Z}$ in dimension zero and zero everywhere else. The homology is thus zero in dimension zero, zero everywhere else. This is also true for singular homology.

4.2 Example. When $n = 1$, the simplex is a line connecting two points. The zero-group (free on the 0-simplices) are the free abelian group on the vertices $[0], [1]$. The 1-group is the free abelian group on $[0, 1]$. The boundary map sends $[0, 1] \to [0] - [1]$. The first homology group is thus $\mathbb{Z} \oplus \mathbb{Z}$ with the first factor identified with the second, hence is isomorphic to $\mathbb{Z}$.

4.3 Example. When $n = 2$, we have a diagram

$$\mathbb{Z}\{[012]\} \to \mathbb{Z}\{[01][12][02]\} \to \mathbb{Z}\{[0][1][2]\}.$$

This is the Δ-chain complex. The cokernel at the zeroth level has the three generators $[0], [1], [2]$ identified so $H_0^\Delta(X) = \mathbb{Z}$.

Consider the kernel at the first level. A typical element of C_1^Δ is given by a formal sum

$$a[01] + b[02] + c[12].$$

The boundary map sends this to

$$-(a + b)[0] + (a - c)[1] + (b + c)[2].$$

For this to be zero, $a = c = -b$. In other words, this (a, b, c) has to look like $(a, -a, a)$. The kernel is generated by $(1, -1, 1)$, which corresponds to the image of $d([012])$. In particular,

$$H_1(X) = 0.$$

It's easy to see that any 1-chain (i.e. element of $C_1^\Delta(X)$) killed by d must go all the way around the boundary, in a loose sense.

4.4 Definition. In a chain complex C_*, we define $Z_n = \ker(d : C_n \to C_{n+1})$; it is called the group of n**-cycles**.[3]. The image $B_n = \mathrm{Im}(d : B_{n+1} \to B_n)$ is called the group of n**-boundaries**.

We know that

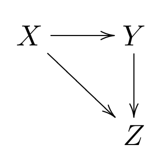
$$Z_n/B_n \simeq H_n(C_*).$$

We now move into some general properties about **singular** homology, which we will see applies to simplicial homology.

1. Singular homology is **functorial**. Suppose $f : X \to Y$ is continuous. Then there is a natural map $C_*(X) \to C_*(Y)$ sending $\Delta^n \to X$ to the composite $\Delta^n \to X \to Y$. This induces maps on homology.

   In particular, there are maps $f_* : H_*(X) \to H_*(Y)$.

   If f is the identity map, so is the map on homology. The other interesting property is that a commutative diagram



[3] The name is based on the remark at the end of the example

leads to a commutative diagram of homology

$$\begin{array}{ccc} H_*(X) & \longrightarrow & H_*(Y) \\ & \searrow & \downarrow \\ & & H_*(Z) \end{array}$$

These are essentially formal properties, and explain the functoriality. The functoriality is immediate from the definitions, since it is essentially all about composing simplices.

2. Homotopy invariance. This is definitely **not** a formal property.

4.5 Proposition. *Suppose $f, g : X \to Y$ are homotopic; this means there is $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x), H(x,1) = g(x)$. Then the induced maps in homology $f_*, g_* : H_*(X) \to H_*(Y)$ are equal.*

We will prove this below.

In particular,

4.6 Corollary. *If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.*

Proof. This is evident. Suppose $g : Y \to X$ is a homotopy inverse. Then fg, gf are homotopic to the identities; so $(fg)_* = f_*g_*, (gf)_* = g_*f_*$ are equal to the identity on the appropriate homology groups. So f_*, g_* are inverses. ▲

4.7 Example. Δ^n and a point Δ^0 are homotopy equivalent. This is because Δ^n is contractible (given a vertex v_0, push any vector v along the line with v, v_0 towards v_0 and deform Δ^n into a point). So the homology is the same. In particular, the singular homology of Δ^n is $\mathbb{Z}$ in degree zero and zero elsewhere.

We now begin our descent towards the proof of homotopy invariance. First, we begin with the notion of **chain homotopy**.

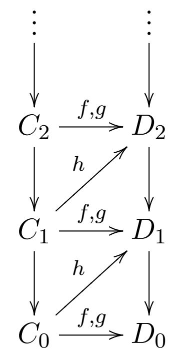
4.8 Definition. Suppose C_*, D_* are chain complexes and $f, g : C_* \to D_*$ are morphisms. A **chain homotopy** between f, g is a collection of maps

$$h : C_n \to D_{n+1}$$

such that

$$dh + hd = g - f.$$

Let's draw a diagram



Now we can do this recursively. Suppose h is defined in dimensions less than n. Suppose $c \in C_n$. Then we want

$$(f - g)c = dhc + hdc,$$

i.e.

$$d(hc) = (f - g)c - h(dc)$$

where the right side is all determined. So to define h in dimension n, we have to pick it as an element hc (and make $c \to hc$ such that its boundary is something pre-determined.

4.9 Lemma. *Suppose f, g chain homotopic; then $f_* = g_* : H_*(C) \to H_*(D)$.*

Proof. If you have a cycle in C_n, then the difference $f - g$ on that will differ by a boundary. So f, g induce the same maps in homology. In detail, if $z \in Z_n \subset C_n$ is a cycle, then

$$f(z) - g(z) = dh(z) + hd(z) = dh(z) \in B_n.$$

So the residue classes of $f(z), g(z)$ in $H_n(C)$ are equal. ▲

We will now prove the theorem that homology is homotopy invariant.

Proof. To prove this, we will show that if $f, g : X \to Y$ are homotopic. We want to prove that they induce the same map in homology. In particular,

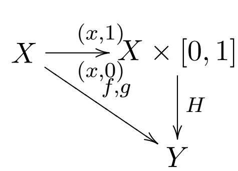
$$f_* = g_* : H_*(X) \to H_*(Y).$$

We will show that the induced maps by f, g

$$C_*(X) \to C_*(Y)$$

are chain homotopic. This will imply the claim. We will write down a formula, but first we will mess around to show something important.

We can reduce this to a specific situation. Namely, it is enough to assume $Y = X \times [0, 1]$ and $f(x) = (x, 0)$ and $g(x) = (x, 1)$. The reason is that there is a diagram



So if the two maps $H_*(X) \to H_*(X \times [0,1])$ are the same, then the two maps $H_*(X) \to H_*(Y)$ are the same by commutativity of the diagram. The point is that we can use functoriality to reduce to universal cases. This is a fundamental point. Later, we will study the theory of categories and this abstract nonsense will get even more fun.

The second reduction is as follows. Namely, we will try to write a chain homotopy between the two maps $f, g : C_*(X) \to C_*(X \times [0,1])$ that works for every X and works for every space. Suppose we have a typical chain $\sum s_\alpha$ where each $s_\alpha : \Delta^n \to X$. We will try to find a "universal formula" hc such that

$$dhc + hdc = fc - gc.$$

Since this is going to be linear, we need only define h on s_α. I claim:

> By naturality, it is enough to define hs where $s \in C_n(\Delta^n)$ is the identity map such that $hs \in C_{n+1}(\Delta^n \times [0,1])$.

This is because any $s_\alpha : \Delta^n \to X$, we get a map

$$s_\alpha : C_*(\Delta^n) \to C_*(X)$$

that sends s to s_α. If we define hs, we map this under s_α to define hs_α. I was messing up the diagrams that I was TeXxing up.

▲

We should do the general formula for how to define hs for s the standard (identity) n-simplex. However, there's no time. Will do this next class; now for an example.

4.10 Example. Consider Δ^0 and $s : \Delta^0 \to \Delta^0$. Then we have that $\Delta^0 \times [0,1] = [0,1]$. The 1-simplex $\Delta^1 \to \Delta^0 \times [0,1]$ which is the obvious homeomorphism has boundary equal to the difference of the two endpoints. So we can send s to this 1-simplex via h to get the chain homotopy.

Lecture 5
9/13

§1 Completion of the proof of homotopy invariance

We're in the middle of something a little technical, but just bear with us. It's more important to understand the style of the argument, since you will see this many more times. It's kind of like when you see δ, ϵ arguments in calculus for the first time; later on they make much more sense. The tools made with this material will be useful.

So, recap: We defined a thing called **singular homology**. We were in the middle of showing that it was homotopy invariant. In other words, if we had a map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x), F(x,1) = g(x)$, then the induced maps on homology

$$f_*, g_* : H_*(X) \to H_*(Y)$$

are the same.

We introduced this notion last time of a **chain homotopy**, which was a purely algebraic notion. Suppose one has two maps of chain complexes $s, t : C_* \to D_*$; a chain homotopy is a map $h : C_* \to D_{*+1}$ (shifting the degree by 1) such that

$$dh + hd = t - s,$$

implying that $t_*, s_* : H_*(C) \to H_*(D)$.

We wanted to show that homotopic maps of topological spaces induce chain-homotopic maps complexes. Today, we will give an explicit construction of the chain homotopy.

Consider the standard simplex Δ^n; label its vertices $e_0, \dots, e_n$. Consider the product $\Delta^n \times I$, the bottom vertices of which we label $v_0, \dots, v_n$, and the top vertices of which we label $w_0, \dots, w_n$. We will divide the product $\Delta^n \times I$ into a simplicial complex. In particular, we decompose this into

$$[v_0 w_0 w_1 w_2 \dots w_n], [v_0 v_1 w_2 \dots w_n], \dots.$$

The geometric idea is to go along the v's for a while, and then make a jump up to the w's and stay there. This forms a simplicial decomposition of the prism into a simplicial complex.

Now let us give a chain homotopy h between the maps $C_*(X) \to C_*(X \times I)$ induced by the two inclusions $X \rightrightarrows X \times [0,1]$. This will be enough to prove homotopy invariance, by what has been discussed in the last lecture.

Let $\sigma : \Delta^n \to X$ be a singular n-simplex. We have to say what $h(\sigma)$ is. We do so as follows:

$$h\sigma = \sum_i (-1)^i (\sigma \times 1)_* [v_0 \dots v_i w_i \dots w_n].$$

First, to make sense of what the heck this is. Now

$$[v_0 \dots v_i w_i \dots w_n]$$

is a $n+1$-simplex in $\Delta^n \times [0,1]$. This when composed with $\sigma \times 1 : \Delta^n \times I \to X \times I$ gives a $n+1$ simplex in $X \times [0,1]$. The alternating sum of these is $h\sigma$.

It can be checked that

$$dh + hd = (i_1)_* - (i_0)_*$$

for $i_0, i_1 : X \rightrightarrows X \times I$. We leave the proof, which is computation, to the reader!

Despite the fact that we only sketched the proof, we will regard that it is now **completely proved** that homology is a homotopy invariant.

§2 Excision

We will lay the groundwork for this today, and state the main technical result. Excision, together with homotopy invariance, lets you compute singular homology. We will start by sketching the heart of this.

Philosophy. Homology can be calculated by only using tiny simplices. "Tiny" will be defined in the sequel.

Here is a mechanism of shrinking simplices. This is called **barycentric subdivision**. We first describe it intuitively.

1. For a 1-simplex, divide into halves.
2. For an n-simplex, subdivide the boundary. Then take the point in the middle of the n-simplex and join it to every vertex on the boundary (in the subdivision).

We now give a formula for barycentric subdivision, which will apply to **any linear simplex**. If $v_0, \dots, v_n \in \mathbb{R}^M$, we will write

$$[v_0 \dots v_n]$$

for the n-simplex spanned by them, as usual.[4] This is the collection of sums $\sum t_i v_i$ where $\sum t_i = 1$ and each $t_i \geq 0$.

5.1 Definition. The **barycenter** is the point

$$\frac{1}{n+1} \sum v_i \in [v_0 \dots v_n].$$

This is where all the coefficients t_i are the same.

5.2 Definition. The vertices of the **barycentric subdivision** of an n-simplex are the barycenters of the sub-simplices. (In particular, to each subset of $\{v_0, \dots, v_n\}$ there is a vertex.)

A simplex in the barycentric subdivision is a collection $[b_0 \dots b_n]$ where b_0 is the barycenter of $[v_0 \dots v_n]$, b_1 a barycenter of an $n-1$-subsimplex, b_2 a barycenter of a $n-2$-subsimplex of the $n-1$-simplex and b_n a 0-simplex. So the simplices are in bijection with the descending sequences of subsimplices of $[v_0 \dots v_n]$.

We now return to the philosophy that homology depends only on "small" simplices.

5.3 Definition. The **diameter** of a simplex $[v_0 \dots v_n]$ is the maximal distance between any two points in it, which is (as one can easily show) $\sup_{i,j} |v_i - v_j|$. (The points that are the furthest apart are the vertices.)

It is clear that when one does the barycentric subdivision, one gets maps out of simplices which are not the regular n-simplex—when you subdivide an equilateral triangle, you don't get equilateral triangles. In fact, they look pretty weird. That's ok, but we want to assure ourselves that they are at least shrinking.

5.4 Lemma. *The diameter of each simplex of the barycentric subdivision of* $[v_0 \dots v_n]$ *is at most* $\frac{n}{n+1} \operatorname{diam}[v_0 \dots v_n]$.

Proof. Left to the reader—straightforward. ▲

Notice that barycentric subdivision need not shrink the simplex a whole lot, but if you keep iterating it, you eventually will get only really small simplices, because $\frac{n}{n+1} < 1$.

[4] Note that all the v_i may be identical!

> "It would be kind of cool to have that lemma the same as that one as that one, but only the last one's right."

Here's where we're headed. We are going to define a subdivision operator S

$$S : C_*(X) \to C_*(X)$$

on the singular chains of any topological space X obtained by restricting each $\sigma : \Delta^n \to X$ to the sum of simplices in the barycentric subdivision of Δ^n. We're going to show

1. S is a chain map, namely it commutes with the differential d.
2. S is chain homotopic to the identity.

Thus S^k for any k is chain-homotopic to the identity. In particular, this gives some substance to the philosophy that homology can be computed in terms of small simplices. But this is the technical heart of the approach.

The current plan for the course is not to go into the technical details, but to work on using excision rather than delving into the proofs of the two items above.

Lecture 6
[Section] 9/13

§1 A discussion of naturality

We will go over in more detail the reduction made before in lecture 4. The main point was in the proof of

6.1 Theorem. *If $f, g : X \to Y$ are homotopic, then the maps induced on homology are the same. More precisely, the maps*

$$C_*(X) \to C_*(Y)$$

are chain homotopic.

The definition of chain homotopy has already been given in these notes, and we do not review it here.

Instead of going right to this problem, let's look at a simpler example of a problem like this.

Problem (Simpler problem)**.** For every abelian group G, suppose given a map t_G

$$t_G : G \to G$$

such that for any map $g : G \to G'$, we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{t_G} & G \\ \downarrow & & \downarrow \\ G' & \xrightarrow{t_{G'}} & G' \end{array} .$$

In other words, we want an endomorphism of the identity functor.

To solve this, we will look at a "universal" example. Consider the maps $\mathbb{Z} \to \mathbb{Z}$; the only maps are multiplication by a. Suppose that our map $t_{\mathbb{Z}}$ is multiplication by n.

I claim that t_G is multiplication by n for any n. Indeed, pick $g \in G$ for a group G. For the $\mathbb{Z} \to G$ sending 1 to g, consider the diagram

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ G & \longrightarrow & G \end{array}$$

The commutativity and the fact that the horizontal row is multiplication by n implies that g must be sent to ng.

Anyway, back to the original problem. For any pair $f, g : X \rightrightarrows Y$ which are homotopic, we want to construct the chain homotopy between the maps on the chain complexes. It would be cool to make this natural, though. So for every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow[g]{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow[g']{f'} & Y' \end{array}$$

with the maps f, g and f', g' homotopic, then the chain homotopies

$$\begin{array}{ccc} C_*(X) & \xrightarrow{h} & C_*(Y) \\ \downarrow & & \downarrow \\ C_*(X') & \xrightarrow{h} & C_*(Y') \end{array}$$

make a commutative diagram. To do this, we look at the universal case. This universal case consists of the two maps

$$i_0, i_1 : X \rightrightarrows X \times [0,1].$$

In this situation, for any pair $X \rightrightarrows Y$ of homotopic maps, we can split it into

$$X \rightrightarrows X \times [0,1] \to Y.$$

So if we define our chain homotopies for the case of i_0, i_1, we can compose it with additional things to define the homotopies for anything.

Next, Hopkins reduced to the case for $X = \Delta^n$. This is because any singular n-simplex $\Delta^n \to X$ can be expressed as the identity simplex $\Delta^n \to \Delta^n$ composed with some map $\Delta^n \to X$. In particular, we have a diagram (commutative by the naturality insistence)

$$\begin{array}{ccc} C_*(\Delta^n) & \xrightarrow{h} & C_{*+1}(\Delta^n \times [0,1]) \\ \downarrow & & \downarrow \\ C_*(X) & \xrightarrow{h} & C_{*+1}(X \times [0,1]) \end{array}$$

and if we have defined the top maps in every case, the bottom maps must be determined (since the maps $\Delta^n \to X$ lead to all the n-simplices). This is why we could make all those reductions.

Lecture 7
9/15

§1 Excision

Last time we sketched something which was a little bit grungy. We're trying to put a little substance to the philosophical comment that **Homology depends only on small chains**. We're going to prove something that gives a real mathematical statement to this, and prove some of the consequences.

Here is the first theorem that we want to prove.

Suppose X is a space and $\mathfrak{A} = \{U_\alpha\}$ is a collection of subsets such that $Int(U_\alpha)$ covers X. In practice, we always want the U_α to be open anyway. We want to consider now, instead of the whole singular chain complex, the subgroup $C_n^{\mathfrak{A}}(X)$, which is the free abelian group on the set of maps $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n)$ is contained in the interior of some U_α. We are looking at simplices which are contained in one of the interiors.

Then $C_*^{\mathfrak{A}}(X)$ is a chain complex since if a chain sits inside one of the U_α, so does its boundary. It is a subcomplex of $C_*(X)$.

7.1 Theorem. *The inclusion* $C_*^{\mathfrak{A}}(X) \to C_*(X)$ *induces isomorphisms on homology.*

In particular, when computing the homology, we can just restrict to small chains. We will prove this today while using the general approach of barycentric subdivision. You can in fact show that the inclusion induces a chain homotopy equivalence, though we won't do this.

Recall first the **subdivision operator** $S : C_*(X) \to C_*(X)$ which sends a chain $\sigma : \Delta^n \to X$ to the sum $\sum (\pm)\sigma|_{\Delta^n}$ over σ being restricted to the various simplices in the barycentric subdivision. The hard thing is to work out the appropriate signs. The signs are forced on you, and they are necessary to make it a chain map.

1. S is a map of chain complexes, i.e. it commutes with the boundary: $Sd = dS$.

2. S is chain homotopic to the identity. In other words, there are maps $h : C_n \to C_{n+1}$ such that
$$dh + hd = 1_C.$$

Using this, we will prove the theorem.

We will now use a simple fact.

7.2 Exercise. Chain homotopy is an equivalence relation on $\mathrm{Hom}(C_*, D_*)$ for C_*, D_* complexes.

It now follows that since $S \sim 1$ (we use $\sim$ to denote chain homotopy), we have $S^2 \sim S \sim 1$, and by induction

$$S^n \sim 1$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. We start by showing the map is **onto**.

Suppose $c \in H_n(X)$. Choose a cycle $z \in C_n(X)$ representing c; i.e., z maps to c in the reduction map $Z_n(X)/B_n(X) \to H_n(X)$. We apply this operator S to z.

Write $z = \sum(\pm)\sigma_i$. Each $\sigma_i : \Delta^n \to X$ is a collection of maps. Since the $Int(U_\alpha)$ cover X, the inverse images $\sigma_i^{-1}(Int(U_\alpha))$ cover Δ^n for each i. So we have a finite collection of covers of Δ^n.

Now choose m so large such that $(n/n+1)^m$ is smaller than the Lebesgue number (see below) of each covering $\sigma^{-1}(Int(U_\alpha))$. For all i, it follows that σ_i restricted to each simplex in the m-th barycentric subdivision—which has diameter at most $(n/n+1)^m$ and consequently is contained in some $\sigma_i^{-1}(U_\alpha)$—belongs to $C_n^{\mathfrak{A}}(X)$, because the image lies in some U_α. In particular, when you subdivide the simplex into a lot of little simplices, each little piece has image lying in one of the $Int(U_\alpha)$.

In particular, $S^m(z) \in C_n^{\mathfrak{A}}(X)$. There exists h such that $dh + hd = S^m - 1$, though, by chain homotopy. This means that

$$S^m z - z = dh(z) + hd(z) = dh(z)$$

so that $S^m z, z$ have the same image in $H_m(X)$, i.e. are in the same homology class. Thus the map $C_*^{\mathfrak{A}} \to C_*$ induces surjections on homology since S^m is in $C_n^{\mathfrak{A}}(X)$.

Now we show that the map induces **injective** maps on homology. Suppose we have a class $c \in H_n(C_*^{\mathfrak{A}}(X))$ which goes to zero in $H_n(X)$; we have to show that c is itself zero. Choose a cycle $z \in C_n^{\mathfrak{A}}(X)$ representing c. Then $z = dw$ for some $w \in C_{n+1}(X)$ because z is a boundary in the full chain complex, but we don't know that $w \in C_{n+1}^{\mathfrak{A}}(X)$.

By the same discussion, for m really large, we have $S^m w \in C_{n+1}^{\mathfrak{A}}(X)$. It follows that $d(S^m w) = S^m dw = S^m z$. since S is a chain map. There is a remark we need to make:

1. S induces a map $C_n^{\mathfrak{A}}(X) \to C_n^{\mathfrak{A}}(X)$. This is obvious; when you subdivide small chains, you get small chains.

2. This map is chain-homotopic to the identity. This isn't as obvious. The chain homotopy preserves the subcomplex. We won't prove this, though.

Anyway, so the homology classes of $S^m z, z$ in $H(C_*^{\mathfrak{A}}(X))$ are the same. It follows that the homology class of $S^m z$ in this homology group is zero because $S^m z = d(S^m w)$, and $S^m w$ is contained in $C_*^{\mathfrak{A}}(X)$ for some m. So z maps to zero homology in this complex too, and we have proved injectivity. The proof the theorem is now complete. ▲

> I should have asked this half an hour ago, but can you people in the back read?

Remark. Suppose T is a compact metric space and the V_α cover T and are open. Then there exists $\epsilon > 0$ such that points $x \in T$, the ball of radius ϵ, namely $B_\epsilon(x)$, is contained in some V_α. This is a theorem about compact metric spaces. Such a number ϵ is called a **Lebesgue number** of T.

§2 Some algebra

We now switch gears to another thing.

7.3 Definition. A **short exact sequence** of abelian groups

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called **exact** if f is injective, g is surjective, and $\ker g = \mathrm{Im} f$.

More generally, a lot sequence of abelian groups

$$\dots \to C_{n+1} \to C_n \to C_{n-1} \to \dots$$

is exact if the kernel of one map is the image of the previous one.

7.4 Definition. A **short exact sequence** of chain complexes

$$0 \to C'_* \to C_* \to C''_* \to 0$$

is a sequence of chain complexes and chain maps such that for each n,

$$0 \to C'_n \to C_n \to C''_n \to 0$$

is a short exact sequence.

The idea is that if you have a short exact sequence and you know two of the three terms in the sequence, then you know the third—in some sense, at least. We will find several different ways of making this precise. This is a philosophy, and it is only approximately true.

7.5 Example. The sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$

is short exact (obvious maps) as is

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0.$$

This shows that the philosophical statement above is not mathematically airtight as it is.

7.6 Example. Suppose $T = T_1 \cup T_2$ as sets (discrete). Then we can take the free abelian group $\mathbb{Z}T_1$ and the free abelian group $\mathbb{Z}T_2$. There is a surjective map

$$\mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \to \mathbb{Z}(T_1 \cup T_2)$$

but it is not an isomorphism; some of the elements got counted twice, namely the ones in T_1, T_2. Namely, the kernel of this map is isomorphic to $\mathbb{Z}(T_1 \cap T_2)$ where the map $\mathbb{Z}(T_1 \cap T_2) \to \mathbb{Z}T_1 \oplus \mathbb{Z}T_2$ sends $t \in T_1 \cap T_2$ to $(t, -t) \in \mathbb{Z}T_1 \oplus \mathbb{Z}T_2$. In particular, we have a short exact sequence

$$0 \to \mathbb{Z}(T_1 \cap T_2) \to \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \to \mathbb{Z}(T_1 \cup T_2) \to 0.$$

Let's go back to the case when X is a space and the interiors of two subsets U_1, U_2 cover X. We consider the set of all n-chains. By the above example, we have a map (for each n)

$$C_n(U_1) \oplus C_n(U_2) \to C_n^{\mathfrak{A}}(X) \to 0$$

where $\mathfrak{A} = \{U_1, U_2\}$. The kernel of this is

$$C_n(U_1 \cap U_2).$$

This follows at once from the previous example. Indeed, we take T_1 as the set of simplices in U_1, T_2 as the set of simplices in U_2, so then $T_1 \cup T_2$ is a basis for $C_n^{\mathfrak{A}}(X)$. There is thus a short exact sequence for each n:

$$0 \to C_n(U_1 \cap U_2) \to C_n(U_1) \oplus C_n(U_2) \to C_n^{\mathfrak{A}}(X) \to 0$$

which leads to a short exact sequence of chain complexes

$$0 \to C_*(U_1 \cap U_2) \to C_*(U_1) \oplus C_*(U_2) \to C_*^{\mathfrak{A}}(X) \to 0.$$

As we will see next time, this leads to an important relationship between the homology of $U_1 \cap U_2$, the homology of U_1, that of U_2, and the homology of $C_*^{\mathfrak{A}}(X)$ (which we just proved is the same as the homology of X). This relationship is what will help you make tons and tons of calculations.

This is yet another expression of the philosophy that homology depends on small simplices.

Lecture 8
9/17

This lecture was given by Eric Wofsey.

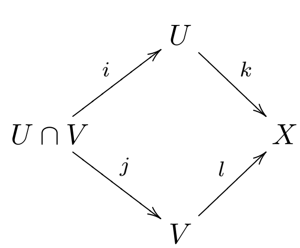
The goal for today is to prove:

8.1 Theorem. *If $n \neq m$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.*

Ad hoc methods suffice to show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^2$ because $\mathbb{R}^1 - 0$ is not homeomorphic to $\mathbb{R}^2 - 0$ (since one is connected and one isn't).

In the previous lecture, Hopkins proved that if $\mathfrak{A}$ is an open cover of a space X, and $C_*^{\mathfrak{A}}$ the complex of singular n-chains lying inside one of the sets of the open cover, then the inclusion $C_*^{\mathfrak{A}}(X) \to C_*(X)$ induces isomorphisms on homology. Moreover, it is even a chain homotopy equivalence.

We will consider today the case when $\mathfrak{A}$ consists of only two sets U, V. We'd like to understand what $C_*^{\mathfrak{A}}(X)$ looks like. First, let's write some maps:



There is a surjection

$$C_*(U) \oplus C_*(V) \to C_*^{\mathfrak{A}}(X)$$

by sending a pair of simplices into their inclusions in X. Thus, this map can be written as

$$C_*(U) \oplus C_*(V) \stackrel{(k,l)}{\to} C_*^{\mathfrak{A}}(X).$$

This map isn't injective, though. If α is a chain in the intersection, we can consider it as either a chain of U and a chain of V. The pair $(\alpha, -\alpha) \in C_*(U) \oplus C_*(V)$ (or more precisely, the push-forwards of these under the inclusion maps) goes to zero. It is also easy to see that this is the only way something in $C_*(U) \oplus C_*(V)$ can get killed. More precisely, we have an exact sequence of *chain complexes*

$$0 \to C_*(U, V) \stackrel{\begin{bmatrix} i \\ j \end{bmatrix}}{\to} C_*(U) \oplus C_*(V) \stackrel{(k,l)}{\to} C_*^{\mathfrak{A}}(X) \to 0.$$

The last guy is chain-homotopic to $C_*(X)$.

The point is now to reason as follows: if we know the homology of U, V, then we should know the homology of X. The way we do this is the very general and extremely important theorem in homological algebra:

8.2 Theorem. *Let* $0 \to A_* \stackrel{f}{\to} B_* \stackrel{g}{\to} C_* \to 0$ *be a short exact sequence of chain complexes. Then there is a long exact sequence in homology*

$$\dots \to H_n(A) \stackrel{f_*}{\to} H_n(B) \stackrel{g_*}{\to} H_n(C) \stackrel{d}{\to} H_{n-1}(A) \stackrel{f_*}{\to} \dots$$

There are "boundary maps" d from $H_n(C) \to H_{n-1}(A)$. It's kind of a horrendous thing to do in certain ways, so we won't prove all of it.

Proof. So what do we need to prove? First, we already have the maps f_*, g_*; we need to define the map d. Then, when we have defined it, we have to check that the sequence is in fact exact. The really interesting part of this is constructing the map, though.

We will construct

$$d : H_n(C) \to H_{n-1}(A).$$

To do this, let us write out the short exact sequence explicitly

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Let us pick the equivalence class of a cycle $\gamma \in Z_n(C)$ representing some element of $H_n(C)$. We'd like to get a representative of some homology class in $H_{n-1}(A)$. We start by going left, since $B_n \to C_n$ is surjective. Find some $\widetilde{\gamma} \in B_n$ which maps to γ in C_n. This is a choice; it is uniquely determined only up to elements of A_n. Next, we move it down, we take the differential $\partial\widetilde{\gamma} \in B_{n-1}$. Since the square commutes, we know that

$$g(\partial\widetilde{\gamma}) = \partial g(\widetilde{\gamma})) = \partial\gamma = 0.$$

In particular, $\partial\widetilde{\gamma}$ actually is an element of A_{n-1} (i.e. lifts uniquely to one), say $\widetilde{\widetilde{\gamma}}$. We now **define** the boundary of the homology class represented by γ to be the class represented by $\widetilde{\widetilde{\gamma}}$.

There are many things we need to check to see that it makes sense. First, we need to check that $\widetilde{\widetilde{\gamma}}$ is a cycle; that's because

$$f(\widetilde{\widetilde{\gamma}}) = \partial(f(\widetilde{\widetilde{\gamma}})) = \partial\partial\widetilde{\gamma} = 0.$$

This is a bit of diagram-chasing, which is best worked out for oneself.

We still have to check a whole bunch of things. For instance, what if we picked a different $\widetilde{\gamma} \in B_n$? We could replace $\widetilde{\gamma}$ with something in A_n. But when we take the boundary to find the new $\widetilde{\widetilde{\gamma}}$, we find the old $\widetilde{\widetilde{\gamma}}$ plus a boundary in A_{n-1}. This means that we get the same homology class in $H_{n-1}(A)$.

We next have to check that this $H_n(C) \to H_{n-1}(A)$ is a homomorphism. But this is clear.

The final big thing to check is that the associated long sequence is actually exact. There are a whole bunch of things to check for this; we'll just check one of them. It is one of the things best done in the privacy of your own home, but it should be done at least once in one's life. For instance, let's check that the kernel of

$$H_n(C) \to H_{n-1}(A)$$

is precisely the image of $H_n(B)$. First, we need to see that the composition is zero. This is easy though. If $\gamma \in Z_n(C)$ is equal in homology to $g_*(\beta)$ for $\beta \in Z_n(B)$, then we could have taken β to have been our $\widetilde{\gamma}$, and consequently $\partial\beta = 0$. Thus our $\widetilde{\widetilde{\gamma}}$ would be zero. In particular, the composition $H_n(B) \to H_n(C) \to H_{n-1}(A)$ is zero.

Now let's go the other way. Suppose that $\partial[\gamma]$ (where $[\gamma] \in H_n(C)$ is the class represented by $\gamma \in Z_n(C)$) is killed by the boundary map to $H_{n-1}(A)$. In particular, our $\widetilde{\widetilde{\gamma}}$ is equal to $\partial\alpha$ for some $\alpha \in A_n$. Also, $\partial f(\alpha) = \partial\widetilde{\gamma}$. We can consider $\beta = \widetilde{\gamma} - f(\alpha)$ which has zero boundary, so represents something in the homology class of B. Thus $\gamma = g(\widetilde{\gamma}) = g(\beta)$ because $g \circ f = 0$. So we get exactness at this step. And that's all we're going to do for the proof; the rest of exactness is left to the reader. ▲

Finally, we can apply this general nonsense to the case we had earlier:

8.3 Theorem (Mayer-Vietoris sequence)**.** *Let $\{U, V\}$ be an open cover of a space X. Then there is a long exact sequence*

$$H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to H_{n-1}(X) \to \dots$$

Proof. This follows at once from the short exact sequence of complexes,

$$0 \to C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*^{\mathfrak{A}}(X) \to 0$$

together with the fact that the homology of $C_*^{\mathfrak{A}}(X)$ is isomorphic to the homology of X. ▲

This will let us compute the homology of a whole bunch of spaces. It is also useful, incidentally, to know what the maps are. The map

$$H_n(U \cap V) \to H_n(U) \oplus H_n(V)$$

is given by

$$\begin{bmatrix} i_* \\ -j_* \end{bmatrix}.$$

Let us now compute the homology of a simple case. The homology of a two-point Hausdorff space is the direct sum of the homology of a point twice. If, more generally, $X = X - 0 \sqcup X_1$ for two spaces X_0, X_1 then

$$C_*(X) \simeq C_*(X_0) \oplus C_*(X_1)$$

inducing the isomorphisms

$$H_*(X) \simeq H_*(X_0) \oplus H_*(X_1).$$

Knowing this, the homotopy invariance, and the Mayer-Vietoris sequence, is enough to compute the homology of any space you'll ever encounter. Let's give some examples.

8.4 Example. Let's compute the homology of S^1. We will use the Mayer-Vietoris sequence, by covering the circle with two arcs, each of which is homeomorphic to an interval (and consequently contractible). Let U, V be these two arcs. Then $U \cap V$ is the union of two arcs, and will be homotopy equivalent to two points.

We know that $H_0(U) = H_0(V) = \mathbb{Z}$ and the higher homology vanishes. This is because both are surjective. For the intersection, we have $\mathbb{Z} \oplus \mathbb{Z}$ in dimension zero, and the higher homology is trivial. The long exact sequence looks like

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(S^1) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(S^1) \to 0.$$

This becomes

$$0 \to 0 \to H_1(S^1) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(S^1) \to 0.$$

We need to know what the map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is. It is easy to see that it sends (x, y) to $(x - y, x - y)$ by the way the maps in the Mayer-Vietoris sequence were defined (by small matrices $(i_*, -j_*)$. The kernel and the cokernel are both $\mathbb{Z}$ so

$$H_1(S^1) \simeq \mathbb{Z}, \quad H_0(S^1) \simeq \mathbb{Z}.$$

In higher dimensions, it's all zero. Indeed, if $n > 1$, we have that

$$H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(S^1) \to H_{n-1}(U \cap V)$$

and since the first, second, and fourth parts of this are zero (since $n-1 \geq 1$), we have that

$$H_n(S^1) = 0 \quad \text{if} \quad n \geq 2.$$

The conclusion is that by using the Mayer-Vietoris sequence, we've computed all the homology of S^1.

Let us start describing how to compute the homology of a 2-sphere. Again, we will use Mayer-Vietoris.

8.5 Example. Let $X = S^2$ be a 2-sphere. We cover it by the upper and lower hemispheres together with a small strip attached in each case. For instance, if z is one of the coordinates, we could take $U = \{z > -\frac{1}{2}\}$ and $V = \left\{z < \frac{1}{2}\right\}$. The intersection deformation retracts to the equator, which is just S^1, and we know the homology of $U \cap V$. But U, V are homeomorphic to the disk and are contractible.

Let's start with H_2. We have an exact sequence:

$$H_2(S^1) \to H_2(U) \oplus H_2(V) \to H_2(S^2) \to H_1(S^1) \to H_1(U) \oplus H_1(V)$$

Filling in the groups we know (from the previous example) yields

$$0 \to 0 \to H_2(S^2) \to \mathbb{Z} \to 0 \to 0$$

so that

$$H_2(S^2) \simeq \mathbb{Z}.$$

We can play the same game as before to show that $H_1(S^2) = 0$.

Lecture 9
9/20

§1 Some algebra

We've come a long way, though it probably doesn't feel like that since we haven't seen how this tool of homology gets used and what it tells you. A couple of more lectures will be necessary to enhance the user interface so we can start using it a little bit.

Most of today will be organizing some algebra to get some tools that we will use. As Eric explained last time, a short exact sequence of chain complexes

$$0 \to A_* \to B_* \to C_* \to 0$$

gives a long exact sequence

$$H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(C) \to \dots$$

on homology.

Remark. Some people like to write short exact sequences as

$$A \rightarrowtail B \twoheadrightarrow C$$

instead of

$$0 \to A \to B \to C \to 0.$$

Hopkins likes the first one, because the second seems as if the two zeros at the ends are sentries stationed at the outside to keep things exact.

We did not learn, however, the snake or the five lemmas. We should probably take a minute and talk about those. This is part of a branch of mathematics called homological algebra. In the old days, there would be a whole course on homological algebra. Nowadays, you just learn episodes of it in a course on algebraic topology. Hopkins says he likes homological algebra, so there will be a fair bit in the course.

9.1 Theorem (Snake lemma)**.** *Suppose given a map of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & R & \longrightarrow & 0\ . \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

We can draw the kernels $\ker f, \ker g, \ker h$*, and their cokernels. Then there is an exact sequence*

$$0 \to \ker f \to \ker g \to \ker h \to \mathrm{coker} f \to \mathrm{coker} g \to \mathrm{coker} h \to 0.$$

The maps between the kernels and between the cokernels are the natural ones.

Proof. The boundary map $\ker h \to \mathrm{coker} f$ is obtained via diagram-chasing. Take something in R which is killed by h. Pull it back to Q; apply g and push it down to M, which goes to zero in N and consequently comes from L. The image of this in the cokernel $L/\mathrm{Im}(P \to L)$ gives the map $\ker h \to \mathrm{coker} f$.

We won't actually prove the exactness, as we leave it to the reader as an exercise. See the movie *It's My Turn.* ▲

One can actually use the snake lemma to prove the long exact sequence in homology. Alternatively, if you think of each of the vertical sequences in the snake lemma as a chain complex (with all other maps zero), then the snake lemma is a special case of the long exact sequence.

The five-lemma is something we're really going to use.

9.2 Theorem (Five-lemma)**.** *Suppose given a morphism of exact sequences*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 & \longrightarrow & E_2\ . \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & D_2 & \longrightarrow & E_2 \end{array}$$

Suppose the maps $A_1 \to A_2, B_1 \to B_2, D_1 \to D_2, E_1 \to E_2$ *are all isomorphisms. Then* $C_1 \to C_2$ *is an isomorphism.*

Proof. This is another diagram chase.

Let's first prove that $C_1 \to C_2$ is a monomorphism. Then the image in D_2 is killed by $D_1 \to D_2$, so the image in D_2 is zero. Thus the thing in C_1 comes from something in B_1 such that the image in B_2 gets killed by $B_2 \to C_2$. So the image in B_2 comes from something in A_2, and it follows that there is something in A_1 whose image in B_2 is the same as the thing in B_1 sent to B_2. Thus the thing in B_1 comes from the thing in A_1. Thus the thing in C_1 comes from something in A_1, so is zero. Similarly one shows it is an isomorphism.

This is just a vague sketch of the actual diagram-chasing. ▲

Remark. One actually has to have a map. For instance, consider the exact sequences

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

There is no morphism between these two exact sequences which induces the identity on the ends, for then it would be an isomorphism by the five-lemma.

§2 Relative homology

Let A be a subspace of X. Then $C_*(A)$ is a subcomplex of $C_*(X)$ in an obvious way. The quotient $C_*(X)/C_*(A)$ is called the complex of **relative chains** and is denoted

$$C_*(X, A).$$

Right now, it is a formal gadget. There is a short exact sequence of complexes

$$C_*(A) \rightarrowtail C_*(X) \twoheadrightarrow C_*(X, A).$$

9.3 Definition. We define the **relative homology**

$$H_n(X, A) = \mathrm{n-th\ homology\ of\ } C_*(X, A).$$

If $A \subset X$, there is a long exact sequence:

9.4 Proposition. *If $A \subset X$, there is a long exact sequence*

$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \dots.$$

Proof. Follows at once from the long exact sequence for a short exact sequence of chain complexes. ▲

9.5 Example. If $H_n(X, A) = H_{n-1}(X, A) = 0$, then $H_{n-1}(A) \simeq H_{n-1}(X)$. To see this, just write out a piece of the exact sequence

$$0 = H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X, A) = 0.$$

Recall that if $0 \to A \to B \to 0$ is exact, then $A \to B$ is an isomorphism.

That's useful at the formal level.

9.6 Example (Reduced homology)**.** Suppose $A = \mathrm{pt} = \{*\}$. In other words, X has a base point.

By definition, the **reduced homology** of X is defined to be

$$\widetilde{H_n(X)} = H_n(X, *).$$

For many purposes, reduced homology is much more convenient than ordinary homology. It doesn't differ very much from ordinary homology, though. We know that there is an exact sequence

$$H_n(*) \to H_n(X) \to \widetilde{H_n(X)} \to H_{n-1}(*) \to \dots.$$

The ending case of this is

$$H_1(*) \to H_1(X) \to \widetilde{H_1(X)} \to H_0(*) \to H_0(X) \to \widetilde{H_0(X)} \to 0.$$

Now we know that $H_n(*) = \mathbb{Z}$ for $n = 0$ but $H_0(*) = \mathbb{Z}$. So we know that, from the first exact sequence that $n > 1$ implies

$$H_n(X) \to \widetilde{H_n(X)}$$

is an isomorphism.

In the last case, it's not so simple. Let's compute $\widetilde{H_1(X)}$. The map $* \to X$ admits a retraction $X \to *$. Applying homology shows that $H_0(*) \to H_0(X)$ admits a retraction, so in particular is injective. In particular, the kernel of $H_0(*) \to H_0(X)$ is zero and $H_1(X, *) \to H_0(*)$ is the zero map. It follows that $H_1(X) \to H_1(X, *)$ is surjective. Since it is also injective (as $H_1(*) = 0$) we find that

$$\widetilde{H_1(X)} \simeq H_1(X).$$

For $n = 0$, we have a split exact sequence

$$0 \to H_0(*) \to H_0(X) \to \widetilde{H_0(X)} \to 0;$$

we know the splitness because $* \to X$ is a split injection. In particular,

$$\widetilde{H_0(X)} \oplus \mathbb{Z} \simeq H_0(X).$$

§3 A substantial theorem

Suppose $A \subset X$ is a subspace. We can form the space $X \cup CA$, which by definition is $X \sqcup A \times [0, 1]$ modulo the relation $a \sim (0, a)$ for $a \in A$ and $(a, 1) \sim *$. In particular, we pinch A off to a point as you go up.

9.7 Theorem. *We have*

$$H_*(X, A) \sim \widetilde{H}_*(X \cup CA).$$

Ideally, one would actually produce a map between these. We have two pairs

$$(X, A), \quad (X \cup CA, *).$$

However, these two don't map to each other nicely. Nevertheless, we have maps of pairs

$$(X, A) \to (X \cup CA, CA)$$

and

$$(X \cup CA, *) \to (X \cup CA, CA).$$

We will show that maps on relative homology induced by these two maps of pairs leads to isomorphisms in homology. This will prove the theorem.

Remark. We should emphasize something not discussed earlier. Suppose $(X, A), (Y, B)$ are pairs of spaces—i.e., $A \subset X, B \subset Y$ are subspaces. Then we get maps of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X,A) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & C_*(B) & \longrightarrow & C_*(Y) & \longrightarrow & C_*(Y,B) & \longrightarrow & 0. \end{array}$$

This induces a morphism of long exact sequences in homology, which I can't typeset now.

Proof of the theorem. Consider the map

$$(X \cup CA, *) \to (X \cup CA, CA).$$

We will show that the maps in relative homology thus induced are isomorphisms.

$$\begin{array}{ccccccccc} H_n(*) & \longrightarrow & H_n(X \cup CA) & \longrightarrow & H_n(X \cup CA, *) & \longrightarrow & H_{n-1}(*) & \longrightarrow & H_{n-1}(X \cup CA) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(CA) & \longrightarrow & H_n(X \cup CA) & \longrightarrow & H_n(X \cup CA, CA) & \longrightarrow & H_{n-1}(CA) & \longrightarrow & H_{n-1}(X \cup CA). \end{array}$$

Now the map $* \to CA$ is a homotopy equivalence because the cone on any space is contractible to the cone point. It follows that all the vertical maps in the diagram except the middle one are isomorphisms because they come either from homotopy equivalences or the identity map. So the maps in the middle, on relative homology, are isomorphisms. That's one step.

Now we need to show that

$$(X, A) \to (X \cup CA, CA)$$

induces isomorphisms in relative homology. This will follow from the excision theorem, since to get (X, A) from $(X \cup CA, CA)$, we cut out an open set from both $X \cup CA$ and CA. We digress to discuss excision.

9.8 Theorem (Excision theorem)**.** *Suppose* $Z \subset B \subset Y$ *and* $\overline{Z} \subset Int(Y)$*. Then the inclusion maps induce isomorphisms in homology*

$$H_n(Y - Z, B - Z) \simeq H_n(Y, B).$$

Proof of excision. We only have one tool, which is the expression of homology depending on small chains. For this we need an open covering of Y. Namely, we will write

$$Y = B \cup (Y - Z).$$

The interiors of these two sets cover Y. If $\mathfrak{A} = \{B, Y - Z\}$, then the $\mathfrak{A}$-chains $C_*^{\mathfrak{A}}(Y)$ inject into $C_*(Y)$; this injection induces isomorphisms in homology. This was sketched earlier, and was the expression of the idea that homology depends on small simplices.

By the way, what's $B \cap (Y - Z)$? It's $B - Z$. There is a diagram of chain complexes

$$\begin{array}{ccc} C_*(B - Z) & \longrightarrow & C_*(Y - Z) \\ \downarrow & & \downarrow \\ C_*(B) & \longrightarrow & C_*^{\mathfrak{A}}(Y). \end{array}$$

There are lots of special properties of this complex, for instance that it is a push-out. Consider the cokernels, and draw them in:

$$\begin{array}{ccccccc} C_*(B - Z) & \longrightarrow & C_*(Y - Z) & \longrightarrow & C_*(Y - Z, B - Z) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ C_*(B) & \longrightarrow & C_*^{\mathfrak{A}}(Y) & \longrightarrow & C & \longrightarrow & 0, \end{array}$$

where C is the cokernel of $C_*(B - Z) \to C_*^{\mathfrak{A}}(Y - Z)$.

In general, if we have sets S_1, S_2 that fill up a set T, then one can draw a diagram

$$\begin{array}{ccccccc} \mathbb{Z}\{S_1 \cap S_2\} & \longrightarrow & \mathbb{Z}\{S_2\} & \longrightarrow & \mathbb{Z}\{S_1 - S_1 \cap S_2\} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}\{S_1\} & \longrightarrow & \mathbb{Z}\{T\} & \longrightarrow & \mathbb{Z}\{T - S_1\} & \longrightarrow & 0 \end{array}$$

where the last downward arrow is an isomorphism. This implies that the last downward arrow in the earlier thing is an isomorphism. More simply, we could argue via the isomorphism theorems in elementary algebra that

$$C_*^{\mathfrak{A}}(Y)/C_*(B) \simeq C_*(Y - Z) + C_*(B)/C_*(B) \simeq C_*(Y - Z)/C_*(B - Z).$$

In any case, this means that $C_*^{\mathfrak{A}}(X)/C_*(B)$ has the same homology as $H_n(Y - Z, B - Z)$. But this first guy has the same homology as $H_*(X, B)$ because $C_*^{\mathfrak{A}}(Y)$ is homotopy equivalent to $C_*(Y)$, and one can use the five lemma. This proves excision. ▲

We will finish up the proof of the main result next time. ▲

Lecture 10
9/22

§1 Finishing up last week

We had a little bit of the proof left over at the end of last class. Namely, last time we proved the **excision theorem**:

10.1 Theorem. *Suppose $Z \subset A \subset X$ and $\overline{Z} \subset Int(A)$. Then the map $(X-Z, A-Z) \to (X, A)$ induces isomorphisms in homology*

$$H_*(X, A) \to H_*(X - Z, A - Z).$$

We deduced this from the same techniques as Mayer-Vietoris.

We were in the middle of proving the important result that

$$H_*(X, A) = \widetilde{H}_*(X \cup CA).$$

Now, we shall finish the proof. In particular, relative homology is a special case of reduced homology. We were going to prove this by the collection of maps

$$\begin{array}{ccc} H_*(X, A) & \longrightarrow & H_*(X \cup CA, CA) \\ & & \uparrow \\ & & H_*(X \cup CA, *) \end{array}$$

We showed that the vertical map was an isomorphism via the five-lemma and the fact that CA deformation retracts onto a point. Now, we want to prove that the horizontal map is an isomorphism. This we will do via excision.

Now it would be nice to take $Z = \{(a, t), t > 0, a \in A\}$, in short everything in CA which is not in X. If we could excise this Z from the pair $X \cup CA, CA$, we would be able to see that the horizontal map is an isomorphism. However, $\overline{Z}$ is too large, and we cannot apply excision directly.

However, we can excise the set

$$Z = \left\{(a, t) : t \geq \frac{1}{2}\right\}$$

which is a closed subspace of CA. The closure is certainly the in the interior of CA. By excision, we find

$$H_*(X \cup CA, CA) \simeq H_*(X \cup A \times [0, 1/2], A \times [1/2])$$

since the remaining non-excised stuff is just a cylinder on A. This last guy, however, is isomorphic to $H(X, A)$, because the cylinder CA deformation retracts onto A. In particular, there is a homotopy equivalence of pairs

$$(X \cup A \times [0, 1/2], A \times [0, 1/2]) \simeq (X, A).$$

10.2 Lemma. *Suppose* $(X, A) \to (Y, B)$ *is a homotopy equivalence of pairs. Then the maps*

$$H_*(X, A) \to H_*(Y, B)$$

are isomorphisms.

Proof. This can be seen using the five-lemma, for instance. ▲

§2 Triples

There is a variation that we want to mention, because it will make things cleaner. Suppose one has three spaces $A \subset B \subset X$. Then we have an inclusion

$$C_*(A) \subset C_*(B) \subset C_*(X)$$

from which one gets an exact sequence

$$0 \to C_*(B)/C_*(A) \to C_*(X)/C_*(A) \to C_*(X)/C_*(B) \to 0.$$

If we write in terms of relative chains, we get an exact sequence

$$C_*(B, A) \rightarrowtail C_*(X, A) \twoheadrightarrow C_*(X, B),$$

which leads to the **long exact sequence for a triple**

$$H_n(B, A) \to H_n(X, A) \to H_n(X, B) \to H_n(B, A) \to \dots.$$

10.3 Example. Suppose A is a base point. Then homology relative to A is reduced homology. So the long exact sequence for a triple becomes

$$\widetilde{H_n}(B) \to \widetilde{H_n}(X) \to \widetilde{H_n}(X, B) \to \widetilde{H_{n-1}}(B),$$

which is a long exact sequence for a pair in reduced homology. Since reduced homology is more convenient, this is useful.

§3 Another variant; homology of the sphere

10.4 Example (Mayer-Vietoris for reduced homology)**.** We now give another variation. Suppose $X = U \cup V$ is the union of two open sets with the basepoint $* \in U \cap V$. Then there is a Mayer-Vietoris sequence in reduced homology. It goes

$$\widetilde{H_n}(U \cap V) \to \widetilde{H_n}(U) \oplus \widetilde{H_n}(V) \to \widetilde{H_n}(X) \to \widetilde{H_{n-1}}(U \cap V) \to \dots$$

An example of Mayer-Vietoris for reduced homology is in the computation of the homology of the sphere.

10.5 Example (Homology of the sphere)**.** Let $X = S^n$, $U_1 = S^n -$ north pole, $U_2 = S^n -$ south pole. The intersection is homotopy equivalent to the $n-1$-sphere. The reduced Mayer-Vietoris sequence goes

$$\widetilde{H_k}(S^{n-1}) \to \widetilde{H_k}(U_1) \oplus \widetilde{H_k}(U_2) \to \widetilde{H_k}(S^n) \to \widetilde{H_{k-1}}(S^{n-1}) \to \dots.$$

Everything involving U_1, U_2 has trivial reduced homology since these are contractible. The exact sequence shows that

$$\widetilde{H_*}(S^n) \simeq \widetilde{H_{*-1}}(S^{n-1}),$$

and by induction one finds

$$\boxed{\widetilde{H_k}(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}}$$

This is a highly useful calculation, which has numerous consequences that we will discuss next time. We have the advantage of a lot of re-thinking of the material, so that it comes out clean; it doesn't seem as substantive as it really is.

Let us do just one simple consequence of this.

If one has two vector spaces V, W, and a linear isomorphism $V \to W$, then $\dim V = \dim W$. The proof very much uses linearity, though. But the same theorem is true if we just had a homeomorphism. In particular, the dimension of a vector space is topological in nature, not just a linear algebra fact.

10.6 Theorem. *$\mathbb{R}^n$ is not isomorphic to $\mathbb{R}^m$ unless $n = m$.*

But let us prove something even stronger.

10.7 Theorem (Invariance of dimension)**.** *If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ and U is homeomorphic to V, then $n = m$.*

The argument really is a generalization of the familiar argument that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$ by removing a point and looking at connected components.

Proof. Suppose we have a homeomorphism $U \to V$, sending some point $x \in U$ via $x \to y$. It follows that $U - x$ is homeomorphic to $V - y$. In particular, we find that

$$H_*(U, U - x) \simeq H_*(V, V - y)$$

under this map of pairs $(U, U - x) \to (V, V - y)$.

Let us compute these relative homology groups. We can find a little set B around x homeomorphic to a ball of radius ϵ. Then we know that

$$H_*(U, U - x) \simeq H_*(B, B - x)$$

by excising $U - B$. By looking at the long exact sequence in relative and reduced homology, we find from contractibility of B:

$$H_k(B, B - x) \simeq \widetilde{H_{k-1}}(B - x)$$

But the reduced homology of $B - x$ is the reduced homology of the sphere S^{n-1}, onto which $B - x$ deformation retracts.

In particular, we find that $H_*(U, U - x) \simeq \widetilde{H_{*-1}}(S^{n-1})$, and similarly $H_*(V, V - x) \simeq \widetilde{H_{*-1}}(S^{m-1})$. But unless $m = n$, the homology of the $m - 1$ and $n - 1$ spheres are different. ▲

§4 Equivalence of simplicial and singular homology

There are a number of algebraic things to be introduced on the fly.

Suppose X is a Δ-complex. We have defined these concepts earlier, so we shall not review them. Then X is filtered by its skeleton

$$X^{(0)} \subset X^{(1)} \subset \dots$$

where $X^{(n)}$ is the n**-skeleton**, i.e the part built out of simplicies of dimension at most n. We defined a while back, without talking about it since, the Δ-chain complex

$$C_*^{\Delta}(X) \subset C_*(X)$$

where $C_n^{\Delta}(X)$ is the free abelian group on the set of characteristic maps $\Delta^n \to X$. ($C_n(X)$ allows all continuous maps; each characteristic map has the special property of being a homeomorphism on the interior.) By assumption, the characteristic maps are stable under the application of the boundary map: this is because of the way Δ-complexes were defined.

The main theorem is

10.8 Theorem. *The map $C_*^{\Delta}(X) \to C_*(X)$ induces isomorphisms in homology. In particular,*

$$H_*^{\Delta}(X) \simeq H_*(X)$$

for a Δ-complex X.

In particular, we only have to use the Δ-chains to compute homology.

Proof. We will induct on n to show that

$$H_*^{\Delta}(X^{(n)}) \to H_*(X^{(n)}),$$

which will prove the theorem in the case for Δ-complexes which have only simplices of bounded dimension. This will cover all the examples we are interested in. The general case follows from a simple argument, but we will explain some variants of this in a later lecture, when we can use more category-theory. For now, we will stick to Δ-complexes of bounded dimension.

The proof is actually pretty easy. The induction starts at $n = 0$. Here $X^{(0)}$ is a discrete set, and we know the homology in that case is just the free abelian group on the same set in dimension zero and is zero in higher dimensions, i.e.

$$H_*(X^{(0)}) \simeq \mathbb{Z}\{X_0\}.$$

This is also true for Δ-homology, as is easy to see.

Now let us look at the long exact sequence of a pair $(X^{(n)}, X^{(n-1)})$. The sequence returns

$$\begin{array}{ccccccccc}
H_{*+1}^{\Delta}(X^{(n)}, X^{(n-1)}) & \longrightarrow & H_*^{\Delta}(X^{n-1}) & \longrightarrow & H_*^{\Delta}(X^{(n)}) & \longrightarrow & H_n^{\Delta}(X^{(n)}, X^{(n-1)} & \longrightarrow & H_*^{\Delta}(X^{(n-1)}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{*+1}(X^{(n)}, X^{(n-1)}) & \longrightarrow & H_*(X^{n-1}) & \longrightarrow & H_*(X^{(n)}) & \longrightarrow & H_n(X^{(n)}, X^{(n-1)} & \longrightarrow & H_*(X^{(n-1)})
\end{array}$$

We have to show the middle downward homology map is an isomorphism. By induction, the second and fifth downward maps are isomorphisms. So it is sufficient to show that

$$H_*^\Delta(X^{(n)}, X^{(n-1)}) \to H_*(X^{(n)}, X^{(n-1})$$

is an isomorphism. (It should be clear what relative Δ-homology; it is the homology of suitable quotients of the usual Δ-chain complexes.) But $C_*^\Delta(X^{(n)}, X^{(n-1})$ is the free abelian group on the n-simplices in the Δ-structure, and is zero elsewhere; its homology is the same.

What about $H_*(X^{(n)}, X^{(n-1)})$? By (a slight variant of) ordinary excision, this isomorphic to

$$H_*(\sqcup \Delta^n, \sqcup \partial \Delta^n)$$

with the coproduct ranging over the various n-simplices. This is not very clear, though; we're out of time, so will clarify next lecture.

Anyhow, this homology $H_*(\sqcup \Delta^n, \sqcup \partial \Delta^n)$ is the same group $\bigoplus_{\text{n simplices}} \mathbb{Z}$; we've guaranteed that the relative homology and relative Δ-homology groups are the same. One has to check that the map is an isomorphism, for which you have to go to Hatcher.

▲

Lecture 11
9/24

Kirsten Wickelgren taught the next couple of lectures.

Last time, we were in the middle of showing that the natural map

$$C_*^\Delta(X) \to C_*(X)$$

was a homology isomorphism. We were stuck in the middle of the proof, and ultimately we have decided not to resolve the issue until we discuss cellular homology.

§1 Degree of a map

Let $f : S^n \to S^n$ be a map. This induces a map $H_n(S^n) \to H_n(S^n)$, and this is $\mathbb{Z} \to \mathbb{Z}$ for $n > 1$ by the homology computation of the sphere. This map must be multiplication by some number d.

11.1 Definition. The **degree** of f is defined to be d.

It follows that f is just multiplication by d on homology.

11.2 Example. Let Δ^n be the n-simplex. You can take two copies Δ_1^n, Δ_2^n, which you can glue to form S^n by identifying the boundaries via the identity map. So

$$S^n = \Delta_1^n \sqcup_{\partial \Delta_1^n = \partial \Delta_2^n} \Delta_2^n.$$

In particular, $f : S^n \to S^n$ can be given by switching the two copies of Δ^n. Let us compute the degree of this map.

Now let us choose an isomorphism $H_n(S^n) \simeq \mathbb{Z}$. The singular class given by $\Delta_1^n - \Delta_2^n$, which is a cycle since the boundaries are identified, is a generator for $H_n(S^n)$. Since f_* sends this to $\Delta_2^n - \Delta_1^n$, it follows that f has degree -1.

Now we prove a few properties of the degree.

1. The identity has degree 1. This is because 1 induces the identity in homology.
2. $\deg(f \circ g) = \deg f \deg g$ for $f, g : S^n \to S^n$. This is because homology is a functor, and $(f \circ g)_* = f_* \circ g_*$.
3. A homotopy equivalence has degree ± 1, because a homotopy equivalence induces an isomorphism on homology.
4. If f, g are homotopic, then $\deg f = \deg g$.

Morally,

Degree counts pre-images with a multiplicity.

In particular, if you consider $z \to z^n, S^1 \to S^1$, you get a map of degree n: there are n pre-images of degree one. (We have not proved this.)

11.3 Example. The antipodal map $-1 : S^n \to S^n$ corresponds to multiplication by -1 on $\mathbb{R}^{n+1}$ restricted to S^n. We compute its degree.

Let the coordinates for $\mathbb{R}^{n+1}$ be $(x_0, \dots, x_n)$. We can define the **reflection** ρ_i in the plane $x_i = 0$ (so it flips the ith coordinate). The antipodal map is the composite of all the ρ_i (in any order you want).

Since each reflection has degree -1 by the first example (the Δ_1^n, Δ_2^n there can be viewed as hemispheres), it follows that -1 has degree $(-1)^{n+1}$.

This has some interesting consequences.

11.4 Proposition. *There is no continuous nowhere-vanishing vector field on an even-dimensional sphere.*

Note that an odd-dimensional sphere S^{2n+1} has a continuous, nowhere-vanishing vector field sending $x = (x_0, x_1, \dots, x_{2n+2}) \to (-x_1, x_0, -x_3, x_2, \dots,)$. This is always orthogonal to x, so it lies in the tangent space of S^n at x. This doesn't work for an even sphere.

Proof. Suppose to the contrary we had a continuous vector field $v : S^{2n} \to \mathbb{R}^{2n+1}$. We can assume, by switching to $v/\|v\|$, that v actually has image in S^{2n}.

So we end up with a map $v : S^{2n} \to S^{2n}$ such that $v(x) \perp x$ for all $x \in S^{2n}$. We now prove two lemmas. Together, they will imply that v has degree -1 and 1 simultaneously, which is a contradiction. It is thus sufficient to prove the lemmas.

11.5 Lemma. *If $f : S^m \to S^m$ has no fixed point, then f is homotopic to the antipodal map and has degree $(-1)^{m+1}$.*

(This implies that the degree of v above is -1.)

Proof. We define
$$H : I \times S^m \to S^m$$
by sending
$$(t, x) \to \frac{f(x)(1-t) + (-x)(1-t)}{\|f(x)(1-t) + (-x)(1-t)\|}$$
and since $f(x), -x$ are never antipodal, the denominator never vanishes. This is the required homotopy.

Geometrically, we are just drawing segments between $f(x)$ and $-x$, and going along these segments, and then normalizing to have norm one. ▲

Next, we do the same thing for a map which never sends a point to its antipode.

11.6 Lemma. *If $f : S^m \to S^m$ is such that $f(x) \neq -x$ for all $x \in S^m$, then f is homotopic to the identity and has degree* 1*.*

(This implies that the degree of v above is 1.)

Proof. We define
$$H : I \times S^m \to S^m$$
by sending
$$(t, x) \to \frac{f(x)(1-t) + x(1-t)}{\|f(x)(1-t) + x(1-t)\|}$$
and since $f(x), x$ are never antipodal, the denominator never vanishes. This is the required homotopy.

Geometrically, we are just drawing segments between $f(x)$ and x, and going along these segments, and then normalizing to have norm one. ▲

▲

Another consequence is that you can limit the groups that freely act on spheres.

11.7 Definition. A group G **acts on a space** X if there is given a homomorphism
$$G \to \mathrm{Homeo}(X).$$
So each $g \in G$ acts as a homeomorphism on X.

G acts **freely** if for all $g \in G - \{1\}$, g has no fixed points as a map $X \to X$.

$\mathbb{Z}/2$ acts freely on any sphere thanks to the antipodal map. But you can't have interesting groups acting on spheres. Namely:

11.8 Proposition. *The only groups that can act freely on an even-dimensional sphere S^{2n} are $\mathbb{Z}/2$ and $\{1\}$.*

Proof. Let G be a group acting freely on S^{2n}. By the lemma, for all $g \in G$ which are not the identity, the degree of g is equal to the degree of the antipodal map, i.e. -1; this is because g has no fixed points. This is impossible unless G has only one element other than the identity. Indeed, $\deg : G \to \{-1, 1\}$ is a homomorphism, so the preimage of -1 is a subgroup of G of index 2. ▲

§2 Computing the degree

The key tool here is the **local degree** of a map $f : S^n \to S^n$ at some point $x \in S^n$. To get a local degree, we use excision. Recall that

$$H_n(S^n, S^n - \{x\}) \simeq \mathbb{Z}$$

because S^n is an n-dimensional manifold, and we can use excision. We used this last time to show invariance of dimension for manifolds. More generally, if U is a small neighborhood of x, then $H_n(U, U - x) \simeq \mathbb{Z}$.

Suppose U is a neighborhood of x and V is a neighborhood of $f(x)$, such that $f(U) \subset V$. Suppose moreover that f does not assume the value $f(x)$ in U except at x. Then we have a map

$$\begin{array}{ccc} H_n(U, U - x) & \xrightarrow{f_*} & H_n(V, V - f(x)) \\ \downarrow \simeq & & \downarrow \simeq \\ H_n(S^n, S^n - x) & & H_n(S^n, S^n - f(x)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

where the vertical arrows are isomorphisms by excision. Note, moreover, that all this depends *only* on a choice of one isomorphism $H_n(S^n) \simeq \mathbb{Z}$. Everything else is induced by an inclusion or a long exact sequence. Indeed, once we choose a generator of $H_n(S^n)$, we get generators of $H_n(S^n, S^n - x)$ and $H_n(S^n, S^n - f(x))$.

11.9 Definition. The induced map $\mathbb{Z} \to \mathbb{Z}$ induced by f as in the above diagram is called the **local degree** of f, written $\deg_x f$. This does not depend on the choice of generator for $H_n(S^n)$.

Let's do an example of these local degrees. This will enable us to compute degrees in interesting cases, since the global degree is always the sum of the local degrees. I didn't live-TeXthis right, but the example is in Hatcher of computing the local degree of a map that starts from a sphere, goes into a "wedge" of spheres, and "crushes" almost all the spheres in the wedge.

Lecture 12
9/27

§1 Recap

Our first task is to do a better job with the example of local degree. We recall that the **degree** (plain old degree) of $f : S^n \to S^n$ is the map $\mathbb{Z} \to \mathbb{Z}$ induced by $f_* : H_n(S^n) \to H_n(S^n)$ (at least when $n \geq 1$).

The local degree was defined as follows. Namely, we pick $y \in S^n$ such that $f^{-1}(y)$ is finite and nonempty, say $\{x_1, \ldots, x_m\}$. Choose a neighborhood V of Y and disjoint neighborhoods U_i of x_i such that $f(U_i - \{x_i\}) \subset V - \{y\}$. Then we have isomorphisms

$$H_n(U_i, U_i - \{x_i\}) \simeq H_n(S^n, S^n - \{x_i\}) \simeq H_n(S^n).$$

But we have maps

$$H_n(U_i, U_i - \{x_i\}) \to H_n(V, V - \{y\}) \simeq H_n(S^n, S^n - \{y\}) \simeq H_n(S^n) = \mathbb{Z}.$$

Once we choose an isomorphism $H_n(S^n) \simeq \mathbb{Z}$, everything else is determined, and we get a map

$$f_{x_i} : \mathbb{Z} \to \mathbb{Z}$$

whose degree is called the **local degree** $\deg_{x_i} f$. This does not depend on the choice of isomorphism $H_n(S^n) \simeq \mathbb{Z}$.

12.1 Example. Consider S^1, which is homeomorphic to the boundary of an octagon. In this octagon, we mark the edges a, b, c, d in such a way that the identification leads to the Riemann surface of degree two. The boundary with the identifications is thus a wedge of four circles a, b, c, d. So we get a map

$$S^1 \to \vee_4 S^1$$

which we compose with the "crushing" map

$$\vee_4 S^1 \to S^1$$

that crushes circles b, c, d. The map $S^1 \to S^1$ thus obtained has local degree -1 and $+1$ when you look at points in the inverse image of a generic point in the a-circle S^1.

I don't feel like writing up the details; this is best actually spoken. Basically, the point is that at one point on edge a, you're preserving the orientation; at the other, you're reversing the orientation. There was a proper discussion in class.

§2 Degree can be calculated locally

12.2 Proposition. *The degree is the sum of the local degrees. That is, if $f : S^n \to S^n$, and y has finitely many preimages $\{x_1, \dots, x_m\}$, then*

$$\deg f = \sum \deg_{x_i} f.$$

Proof. To do this, we're going to look at the relative homology of

$$H_n(S^n, S^n - \{x_1, \dots, x_m\}).$$

We have a canonical map $H_n(S^n) \to H_n(S^n, S^n - \{x_1, \dots, x_m\})$, though it is **not an isomorphism**.

As before, we choose V a nbd of y and the U_i disjoint nbds of x_i. We have maps

$$H_n(U_i, U_i - x_i) \longrightarrow H_n(S^n, S^n - x_i) \longrightarrow H_n(S^n, S^n - \{x_1, \dots, x_m\}) \longleftarrow H_n(S^n).$$

By excision,[5] $H_n(S^n, S^n - \{x_1, \dots, x_m\})$ is isomorphic to the direct sum of

$$\bigoplus H_n(U_i, U_i - x_i)$$

under the injections

$$H_n(U_i, U_i - x_i) \to H_n(S^n, S^n - \{x_1, \dots, x_m\})$$

induced by the map of pairs $(U_i, U_i - x_i) \to (S^n, S^n - f^{-1}(y))$. The map $f_* : H_n(S^n) \to H_n(S^n)$ factors through

$$H_n(S^n) \simeq H_n(S^n - f^{-1}(y)) \simeq \bigoplus H_n(U_i, U_i - x_i) \stackrel{f_*}{\to} H_n(S^n, S^n - y) \to H_n(S^n).$$

Note that the thing marked f_* above actually corresponds to the local degree.

Now the generator $1 \in H_n(S^n)$ is going to go to the element $(1, 1, \dots, 1)$ in the direct sum $H_n(U_i, U_i - x_i)$. That's because the projection

$$H_n(S^n, S^n - \{x_1, \dots, x_m\}) = \bigoplus H_n(U_i, U_i - x_i) \to H_n(U_j, U_j - x_j)$$

can be described geometrically, just by using the inclusion $(S^n, S^n - f^{-1}(y)) \to (S^n, S^n - x_j)$. Note that $H_n(S^n, S^n - x_j) \simeq H_n(U_j, U_j - x_j)$. This gives the projection and shows that the element 1 generating $H_n(S^n)$ goes to the collection of ones in $\bigoplus H_i(U_i, U_i - x_i)$.

From this, and the above factorization of f_*, the result follows. This is confusing and should be thought through carefully. The key point is the resolution of $H^n(S^n, S^n - f^{-1}(y))$ into the direct sum $\bigoplus H_n(U_i, U_i - x_i)$ and the fact that 1 in H_n was sent into the bunch of 1's in the latter direct sum. ▲

§3 Cellular homology

This gives a very computable way of doing homology for a CW complex or a Δ-complex. We will do this for finite complexes. (Later we will discuss how this generalizes to infinite ones.)

Recall that a finite CW complex X has a filtration $X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(d)} = X$. For all n, the nth step in the filtration is built by attaching various n-cells D^n to $X^{(n-1)}$ via boundary maps $S^{n-1} \to X^{(n-1)}$.

In other words, given $X^{(n-1)}$, we obtain $X^{(n)}$ from a set A_n of maps $f : S^{n-1} \to X^{(n-1)}$ and let

$$X^{(n)} = X^{(n-1)} \cup_{f:\partial D^n \to X^{(n-1)}} \sqcup_{A_n} D^n.$$

So $X^{(0)}$ is just a bunch of discrete points, $X^{(1)}$ has a few intervals attached (with the boundaries identified to some of the discrete points), and so on.

Cellular homology allows homology to be computed via a highly understandable chain complex. The chain complex will in the nth stage will be $\mathbb{Z}$-free on the n-cells. The different maps between the various free abelian groups will be matrices of degrees.

The reason we can do this is that

$$H_n(X^{(n)}, X^{(n-1)})$$

is a free abelian group. This is what we will prove in the remainder of class time.

[5] Excise everything not in the U_j.

12.3 Proposition. *If X is a finite CW complex, then $H_n(X^{(n)}, X^{(n-1)})$ is free on the set of n-cells.*

Proof. In order to do this, we will need a technical proposition:

12.4 Definition. If $A \subset X$ is a subspace, then A is a **deformation retract** if there exists a retract $r : X \to A$ which is homotopic relative to A to the identity $1 : X \to X$. As a result, $A \to X$ is a homotopy equivalence, and so is $A/A \to X/A$.

This means that there is $H : X \times I \to X$ such that $H(x, 0) = x$, $H(X \times 1) \subset A$, and $H(a, t) = a$ for all $a \in A, t \in I$.

The technical proposition says that you can quotient out by certain closed spaces to compute relative homology.

12.5 Proposition. *Let $A \subset X$ be a closed subspace such that there exists an open neighborhood $V \supset A$ that deformation retracts onto A. Then*

$$H_n(X, A) \simeq \widetilde{H_n}(X/A).$$

This is exactly what we want for computing relative homology of CW-complexes. Indeed, $X^{(n)}/X^{(n-1)}$ is a wedge sum of n-spheres. As a result, the proposition implies that $H_n(X^{(n)}, X^{(n-1)})$ is the reduced n-homology of a wedge of these spheres, so is free abelian on the set of attached n-cells. As a result, we just need to prove this technical proposition.

Proof. Now V deformation retracts onto A. In particular,

$$H_n(X, A) \simeq H_n(X, V)$$

because the maps $H_n(A) \to H_n(V)$ are isomorphisms. Let us show this. Indeed, when one draws the two long exact sequences of pairs

$$\begin{array}{ccccccccc}
H_*(A) & \longrightarrow & H_*(X) & \longrightarrow & H_*(X, A) & \longrightarrow & H_{*-1}(A) & \longrightarrow & H_{*-1}(X) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
H_*(V) & \longrightarrow & H_*(X) & \longrightarrow & H_*(X, V) & \longrightarrow & H_{*-1}(V) & \longrightarrow & H_{*-1}(X)
\end{array}$$

where everything but possibly the center is an isomorphism (since $A \to V$ is a homotopy equivalence). So the middle maps are isomorphisms. We have used the five lemma.

Now $A/A \to V/A$ is a homotopy equivalence. So we can do the same exact sequence argument to find that

$$\widetilde{H_*}(X/A) \simeq H_*(X/A, A/A) \simeq H_*(X/A, V/A).$$

But $H_*(X/A, V/A)$ is isomorphic to $H_*(X/A - *, V/A - *)$ by excision. Similarly, $H_*(X, V)$ is isomorphic to $H_*(X - A, V - A)$. Now the two pairs

$$(X/A - *, V/A - *), \quad (X - A, V - A)$$

are homeomorphic. So they have the same homology.

It follows that $\widetilde{H_*}(X/A) \simeq H_*(X/A, V/A) \simeq H_*(X, A)$. That completes the proof. ▲

▲

Lecture 13
[Section] 9/27

§1 Jordan curve theorem

Some people have heard about the Jordan curve theorem.

13.1 Theorem (Jordan curve theorem)**.** *Let C be a simple plane curve. Then there are two points in $\mathbb{R}^2 - C$ such that every point in $\mathbb{R}^2 - C$ can be joined to one of them in a path not hitting C.*

In other words, $\mathbb{R}^2 - C$ has two components. The boundary of each is C.

Another way of saying this is that if $\gamma : S^1 \to \mathbb{R}^2$ is an injective map, then

$$H_0(\mathbb{R}^2 - \gamma(S^1)) = \mathbb{Z}^2.$$

Since by adding a point at infinity we do not change the connected components, we can replace $\mathbb{R}^2$ by S^2. In particular, we'd like to show that $H_0(S^2 - \gamma(S^1)$ is $\mathbb{Z}^2$.

Start with a different problem. Let's look at maps out of an interval.

13.2 Lemma. *Let $\gamma : I \to S^2$ be an injective map. Then $\widetilde{H_*}(S^2 - \gamma) = 0$.*

Proof. Suppose the contrary. Consider $X_{a,b} = S^2 - \gamma([a, b])$. Then $X_{a,b} \cup X_{b,c}$ is just $S^2 - \gamma(b)$ and $X_{a,b} \cap X_{b,c} = X_{a,c}$. So we can apply Mayer-Vietoris to this cover. We get a long exact sequence

$$\widetilde{H_{*+1}}(S^2 - \gamma(b)) \to \widetilde{H_*}(X_{a,c}) \to \widetilde{H_*}(X_{a,b}) \oplus \widetilde{H_*}(X_{b,c}) \to \widetilde{H_*}(S^2 - \gamma(b)).$$

However, $S^2 - \gamma(b)$ is contractible. So the middle map is an isomorphism.

Let us suppose that $\omega \in \widetilde{H_n}(X_{0,1}) - 0$. Then ω must be nontrivial in one of $\widetilde{H_n}(X_{0,1/2})$ or $H_n\widetilde{(X_{1/2,1})}$. Then keep going. We get that ω is not zero in $\widetilde{H_n}(X_{a,b}))$ where a, b get closer and closer to some point x, since a nested sequence of compact intervals must have a nonzero intersection.

Now $S^2 - x$ has trivial homology. In particular, ω is a boundary ∂B in $S^2 - x$. But the boundary must have compact image, i.e. B must have compact image. So ω must be a boundary ∂B in some $\widetilde{H_n}(X_{a,b})$ where a, b are close enough to x, contradicting the assumption that it is nontrivial there. ▲

Now we are almost done in our attempt to do this for $\gamma : S^1 \to S^2$. Let $\gamma : S^1 \to S^2$ be an injective map. We can decompose S^1 into two halves A, B, each of which is an arc.

We have a long exact sequence (Mayer-Vietoris with $S^2 - \gamma(A), S^2 - \gamma(B)$—the union is S^2 minus two points):

$$\widetilde{H_{n+1}}(S^2-\gamma(A))\oplus\widetilde{H_{n+1}}(S^2-\gamma(B)) \to \widetilde{H_{n+1}}(S^2-*,*') \to \widetilde{H_n}(S^2-\gamma) \to \widetilde{H_n}(S^2-\gamma(A))\oplus\widetilde{H_n}(S^2-\gamma(B))$$

But the two ends are zero since S^2 minus an interval has trivial reduced homology as shown. It follows that

$$\widetilde{H_0}(S^2 - \gamma(S^1)) \simeq \widetilde{H_1}(S^2 - *, *') \simeq \widetilde{H_1}(S^1) \simeq \mathbb{Z}$$

as $S^2 - *, *'$ deformation retracts onto a circle S^1.

§2 Suspensions

Let X be a space. Then we can consider the **suspension** ΣX obtained by taking the cylinder $X \times I$ and collapsing $X \otimes \{0\}$ and $X \otimes \{1\}$ to a point. In general,

$$\Sigma S^n \simeq S^{n+1}.$$

There is a nice relation between the homology of X and ΣX.

13.3 Proposition. $\widetilde{H_n}(X) \simeq \widetilde{H_{n+1}}(\Sigma X)$.

Proof. Indeed, there is an open cover $\Sigma X = U \cup V$ where U, V are open sets whose intersection deformation retracts onto X and such that U, V are contractible. This is because U can consist of pairs (x, t) with $t > \frac{1}{4}$ and V can consist of points (x, t) with $t < \frac{3}{4}$. The Mayer-Vietoris sequence in reduced homology now immediately gives the result. ▲

In general, there is a similar relation between homotopy groups:

$$\pi_n(X) \simeq \pi_{n+1}(\Omega X).$$

This follows because Σ and Ω are adjoint functors.

§3 Example of cellular homology

Let us compute the homology groups of the sphere (not with the very simple cell structure). We already know the answer, of course.

Never mind, can't possibly liveTEX this. The homology of S^2 was computed via cellular homology. Then, $\mathbb{R}P^2$ was done via the canonical cellular structure. More generally, $\mathbb{R}P^n$ can be done in this way.

Then there was a discussion of what the lens space looks like.

Lecture 14
9/29

§1 An application of degree

There is an important application of the degree of a map that is highly useful. We have to show that the homotopy invariance of degree implies the fundamental theorem of algebra.

14.1 Theorem. *Every nonconstant polynomial in* $\mathbb{C}[z]$ *has a complex root.*

Proof. When the absolute value of the input is very large, the polynomial map looks like $z \to z^d$.

So to begin with the proof, we consider the degree of

$$z \to z^d,$$

$S^1 \to S^1$. If you view S^1 as a d-gon, then each side goes around the whole circle once. Since a generator of $H_1(S^1)$ is given by the sum of the sides of the d-gon, it follows that $z \to z^d$ takes a homology generator to d times it. In particular, the degree is d.

Consider a hypothetical monic nonconstant polynomial

$$f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0,$$

which has no root in $\mathbb{C}$. Then the paths γ_r defined by $\gamma_r(t) = f(re^{it})$ produces a family of maps from $S^1 \to \mathbb{C} - \{0\}$. Projection to S^1 gives a family of maps

$$\Gamma_r : S^1 \to S^1, \quad \Gamma_r(t) = \frac{f(re^{it})}{|f(re^{it})|}$$

which are obviously homotopic to each other. When r is large, then f is approximately z^d so γ_r is approximately $t \to r^d e^{idt}$, implying Γ is basically $z \to z^d$. Thus all the Γ_r have degree d. But Γ_0 is constant so has degree zero, contradiction.

To be clear, when $r \gg 0$, the point is that Γ_r is very close to $z \to z^d$, and any two very close maps $S^1 \to S^1$ are homotopic. ▲

§2 Cellular homology

We now discuss a way to compute the homology of CW complexes.

Let X be a CW complex. Then there is a filtration on X of the form $X^{(0)} \subset X^{(1)} \subset \ldots$ such that $X^{(n)}$ is obtained from attaching n-cells to the $n-1$-skeleton $X^{(n-1)}$. Let the set of n-cells used be A_n.

We are going to work with finite complexes, so we have $X = X^{(d)}$ for some d. The goal is to get a chain complex

$$C_*^{CW}(X)$$

whose homology is the singular homology $H_*(X)$. $C_*^{CW}(X)$ will be free in dimension k on A_k.

Last time, we saw that $H_n(X^{(n)}, X^{(n-1)}) \simeq \mathbb{Z}^{A_n}$. Using this fact, we will get the above fact. More generally, since $X^{(n-1)}$ is locally a deformation retract, we have that

$$H_k(X^{(n)}, X^{(n-1)}) \simeq \widetilde{H_k}(X^{(n)}, X^{(n-1)}) \simeq \begin{cases} \mathbb{Z}A_n & \text{for } k = n \\ 0 & \text{otherwise} \end{cases}$$

because $X^{(n)}/X^{(n-1)}$ is the wedge sum of various n-spheres indexed by A_n.

14.2 Definition. We define $C_*^{CW}(X)$ as follows. The group $C_n^{CW}(X)$ is defined to be $H_n(X^{(n)}, X^{(n-1)})$. To get the boundary map, we take the composition

$$H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

where the first map is the boundary map in the long exact sequence and the second map comes from the inclusion.

This is indeed a chain complex. If we compose two consecutive maps, we find

$$H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}), X^{(n-2)}) \\ \to H_{n-2}(X^{(n-2)}) \to H_{n-2}(X^{(n-2)}, X^{(n-3)})$$

where the middle two are just consecutive maps in a suitable long exact sequence. So the whole composition is zero.

14.3 Proposition. *The homology $H_*(C_*^{CW}(X))$ is isomorphic to singular homology $H_*(X)$.*

To see this, we will prove

14.4 Lemma. $H_k(X^{(n)}) = 0$ *if* $k > n$.

This would be necessary to show that cellular homology worked, because $X^{(n)}$ has no k-cells.

Proof. Induction. When $n = 0$, then $X^{(0)}$ is just a bunch of discrete points, and evidently $H_k(X^{(0)}) = 0$ when $k > 0$.

Assume the lemma holds for $n - 1$. Consider the piece of the long exact sequence

$$H_k(X^{(n-1)}) \to H_k(X^{(n)}) \to H_k(X^{(n)}, X^{(n-1)}).$$

By the inductive hypothesis, the left is zero. The right is zero because of the computation of $H_*(X^{(n)}, X^{(n-1)})$, which is the reduced homology of a wedge of spheres. ▲

Remark. By Mayer-Vietoris, the homology of a wedge of spaces Y_i such that the basepoints have contractible neighborhoods satisfies

$$H_*(\vee Y_i) = \bigoplus H_*(Y_i).$$

14.5 Lemma. *The inclusion* $X^{(n+1)} \to X$ *induces an isomorphism* $H_k(X^{(n)}) \to H_k(X)$ *if* $n \geq k + 1$.

Proof. Also by induction. This time, however, we use descending induction. For $n = d$, the claim is true because $X^{(n)} = X$. Assume that the lemma holds for $n + 1$; we prove it for n.

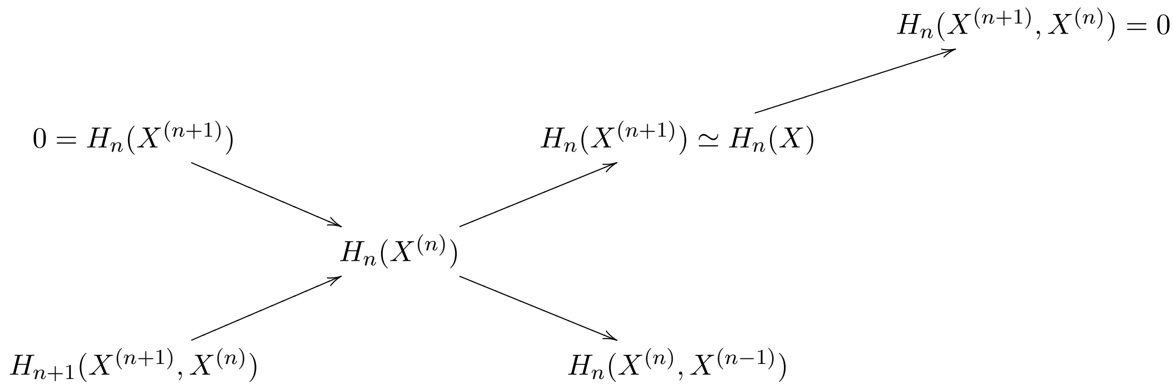
By the inductive hypothesis, the inclusion $X^{(n+1)} \to X$ induces isomorphisms on H_k, $k \leq n$. Now we want to show something similar for $X^{(n)} \to X$. By this hypothesis, it is sufficient to show that $X^{(n)} \to X^{(n+1)}$ induces isomorphism in homology for $k + 1 \leq n$. But we can use the long exact sequence in homology, as usual:

$$H_k(X^{(n+1)}, X^{(n)}) \to H_k(X^{(n)}) \to H_k(X^{(n+1)}) \to H_k(X^{(n+1)}, X^{(n)}).$$

Since in this exact sequence we have $k < n$, we also have that the two ends are zero, and the map on H_k is an isomorphism. The lemma is proved. ▲

Now, let us prove the proposition.

Proof. Now one writes some exact sequences. I will try to liveTEXthis up properly later. At level n, we have $H_n(X^{(n)}, X^{(n-1)})$. We recall that the boundary maps were given by



In particular, we can write $H_n(X)$ as the cokernel of $H_{n+1}(X^{(n+1)}, X^{(n-1)}) \to H_n(X^{(n)})$. Now, the kernel of the map out of $H_n(X^{(n)}, X^{(n-1)})$ can be computed by drawing more exact sequences and extending this diagram by symmetry, together with the same tricks used above. I find it easier to think about than write about. ▲

Lecture 15
10/1

§1 The cellular boundary formula

The first thing we have to do today is to talk about cellular homology. Well, we know what the cellular complex's groups are. Recall that if we have a CW complex X with a skeletal filtration

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(d)} = X$$

such that each $X^{(n)}$ is constructed from $X^{(n-1)}$ by gluing n-cells, which form a set A_n. Suppose that the attaching maps are given by $i_\alpha : D^n \to X^{(n)}$.

The **cellular chain complex** was

$$H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})$$

where the boundary map was the composition of a boundary map for the exact sequence of a pair and the map $X^{(n-1)} \to (X^{(n-1)}, X^{(n-2)})$. Here the nth group turns out in fact to be free on A_n. In other words, A_n is a basis for $H_n(X^{(n)}, X^{(n-1)})$, because this latter guy is isomorphic to $\widetilde{H_n}(X^{(n)}, X^{(n-1)})$, and this quotient is just a wedge of spheres.

The question is what the boundary of the element $\alpha \in A_n$ is as an element of the $n-1$st group, i.e. as a sum of elements $\beta \in A_{n-1}$. The definition of the boundary map sends α (given by i_α) to $i_\alpha|_{\partial D^n}$, which becomes an element of the $n-1$st group. For finite free modules, we have projections. So $H_{n-1}(X^{(n-1)}, X^{(n-2)})$ projects onto each of the direct summands $\mathbb{Z} i_\beta$ for $\beta \in A_{n-1}$. This comes from taking the projection onto the sphere S^{n-1}_β and using the fact that the homology of the sphere is $\mathbb{Z}$. The map $X^{(n-1)}/X^{(n-2)} \to S^{n-1}_\beta$ just crushes all the spheres except the βth one.

The long story is that α gets sent to the sum $\sum c_\alpha \beta$, where c_α is the integer that describes the map $i_\alpha|_{\partial D^n} \to S^{n-1}_\beta$. In particular, c_α is just the degree of $i_\alpha|_{\partial D^n} \to S^{n-1}_\beta$. (This takes a little amount of self-convincing since I am sketching the argument; Hatcher has a thorough proof.)

In particular, the matrix between

$$\mathbb{Z}A_n \to \mathbb{Z}A_{n-1}$$

is just the matrix of degrees of the n-attaching maps restricted to S^{n-1} and crushed to the various $n-1$ spheres.

15.1 Proposition (Cellular homology)**.** *The (β, α)th entry in the matrix of the boundary map $\mathbb{Z}A_n \to \mathbb{Z}A_{n-1}$ is*

$$\deg q_\beta i_\alpha|_{\partial D^n}$$

where $q_\beta : X^{(n-1)} \to S^{n-1}_\beta$ is the βth "crushing."

§2 Examples

15.2 Example. Let us compute the homology of real projective space $\mathbb{RP}^n$. Recall that this is made from the set of nonzero vectors $v \in \mathbb{R}^{n+1} - \{0\}$ up to equivalence $v \sim \lambda v, \lambda \in \mathbb{R}^*$ under scaling. It is equivalently

$$S^n / \{x \sim -x\}$$

since everything can be scaled down to the sphere. So there is a canonical map

$$S^n \to \mathbb{RP}^n.$$

The cell structure of $\mathbb{RP}^n$ has one cell in each dimension from 0 to n. In particular, $\mathbb{RP}^n$ is a union $D^0 \cup D^1 \cup D^2 \cup \cdots \cup D^n$. The reason is that $\mathbb{RP}^n - \mathbb{RP}^{n-1}$ is a single disk. Indeed, the collection of classes of $\mathbb{RP}^n$ represented by $(x_0, \dots, x_n, 0)$ (where the last coordinate is zero) is isomorphic to $\mathbb{RP}^{n-1}$. The complement of this, namely the set of classes of the form

$$(x_0, \dots, x_n) \in S^{n-1} \quad x_n \neq 0$$

is equivalently the set of classes of the form

$$(x_0, \dots, x_{n-1}, x_n),\ x_n > 0$$

This set of classes is homeomorphic to D^{n-1} (because the first $n-1$ coordinates can be anything). The cellular complex is all $\mathbb{Z}$ as a result.

What are the boundary maps? Well, recall that the top disk is S^n intersected with the upper half-space. The boundary $\mathbb{RP}^{n-1}$ is the quotient of the boundary S^{n-1} on the plane $x_n = 0$ under the antipodal map. Let i_k be the attaching map $D^k \to \mathbb{RP}^n$

which sends D^k to the upper half-space of S^k. We need to compute the degrees of i_k restricted to the boundary and crushed

$$\mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \simeq S^{k-1}.$$

This composition turns out to be best drawn visually, but it is the coproduct

$$S^{k-1} \stackrel{1\vee -1}{\to} S^{k-1} \vee S^{k-1} \to S^{k-1}.$$

But, well, the antipodal map has degree $(-1)^k$. Thus the degree is $1+(-1)^k$. So it follows that the cellular chain complex has the boundary maps which are alternatively zero and multiplication by 2.

It follows that, for n even, $H_*(\mathbb{RP}^n)$ is $\mathbb{Z}$ at zero, $\mathbb{Z}/2$ in odd degrees, and zero otherwise. (For n odd, there is also a $\mathbb{Z}$ in the top degree.)

More generally, one can define the **Stiefel manifolds** as parameterizing not just lines (like projective space) but planes.

15.3 Definition. The **Stiefel manifold** $V_{n,k}$ parameterizes ordered tuples $(v_1, \dots, v_k) \in \mathbb{R}^n$ of orthonormal vectors. Each such element is called a k**-frame**. There are a bunch of different (equivalent) ways to give it a topology. We will view it as a coset space of $SO(n)$. Recall that $SO(n)$ is the set of orthogonal matrices $\mathbb{R}^n \to \mathbb{R}^n$ and determinant one.

If $e_1, \dots, e_n$ is an orthonormal basis for $\mathbb{R}^n$, then there is a set map

$$SO(n) \to V_{n,k}$$

sending M to

$$M \to (Me_n, Me_{n-1}, \dots, Me_{n-k+1}).$$

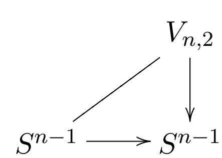
The subgroup $SO(n-k)$ is a subgroup of $SO(n)$ consisting of matrices that fix the last k vectors $e_j, j \geq n-k+1$. In particular, we get a map

$$SO(n)/SO(n-k) \to V_{n,k}.$$

By definition, this map is bijective; we give $V_{n,k}$ a topology such that this is a homeomorphism (where $SO(n)/SO(n-k)$ is given the quotient topology).

15.4 Example. Now $V_{n,2}$ is equal the set of pairs of vectors that are orthogonal and each is norm one, so it is really the subspace of $T(S^{n-1})$ where the tangent vector has length one. The map is (v_1, v_2) to v_1 with tangent v_2.

What we showed earlier, the nonexistence of nonvanishing vector fields on even-dimensional spheres, is that there is no section



This nonexistence of sections has to do with characteristic classes. By calculating the homology of the Stiefel manifold, we can reprove the earlier result. The thing we will show is that

15.5 Proposition. *$H_n(V_{n+1,2}) \simeq \mathbb{Z}/2$ for n odd.*

This will imply that there is no section out of S^n for n even, because there is no map $\mathbb{Z}$ to $\mathbb{Z}/2$ that could complete the above diagram. But, um, we're out of time again. So actually we are not going to show this. Sorry about that.

Lecture 16
10/4

In the traditional courses on algebraic topology, you learn your way around real projective space real good. We want to look at more important spaces. Kirsten said something about Stiefel manifolds last time, but she didn't quite finish explaining the cell structure. Actually, she didn't start explaining the cell structure. We're going to come back to the topic on Wednesday.

What we want to talk about today is the **Lefschetz formula**.

§1 Lefschetz fixed point formula

There're lots of variations on this very beautiful idea. Let us start with a simple situation. Suppose S is a finite set. Let $T : S \to S$ be a map. A **fixed point** is a point $s \in S$ such that $T(s) = s$. We can try to get a measure of the number of fixed points in the following way. Let $\mathbb{Z}\{S\}$ be the free abelian group on S. There is a map

$$T_* : \mathbb{Z}\{S\} \to \mathbb{Z}\{S\}$$

where the matrix of T_* is given by how T acts (e.g. a permutation matrix if T is a bijection).

16.1 Example. Let $S = \{1, 2, 3\}$ and T swaps $(2, 3)$ while fixes 1. Then the matrix of T_* is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

the fixed points correspond to the ones in the diagonal. In particular, we can count the fixed points of T by counting the ones down the diagonal of the T_* matrix. That is, finding the **trace** of T_*.

$$|\{\text{fixed points}\}| = \operatorname{tr} T_*$$

The Lefschetz formula generalizes this to spaces. It replaces this map with the map on homology.

Suppose X has the homotopy type of a finite CW complex. This implies that the homology groups $H_n(X)$ are finitely generated by cellular homology. They are also nonzero for only finitely many n.

Let $f : X \to X$ be a map. Then it induces a map

$$f_* : H_*(X) \to H_*(X)$$

and we will define a number, called the **Lefschetz number** $\tau(f)$, which will be a kind of trace. This is one of those things that's obvious after someone has told them to you. You might think that the right thing to do would be to take the trace of the map on homology. But you actually take the alternating sum.

16.2 Definition. The **Lefschetz number** $\tau(f)$ is

$$\tau(f) = \sum_{i=0}^{\infty} (-1)^i \mathrm{Tr}(f_*|_{H_n(X)}).$$

Note that the sum is finite.

Technically, we have to define the **trace**. We do this now. Let A be a f.g. abelian group. Let $T : A \to A$ be a homomorphism. Then T induces a map

$$T : A/A_{\mathrm{tor}} \to A/A_{\mathrm{tor}}.$$

Note that a f.g. abelian group can be represented as $\mathbb{Z}^r$ (for r the **rank**) plus a finite group. The decomposition is not canonical, but the subgroup A_{tor} is canonical, so the quotient is canonical. Anyway, the map on the torsion quotients leads to

$$\overline{T} : \mathbb{Z}^r \to \mathbb{Z}^r,$$

whose trace is defined to be the **trace** of T.

16.3 Example. If $f = 1_X$, then $\tau(f)$ is the Euler characteristic $\chi(X) = \sum (-1)^i \mathrm{rank} H_i(X)$.

We'd like a theorem like:

> The number of fixed points of f is the Lefschetz number.

We'd like to say that this is true, and we can say that; it's a free country. But we may be wrong. And indeed, this is **not true**. It is actually true that f is always homotopic to a map with $\tau(f)$ fixed points.

We will discuss a special case of this which is useful. The theorem is:

16.4 Theorem. *Let K be a finite simplicial complex, and X the underlying space (the geometric realization). If $f : X \to X$ has no fixed points, then*

$$\tau(f) = 0.$$

We will start proving this today. But first, a comment. Note that the Lefschetz number depends only on the effect in homology. So it depends only on the homotopy class of f. So if $\tau(f) \neq 0$, then f is not homotopic to a map without fixed points.

16.5 Corollary. *If $f : X \to X$ satisfies $\tau(f) \neq 0$, then f has a fixed point.*

§2 Simplicial approximation

16.6 Theorem. *Let $X = |K|$ is the space underlying a finite simplicial complex K, and $Y = |L|$ the space underlying another one L. Let $f : X \to Y$ be a continuous one. After sufficient barycentric subdivision of X, f is homotopic to a simplicial map.*

A simplicial map is a purely combinatorial map of simplicial complexes, i.e. a map of vertices that sends simplices into simplices. This leads by linear extension to a map on the spaces.

We'll describe the proof of this in the next lecture.

16.7 Corollary. *Let $X = |K|$, K finite. Let $f : X \to X$. If f has no fixed points, then after sufficient barycentric subdivision of X, f is homotopic to a simplicial map $\widetilde{f}$ which fixes no simplices (of any dimension).*

16.8 Example. Take the circle with its simplicial structure (two 0-simplices, two 1-simplices). Then rotation by 90 degrees is not simplicial. However, barycentric subdivision yields a simplicial map.

OK. We'll come to grips with all this later. We'll prove this next time.

§3 Proof of the theorem

For now, assume the simplicial approximation theorem and all that.

Pf of Lefschetz. As usual, $X = |K|$, and $f : X \to X$. K is finite. f has no fixed points.

The corollary implies that we can subdivide X to get a new finite simplicial complex $X' = |K'|$, we may suppose that f is homotopic to a simplicial map $\widetilde{f}$ with no fixed simplices.

To set some notation, let K'_n be the set of n-simplices in K'. Then $\widetilde{f} : K'_n \to K'_n$ has no fixed points. We can use the simplicial chains to compute the homology. So $C_n^{\mathrm{simp}}(X) = \mathbb{Z}\{K'_n\}$ is the simplicial chain complex. The map

$$\widetilde{f} : C_n^{\mathrm{simp}}(X) \to C_n^{\mathrm{simp}}(X)$$

has trace zero since $\widetilde{f}$ has no fixed points on K'_n. We find that the alternating sum

$$\sum (-1)^i \mathrm{Tr}(\widetilde{f}_*)|_{C_i^{\mathrm{simp}}(X)} = 0.$$

Now we use some algebra.

16.9 Lemma. *Let C_*, d be a finite chain complex of finitely generated abelian groups. Let $T : C_* \to C_*$ be a chain map. Then*

$$\sum (-1)^i \mathrm{Tr} T|_{C_i} = \sum (-1)^i \mathrm{Tr} T|_{H_i(C)}.$$

Proof. Exercise. (It is in the homework.) ▲

So you can compute the Lefschetz number either using the chain complex or the homology. By the algebraic lemma, the Lefschetz number $\tau(\widetilde{f}) = 0$, so the same is true for the homotopic map f. ▲

(Incidentally, we have used the equivalence of simplicial and singular homology.)

16.10 Example. Let $X = D^n$, which is the realization of a suitable simplicial complex. Then any map $f : X \to X$ has a fixed point. The reason is that the Lefschetz number must automatically be one, so the homology is just $\mathbb{Z}$ in degree zero and zero everywhere else. We have proved:

16.11 Theorem (Brouwer fixed point theorem)**.** *Let $f : D^n \to D^n$ be a continuous map; then it has a fixed point.*

There's another proof just directly by singular homology. It's usually done as a corollary of the computation for the sphere's homology. The usual argument: if f has no fixed points, then drawing the ray through $f(x), x$ and intersecting it with S^{n-1} gives a retraction $D^n \to S^{n-1}$. This is impossible since D^n has trivial homology and $H_{n-1}(S^{n-1}) = \mathbb{Z}$.

16.12 Example. Take a torus $\mathbb{T}^2$ and map it to itself $f : \mathbb{T}^2 \to \mathbb{T}^2$ by flipping it around the vertical axis. The map has four fixed points. Let's calculate the Lefschetz numbers. Then f induces the identity on H_0. On H_1, we have a group generated by cycles a, b. If you draw some pictures, you can arrange it so that f sends a to $-a$ and b goes to $-b$. So the trace of f on H_1 is -2. Moreover, $x \in X$ is the top fixed point, then since $H_2(X) \simeq H_2(X - x)$ and f is homotopic to the identity in a neighborhood D of x, we can compute the Lefschetz number to be 4.

The theorem can often be used to count the number of fixed points.

Lecture 17
[Section] 10/4

Recall that the **Stiefel manifold** $V_{k,n}$ is the set of tuples $(e_1, \dots, e_k) \in \mathbb{R}^n$ of unit length vectors which are mutually orthogonal. A priori, this is just a set. However, we have an inclusion

$$V_{k,n} \to S^{n-1} \times \cdots \times S^{n-1},$$

and we give it the subspace topology.

Another way to give $V_{k,n}$ a topology is to use the map

$$SO(n)/SO(n-k) \to V_{n,k}$$

and to give $V_{n,k}$ the quotient topology. (Here $SO(n)$ has a subspace topology as a subspace of $M_n(\mathbb{R})$.) This map sends a matrix $M \in SO(n)$ to

$$(Me_1, \dots, Me_k) \in V_{n,k}.$$

The elements of $SO(n)$ that fix all these $\{e_1, \dots, e_k\}$ are precisely the ones in $SO(n-k)$, so this map is well-defined.

These two topologies are the same.

17.1 Example. If you have a map $f : X \to Y$ between CW complexes such that $f_* : H_*(X) \to H_*(Y)$ is trivial in all degrees > 0, then is necessarily f homotopic to the constant map?

The answer is **no**. An example is the **Hopf fibration**. Take the 3-sphere $S^3 \subset \mathbb{C}^2$. There is a natural map $\mathbb{C}^2 \to \mathbb{CP}^1 = S^2$ sending $(z, w) \to [z, w]$. We get a map

$$S^3 \to S^2$$

which induces zero in homology because the homology groups don't simultaneously nonvanish outside dimension zero. However, it is not null-homotopic. We don't know enough to prove this though.

There is a proof that sounds reasonable. This map $S^3 \to S^2$ is a fibration. We don't know what this is though. It sort of looks like a covering space. If I take a point in S^2 and takes its preimage in S^3, that pre-image is just going to correspond to a (complex) line in $\mathbb{C}^2$, which intersects with S^3 in a circle S^1. In particular, the fibers are all homeomorphic to S^1. It turns out that many of the properties we know about covering spaces work for this map as well. In particular, we have the **homotopy lifting property**.

If $S^3 \to S^2$ were homotopic to a constant, then the identity $S^3 \to S^3$ would be homotopic to a constant as well by the homotopy lifting property. This is impossible since the sphere is not contractible.

Lecture 18
10/6

We will talk today about the simplicial approximation theorem.

§1 Simplicial approximation theorem

18.1 Theorem. *Let K be a finite simplicial complex and L a simplicial complex. Suppose given a continuous map*

$$f : |K| \to |L|.$$

After subdividing K, f is homotopic to a simplicial map.

Remark. L is the union of its finite subcomplexes. The image of f is compact, so its image lands inside some finite subcomplex. As a result, we don't gain anything by allowing L to be infinite; we might as well assume that L is finite as well.

Remember that a simplicial complex K is a combinatorial object: has a set K_0 of vertices and for each n, a set K_n of $n+1$-element subsets of K_0 called the n-simplices. Every subset of a simplex is a simplex. This does **not** mean that every subset of the geometric realization is a simplex! The point is that a collection of vertices spans a simplex in the geometric realization, then every subset of that collection does so as well.

We start with some generalities on simplicial complexes.

§2 Stars

18.2 Definition. Let $\sigma \in K_n$ be an n-simplex of K. Then we define the **star** of σ to be the subset of $|K$ given by:

$$\mathrm{St}(\sigma) = \bigcup_{\sigma' : \sigma \subset \sigma'} \sigma'.$$

For instance, the star of a vertex v is the collection of all the closed simplices containing v. Note that not every simplex contained in the star has to contain the vertex! This is because we allow faces of simplexes in the star to be in the star.

18.3 Definition. The **open star** of σ is the union

$$\text{open star}(\sigma) = \bigcup_{\sigma \subset \sigma'} \sigma' - \partial \sigma'.$$

The open star is an open subset of the star.

18.4 Example. The open star of a vertex v is the collection of points in the realization $|K|$ which contain v with nonzero coordinate. The open star of a simplex σ which isn't a zero-simplex does not contain the simplex, though. (We always have $\mathrm{openstar}(\sigma) \supset \mathrm{Int}(\sigma)$.)

18.5 Proposition. *The open star* $\mathrm{openstar}(\sigma)$ *is always an open set containing* $\sigma - \partial\sigma$.

Proof. One can prove this by checking it for the standard n-simplex. When K is finite, the realization $|K|$ is a subcomplex of some Δ^N (e.g. take $N = K_0$). One checks that the act of taking the open star commutes with intersections, so the fact that the open star is open in a standard simplex implies that it is open in any finite simplicial complex. ▲

Another fact we will need is:

18.6 Proposition. *The collection* $\mathrm{St}(v), v \in K_0$ *of stars of vertices is a collection of subsets of* $|K|$ *whose interiors cover* $|K|$.

Proof. Indeed, any point $x \in |K|$ is in the interior of some simplex, so it is in the open star of any vertex of that simplex. ▲

§3 Proof of the simplicial approximation theorem

Let $f : |K| \to |L|$ be continuous.

Proof. We can cover $|L|$ by the open stars $\mathrm{openstar}(v)$ for $v \in L_0$. Choose ϵ so small that ϵ is bigger than the Lebesgue number of the covering of K:

$$f^{-1}(\mathrm{openstar}(v)), v \in L_0.$$

We are regarding $|K|$ as a subset of some standard simplex. In particular, $|K|$ is a metric space. Now subdivide $|K|$ so that the diameter of each simplex in the subdivision is $< \epsilon/2$. Call the new simplicial complex K'. It follows then that f maps each simplex in $|K'|$ into an open star of something in $|L|$. For each $v \in K'_0$, choose $g(v) \in L_0$ such that

$$f(v) \in \text{openstar}(g(v)).$$

We get a map

$$g : K'_0 \to L_0.$$

I claim that this map extends to a simplicial map $|K'| \to |L|$. What's the content of that? The content is that if a collection $v_0 \dots v_m$ span a simplex in $|K'|$, then $g(v_0), \dots, g(v_m)$ span a simplex in L. The proof is fairly simple, but we need a lemma.

18.7 Lemma. *In an arbitrary simplicial complex W, vertices $w_0, \dots, w_n$ span a simplex of W if and only if*

$$\bigcap \text{openstar}(v_i) \neq \emptyset.$$

Proof. We might as well suppose that the vertices are distinct. If they span a simplex, then the interior of that simplex is in all their stars. Conversely, if $\bigcap \text{openstar}(v_i)$ is nonempty, then pick a point in that intersection. The smallest simplex containing it contains all the v_i.

This is "easy." But if one doesn't prove it, one might get the statement wrong. ▲

OK. Now, we have to show that $g(v_0), \dots, g(v_m)$ spans a simplex in L if $v_0, \dots, v_m$ spans a simplex in $|K'|$. Let x be in the interior of the simplex containing the $\{v_i\}$. Then $f(x)$ is contained in openstar$(g(v_i))$ for each i. This implies that

$$\bigcap \text{openstar}(g(v_i)) \neq \emptyset.$$

So g extends to a simplicial map. We now have to construct the homotopy between f, g. Again, we have two maps

$$|K'| \xrightarrow{f,g} |L| \subset \mathbb{R}^N.$$

We can just consider the linear homotopy $t \to (1-t)f + tg$ between f, g. Of course, one has to check that these linear combinations actually lie in $|L|$ and not simply in $\mathbb{R}^N$. That's because of the Lebesgue number business. For time constraints, we can't actually go through the proof of that.

▲

§4 Lefschetz fixed point theorem

Last time there was a mistake. Let's go back to the Lefschetz fixed point theorem.

Let

$$|K| \xrightarrow{f} |K|$$

be a continuous endomap of the realization of a finite simplicial complex to itself. We claimed that if f has no fixed points, then $\tau(f) = 0$.

Suppose f has no fixed points. Put a metric d on $|K|$. We can choose $\epsilon > 0$ such that $d(x, f(x)) \geq \epsilon$. We can subdivide $|K|$ so that every open star of a vertex in v has diameter $< \epsilon$. Taking the subdivision doesn't change the topology of the geometric realization. So we can just assume that every simplex in K has diameter $< \epsilon$.

If we subdivide further, we can use simplicial approximation to get a *simplicial* map

$$g : |K'| \to |K|$$

for K' a further subdivision of K, such that $g \simeq f$. The thing about the open stars tells you that every simplex in K', which came from a simplex σ in K, gets moved under g to a simplex which is disjoint from σ. This is because f moves everything a whole bunch and the simplices are very small.

We now have to check that $\tau(g) = 0$. We have a chain map

$$C_*(K') \to C_*(K) \to C_*(K')$$

thanks to the chain homotopy involving subdivision. With respect to this chain map, the associated $C_*(K') \to C_*(K')$ associated to g has zeros on the diagonal when represented as a matrix. So the trace is zero. Thus $\tau(g) = 0$. Alright, sorry. We were rushing at the end.

Lecture 19
10/8

We now want to add in a bunch of topics that aren't particularly well represented in Hatcher's book. Hatcher is a geometric topologist, but we like algebra, so we will emphasize some algebra. The first thing to talk about is the tensor product.

§1 Tensor products

Everyone in the class already knows about them, so it is a bit of a review.

Recall that if A, B are abelian groups, then we can define a new group $A \otimes B$ which we can characterize by its mapping properties. A homomorphism

$$A \otimes B \to C$$

is the same thing as a bilinear map

$$A \times B \xrightarrow{f} C.$$

The bilinearity means precisely that

1. $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$.
2. $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$.

Given this, we can give a definition of $A \otimes B$. This is the quotient of the free abelian group on generators $\{a \otimes b, a \in A, b \in B\}$ modulo the relations $(a+a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$.

19.1 Definition. $A \otimes B$ is called the **tensor product** of A, B.

19.2 Example. I can't resist telling you about this one great example. Consider the tensor product of abelian groups:

$$\mathbb{R} \otimes \mathbb{R}/\mathbb{Z}.$$

Start by considering an element $r \otimes \frac{x}{y}, x, y, \in \mathbb{Z}$. If we iterate the rules of the tensor product, it follows that

$$na \otimes b = n(a \otimes b),$$

so that

$$r \otimes \frac{x}{y} = (y\frac{y}{r}) \otimes \frac{x}{y} = \frac{y}{r} \otimes x = 0 \in \mathbb{R} \otimes \mathbb{R}/\mathbb{Z}.$$

It follows from this reasoning that $\mathbb{R} \otimes \mathbb{R}/\mathbb{Z} \simeq \mathbb{R} \otimes \mathbb{R}/\mathbb{Q}$.

A similar example would show that

$$\mathbb{Q} \otimes \mathbb{Q} \simeq \mathbb{Q}$$

under the bilinear multiplication map $(x, y) \to xy$.

Another really general fact is the tensor product commutes with direct sums.

19.3 Proposition. *We have*

$$(\bigoplus A_\alpha) \otimes B \simeq \bigoplus (A_\alpha \otimes B).$$

Proof. This follows from the universal property of the tensor product. This is also true in the other variable as well, namely

$$A \otimes \bigoplus B_\alpha \simeq \bigoplus A \otimes B_\alpha.$$

We leave a lot of these kinds of things to the reader, because they are best thought through for oneself. ▲

19.4 Example. Let's go back to the earlier example. As a $\mathbb{Q}$-vector space, $\mathbb{R}$ is uncountably-dimensional. We have

$$\mathbb{R} \simeq \bigoplus_{\text{uncountable}} \mathbb{Q}.$$

So $\mathbb{R} \otimes \mathbb{R}/\mathbb{Q}$ is a big sum of $\mathbb{Q}$'s tensored with a big sum of $\mathbb{Q}$'s. Since $\mathbb{R}/\mathbb{Q}$ is a direct sum of uncountably many copies of the rationals, $\mathbb{R}/\mathbb{Q} \simeq \bigoplus_{\text{uncountable}} \mathbb{Q}$, and we have

$$\mathbb{R} \otimes \mathbb{R}/\mathbb{Q} \simeq \bigoplus_{\text{uncountable}} \mathbb{Q} \simeq \mathbb{R}.$$

Remark. That's the space where a really beautiful invariant lives. This is called the Dehn invariant. There was an old problem of scissors congruence. If you take two polygons in $\mathbb{R}^2$, one defines them to be **scissors congruent** if you can cut the interior of the first one up to make the other. If two polygons P_1, P_2 are scissors congruent, then the areas are obviously the same. The converse is also true. If two polygons bound the same area, they are scissors congruent.

As someone pointed out loudly in class, this is **not** true in three dimensions. One looks at polyhedra and defines a corresponding notion of scissors congruence. One of the Hilbert problems was to determine whether there were other invariants for scissors congruence on polyhedra.

The answer is no. Dehn introduced something called the **Dehn invariant**. Given a polyhedron, one defines

$$D(P) = \sum_{\text{edges } e} \text{length}(e) \otimes \frac{\text{dihedral angle}}{2\pi} \in \mathbb{R} \otimes \mathbb{R}/\mathbb{Z}.$$

One can check that the Dehn invariant is an invariant of scissors congruence. One can calculate the Dehn invariant of a cube is zero, because the angles are all rational multiples of π, while the Dehn invariant of a tetrahedron is nonzero, because the angles are irrational multiples of π.

§2 Torsion products

I don't think anyone really says that all the way out.

Tensor products are great, except for the following. Suppose we have a surjective map

$$M \twoheadrightarrow N$$

of abelian groups, and an abelian group A; then

$$M \otimes A \twoheadrightarrow N \otimes A$$

is also surjective (check on the generators, for instance). The other relation is **false**. Suppose $M \hookrightarrow N$ is a subgroup. It is not necessarily true that $A \otimes M \to A \otimes N$ is injective.

19.5 Example. Suppose we have the map $\mathbb{Z} \overset{2}{\hookrightarrow} \mathbb{Z}$ of multiplication by two. Let us tensor by $\mathbb{Z}/2$. Then we get the map

$$\mathbb{Z}/2 \overset{2=0}{\to} \mathbb{Z}/2.$$

This is not injective, because multiplication by two is not injective.

So tensor products don't preserve exact sequences, in general.

Tensor products are not that far from doing so, though. There is a nice remedy for its not preserving an exact sequences. Let M be an abelian group, and A an abelian group. Choose a surjection $F_1 \twoheadrightarrow M$ for F free; the kernel F_2 is also free abelian. (The subgroup of a free abelian group is free abelian.) So we have an exact sequence

$$F_2 \hookrightarrow F_1 \twoheadrightarrow M.$$

We have a projective resolution of length two.

Tensor this with A: we get a sequence

$$A \otimes F_2 \to A \otimes F_1 \to A \otimes M \to 0.$$

Tensor products commute with cokernels, so this sequence is exact. The kernel of this sequence is not necessarily zero, though.

19.6 Definition. The kernel of $A \otimes F_2 \to A \otimes F_1$ is called $\mathrm{Tor}(A, M)$.

Remark. For modules over a ring other than $\mathbb{Z}$, you generally get higher tors.

The point is that the Tor takes care of the lack of exactness. This is best checked for oneself.

19.7 Proposition. $\mathrm{Tor}(A, M)$ *is independent of the free resolution of* M*.*

Proof. Remember that any two resolutions of M are chain homotopy equivalent—this is standard homological algebra. ▲

19.8 Proposition. $\mathrm{Tor}(A, M)$ *is symmetric in* A, M*. In particular,* $\mathrm{Tor}(A, M) \simeq \mathrm{Tor}(M, A)$*.*

Proof. Omitted. ▲

19.9 Example. Let us compute $\mathrm{Tor}(\mathbb{Z}/4, \mathbb{Z}/6)$. We use the resolution

$$0 \to \mathbb{Z} \xrightarrow{6} \mathbb{Z} \to \mathbb{Z}/6 \to 0.$$

Tensor this with $\mathbb{Z}/4$:

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \mathbb{Z}/6 \otimes \mathbb{Z}/4 \to 0.$$

The kernel is isomorphic to $\mathbb{Z}/2$, so this is the torsion product.

There is an important point next to save for later; there are other things to mention later.

19.10 Proposition. *If* M *is a torsion-free abelian group, then* $\mathrm{Tor}(A, M) = 0$ *for any abelian group* A*. Moreover, tensoring with* M *preserves exact sequences.*

Proof. To come back to later. ▲

§3 Homology with coefficients

There is a variation on the theme of homology that turns out to be extremely useful. It just reorganizes some of the information already done, but the reorganization can make things less opaque. Let A be an abelian group. X will be a topological space. Consider the chain complex $C_*(X)$ of X and tensor it everywhere with A. We define

$$C_*(X, A) = C_*(X) \otimes A.$$

This is a chain complex.

19.11 Definition. The homology groups of this tensored chain complex $H_*(C_*(X, A))$ are called the **homology groups with coefficients in** A**.** We can do this for cellular, singular, or simplicial chains.

Homology with coefficients satisfies all the same properties of regular homology. Namely,

1. Homotopy invariance.
2. Mayer-Vietoris
3. Excision

These are the analogous properties as in ordinary homology. These are actually all easy to prove. The homotopy invariance was an explicit chain homotopy, which you can just tensor with A. There is nothing deep there. Mayer-Vietoris and excision are really the same statement: the local nature of homology, namely the idea that homology depends only on small chains.

19.12 Example. Let us deduce excision for homology with coefficients. Recall that we deduced this from the short exact sequence

$$C_*(U \cap V) \hookrightarrow C_*(U) \oplus C_*(V) \twoheadrightarrow C_*^{\mathfrak{A}}(X) \subset C_*(X)$$

such that the last inclusion was a chain homotopy equivalence.

But every one of these things was a free abelian group. So it follows from the previous proposition that when you tensor this with A, you still get a short exact sequence

$$C_*(U \cap V) \otimes A \hookrightarrow (C_*(U) \oplus C_*(V)) \otimes A \twoheadrightarrow C_*^{\mathfrak{A}}(X) \otimes A.$$

The last thing is still chain homotopic to $C_*(X) \otimes A$ because you just tensor the chain homotopy with A.

The key point to note here is that the preservation of exactness in this case **does happen** precisely because we are working with free abelian groups.

19.13 Example. Let us look at $H_*(\mathbb{RP}^n, \mathbb{Z}/2)$. We can compute this via the cellular chain complex (with the standard CW structure on $\mathbb{RP}^n$), which we recall was a bunch of $\mathbb{Z}$'s such that the maps are either 2 or zero. When you tensor this with $\mathbb{Z}/2$, you get a cellular chain complex of copies of $\mathbb{Z}/2$ where all the maps are zero. In particular, all the homology groups of real projective space with coefficients in $\mathbb{Z}/2$ are just $\mathbb{Z}/2$, i.e.

$$H_*(\mathbb{RP}^n, \mathbb{Z}/2) \simeq \mathbb{Z}/2 \quad \text{for} \quad 0 \leq * \leq n.$$

The homology is still zero in higher dimensions. This is nice. There is less to remember.

§4 A loose end: the trace on a f.g. abelian group

There was something to mention that came up the problem set. If A is a f.g. abelian group, then A looks like $A_{\mathrm{tor}} \oplus \mathbb{Z}^r$ for some r. Note that

$$A \otimes \mathbb{Q} \simeq A_{\mathrm{tor}} \otimes \mathbb{Q} \oplus \mathbb{Q}^r.$$

However, $A_{\mathrm{tor}} \otimes \mathbb{Q} \simeq 0$ because if $a \in A$ satisfies $na = 0$, then

$$a \otimes x = na \otimes \frac{n}{x} = 0 \in A_{\mathrm{tor}} \otimes \mathbb{Q}.$$

We find that $A \otimes \mathbb{Q} \simeq \mathbb{Q}^r$. Suppose we have a map

$$f : A \to A.$$

We defined the **trace** of f to be the trace of the induced map $\overline{f}$ on $A/A_{\mathrm{tor}} \to A/A_{\mathrm{tor}}$. This, however, can be a mess.

The point is that we could have also defined the **trace** of f to be the trace on the map $A \otimes \mathbb{Q} \stackrel{f\otimes 1}{\to} A \otimes \mathbb{Q}$ in the sense of linear algebra. These are the same.

Lecture 20
10/11 [Section]

§1 Problems

Here is a hint for problem number 6:

20.1 Exercise. Consider the configuration space C of pairs of distinct points in $\mathbb{R}^n$, which is the quotient of $\mathbb{R}^n \times \mathbb{R}^n - \Delta$ modulo the identification $(x, y) \sim (y, x)$.

Solution. We have a linear map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ sending a point $(x, y) \to (x + y, x - y)$. It sort of descends to the quotient. In particular, it becomes a map

$$\mathbb{R}^n \times (\mathbb{R}^n - 0)/((x, y) \sim (x, -y)) \to C.$$

This is the hint.

Note that C is **not** homeomorphic to the set of pairs (x, y) such that $x < y$ in the lexicographic ordering. The lexicographic ordering is not continuous.

§2 The Riemann-Hurwitz formula

Let $f(x, y, z)$ be a homogeneous polynomial of degree d. So $f(x, y, z) \in \mathbb{C}[x, y, z]$. Then the zero locus $V(f) \subset \mathbb{CP}^2$. This is some closed subset. Suppose, moreover, that f is nonsingular, so $V(f)$ is a manifold. What is its genus?

Someone suggested that one can use the "arithmetic genus," but we don't need to talk about all that.

We have a map $V(f) \to \mathbb{CP}^1$ obtained by projection from a point $P \notin V(f)$. Now $\mathbb{CP}^1 = S^2$. So we have a map

$$M \to S^2.$$

Since a random line will intersect $V(f)$ in d points generally, we know that the generic inverse image of a point in S^2 has d points.

More generally:

> Suppose $f : M \to N$ is a mapping of manifolds which has degree d preimages most of the time (except outside a finite subset). What's the relation between $\chi(M), \chi(N)$, and d?

Let $Q = \operatorname{card}\{\text{points where } f \text{ fails to have degree d}\}$, where N is counted with multiplicities.

Then, with the above notation:

20.2 Proposition (Riemann-Hurwitz formula)**.** $\chi(M) = d\chi(N) - Q$.

Proof. We can use the simplicial approximation theorem talked about in class. First, we can approximate this map so it is a simplicial map. Then we can further subdivide our simplices such that all the points where f fails to have degree d are zero-simplices. Every k-cell in N is going to show up d times in M except in the case $k = 0$ when one has a point among the Q bad points. From this, one can get the formula.

There is a point that the subdivided map might not have the same condition of generically d preimages. But we'll avoid that point for the moment. ▲

The minus n is sometimes written as $-\sum(e_P - 1)$, which is the same thing as saying "compute with multiplicity."

Let us now apply this to the case of $V(f) \to \mathbb{CP}^1$. We find that

$$\chi(V(f)) = d\chi(S^2) - Q.$$

What's Q? We reduce to the question:

> Given a homogeneous polynomial in x, y of degree d, for how many y does the polynomial have a multiple root?

To do this, we find the discriminant, which is a polynomial in y of degree $d(d-1)$. For a generic polynomial, we find that the number of y's is $d(d-1)$. Thus for a generic nonsingular polynomial, we find

$$\chi(V(f)) = d - 2 - d(d-1).$$

Since the genus is $(2-\chi)/2$, we compute that the genus of $V(f)$ is $\binom{d-1}{2}$.

§3 Cellular homology

If we have two CW complexes X, Y and a map $f : X \to Y$, we say that f is **cellular** if $f(X^{(n)}) \subset Y^{(n)}$ for each n. If we have a cellular map, it induces a map in cellular homology. Given a cell $e_n^\alpha \in X$ and a cell $e_n^\beta \in Y$, we consider the map

$$e_n^\alpha \to X^{(n)} \to Y^{(n)} \to S_\beta^n$$

obtained by collapsing everything in $Y^{(n)}$ other than e_n^β to a point. This induces a map $S_n^\alpha \to S_n^\beta$. Taking the multiplicities lets you define the image of the class of e_n^α as a sum of e_n^β's with multiplicities corresponding to the degrees.

20.3 Theorem (Cellular approximation theorem)**.** *Any map $f : X \to Y$ between CW complexes is homotopic to a simplicial map.*

Proof. It's kind of a pain. ▲

As an application of this, we can show that

$$\pi_n(S^m) = 0$$

if $n < m$. Indeed, any map is homotopic to a cellular map, and a cellular map $S^n \to S^m$ must be a constant map (if we use cell structure on S^n, S^m with a cell in zero dimensions and a cell in the highest dimension).

You can also prove that

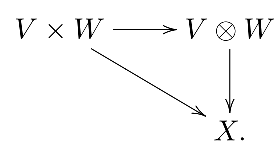
$$\pi_n(S^n) \simeq \mathbb{Z}$$

in general. After this, homotopy groups get extremely complicated, which is why we're doing homology first.

20.4 Example. $\pi_3(S^2) \simeq \mathbb{Z}$. This is generated by the Hopf fibration. One can prove this by the long exact sequence of a fibration.

§4 Tensor products

The universal property of a tensor product is as follows. Suppose V, W are modules over a ring R. If you want, you can think of R as a field, but we're going to call it R. Then we say that $V \otimes W$ is something which satisfies the following universal property.

> There is a bilinear map
>
> $$V \times W \to V \otimes W$$
>
> such that for any module X and bilinear map $V \times W \to X$, there exists a unique map $V \otimes W \to X$ making the diagram
>
> 
>

The uniqueness of the object $V \otimes W$ follows from general nonsense. Existence is not so trivial, though.

We have to actually construct a tensor product. There are several ways to do this. The most comprehensible way is to define $V \otimes W$ to be the free R-module generated by symbols $v \otimes w, v \in V, w \in W$ quotiented by the relations

$$v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$$

$$v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$$

and

$$r(v \otimes w) = (rv \otimes w) = v \otimes rw.$$

Lecture 21
10/13

We're going to deviate from Hatcher for a while. The natural follow-up as in Hatcher would be talk about cup products and cohomology. However, we are going to talk about categories.

§1 Categories

Categories are supposed to be places where mathematical objects live.

21.1 Definition. A **category** $\mathcal{C}$ consists of a collection of **objects**, ob$\mathcal{C}$, and for each pair of objects $X, Y \in \text{ob}\mathcal{C}$, a set of **morphisms** $\mathcal{C}(X, Y)$.

For each object $X \in \text{ob}\mathcal{C}$, there is an **identity morphism** $1 \in \mathcal{C}(X, X)$. Next, there is a **composition law** $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z) \to \mathcal{C}(X, Z), (g, f) \to g \circ f$ for every triple X, Y, Z of objects, which is unital and associative.

We write $X \to Y$ to denote an element of $\mathcal{C}(X, Y)$.

$\mathcal{C}$ is the storehouse for mathematical objects: groups, Lie algebras, rings, etc.

Remark. Some people use $\text{Hom}(X, Y)$ to denote the set of morphisms between X, Y.

Remark. When we write $X \in \mathcal{C}$, it means $X \in \text{ob}\mathcal{C}$. This is a convenient convention.

Remark. The objects don't have to form a set; they can be large. But the things $\mathcal{C}(X, Y)$ are sets.

Now, we do a bunch of examples.

21.2 Example.

1. $\mathcal{C} =$ **Sets**; the objects are sets, and the morphisms are maps of sets.
2. $\mathcal{C} =$ **Grps**; the objects are groups, and the morphisms are maps of groups (i.e. homomorphisms).
3. $\mathcal{C} =$ **LieAlg**; the objects are Lie algebras, and the morphisms are maps of Lie algebras (i.e. homomorphisms).

There is **nothing** in the language of categories that lets you look *inside* an object. We think of vector spaces having elements, spaces having points, etc. Categories treat these kinds of things as invisible. There is nothing "inside" of $X \in \mathcal{C}$. The only way to understand X is to understand the homs into and out of X.

We will elaborate on this in the future.

21.3 Example. Let G be a finite group. Then we an make a category B_G where the objects just consist of one point $*$ and the maps $* \to *$ are the elements of G. The identity is the identity of G and composition is multiplication in the group.

In this case, the category doesn't represent so much of a class of objects, but instead we think of the composition law as the key thing. So a group is a special kind of category.

21.4 Example. A monoid is precisely a category with one object. Recall that a **monoid** has an associative and unital multiplication (not necessarily inverses).

Figures... ADD THEM

Remark. A lot of people said that they've seen this before in commutative algebra, but not necessarily thoroughly. So we will keep talking about this.

§2 Functors

Let $\mathcal{C}, \mathcal{D}$ be categories.

21.5 Definition. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a function $F : \text{ob}\mathcal{C} \to \text{ob}\mathcal{D}$ and, for each pair $X, Y \in \mathcal{C}$, a map $F : \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$, which preserves the identity maps and composition.

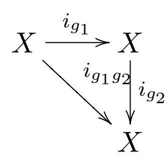
21.6 Example. There is a functor from **Sets** $\to$ **AbelianGrp** sending a set S to a free abelian group on the set.

21.7 Example. There is a functor from **TopSpaces** $\to$ **GradedAbGrp** (categories of topological spaces and graded abelian groups) sending a space X to its homology groups $H_*(X)$. We know that given a map of spaces, we get a map of graded abelian groups.

21.8 Example. What is a functor $B_G \to$ **Sets**? Here B_G is the category alluded to above.

The unique object $*$ goes to some set X. For each element $g \in G$, we get a map $g : * \to *$ and thus a map $X \to X$. This is supposed to preserve the composition law (which in G is just multiplication), as well as identities.

In particular, we get maps $i_G : X \to X$ corresponding to each $g \in G$, such that the following diagram commutes:



So a functor $B_G \to$ **Sets** is just a left G-action on a set X.

"I never liked the idea of left and right action. What about aliens on another planet that didn't have left and right hands?"

Sometimes these are called **covariant functors**. Indeed:

21.9 Definition. A **contravariant functor** from $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is similar data except that now a map $X \to Y$ now goes to a map $FY \to FX$. Composites are required to be preserved, albeit in the other direction.

"The reason we have shiny objects in nature is so that we will think of contravariant functors. If you look in a mirror, it's like applying a contravariant functor to yourself. This is kind of a myopic view of mankind."

As you might guess:

21.10 Example. A **contravariant** functor from B_G to **Sets** corresponds to a set with a *right* G-action.

We will, in a week or so, define a contravariant version of homology when we start studying cohomology.

21.11 Example. On the category **Vect** of vector spaces, we have the contravariant functor

$$V \to V^*.$$

21.12 Example. If we map $B_G \to B_G$ sending $* \to *$ and $g \to g^{-1}$, we get a contravariant functor.

There is room, nevertheless, for something else. You could have something that sent an object to a map. This is, I think, the reason for Maclane and Eilenberg to describe the next property.

§3 Natural transformations

The original paper of Eilenberg and Maclane was called "On a general theory of natural transformations." Maclane was a great guy, incidentally, besides inventing homological algebra; there was a picture in his office of someone holding a ray gun like a sci-fi movie yelling Tor, Tor, Tor.

Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are functors.

21.13 Definition. A **natural transformation** $T : F \to G$ consists of the following data. For each $X \in C$, there is a morphism $TX : FX \to GX$ satisfying the following condition. Whenever $f : X \to Y$ is a morphism, the following diagram must commute:

$$\begin{CD} FX @>>> FY \\ @V{TX}VV @VV{TY}V \\ GX @>>> GY \end{CD} \quad .$$

When we say that things are "natural" in the future, we will mean that the transformation between functors is natural in this sense.

21.14 Example. The **connecting homomorphism** $H_n(X, A) \to H_{n-1}(A)$ is natural. This is going to be a little rocky, but let's say what this means.

If we have pairs $(X, A) \to (Y, B)$, then the following diagram commutes

$$\begin{CD} H_n(X,A) @>>> H_{n-1}(A) \\ @VVV @VVV \\ H_n(Y,B) @>>> H_{n-1}(B) \end{CD} \quad .$$

This identity is very important in the axiomatic characterization of homology, due to Eilenberg-Steenrod.

This is a little funny, and one has to think about which category we're talking about. We can use the category of **pairs of topological spaces**. So the objects here are pairs (X, A) and morphisms are morphisms of pairs.

Remark. "I'm here to put the 'funk' in functor." (Dick Gross came in to say hello.)

Some people don't like this. They don't like to use the language of categories. If you really try to go in and examine things, it can be hard to figure out what things really mean. However, we will use it to state theorems conveniently.

21.15 Exercise. Work this out for yourselves. Suppose you have two functors $B_G \to$ **Sets**, i.e. G-sets. What's a natural transformation between them?

Now I want to prove a theorem.

21.16 Theorem. *If $f : X \to Y$ is a map in $\mathcal{C}$, and $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F(f) : FX \to FY$ is an isomorphism.*

This is going to have a really stupid proof, but there is an important point lurking here.

21.17 Example. Let $\mathcal{C}$ be the homotopy category of topological spaces **hoT**. The objects are topological spaces and the morphisms between X, Y are the continuous maps $X \to Y$ modulo the relation of being homotopic. Homology is actually a functor from **hoT** to the category of graded abelian groups.

An isomorphism in **hoT** is a homotopy equivalence, by definition. We thus see:

21.18 Corollary. *A homotopy equivalence induces isomorphisms in homology.*

It is, incidentally, harder to show that the same is true for the fundamental group. Note that the argument is made very slick and convenient by the above theorem. We don't have to think about cycles or boundaries. Because of the generality of the theorem, we have no choice but to give a slick proof. We can't talk about one-to-one and onto maps are.

Hold on. Wait a second. Do we even know what an isomorphism in a category even is? No, we don't.

21.19 Definition. An **isomorphism** between objects X, Y in a category $\mathcal{C}$ is a map $f : X \to Y$ such that there exists $g : Y \to X$ with

$$g \circ f = 1_X, \quad f \circ g = 1_Y.$$

This is more correct than the idea of being one-to-one and onto. A bijection of topological spaces is not necessarily a homeomorphism.

Proof. If we have maps $f : X \to Y$ and $g : Y \to X$ such that the composites both ways are identities, then we can apply the functor F to the whole dog and pony show, and we find that since

$$f \circ g = 1_Y, \quad g \circ f = 1_X,$$

that

$$F(f) \circ F(g) = 1_{F(Y)}, \quad F(g) \circ F(f) = 1_{F(X)}.$$

We have used the fact that functors preserve composition and identities. This implies that $F(f)$ is an isomorphism. ▲

Categories have a way of making things so general that they're trivial. Hence, it is called general abstract nonsense. The things that become meaningful in category theory are **not** the proofs. They are the **definitions**. What we just did is very much in the spirit I was describing of categories. The notion of isomorphism was defined in terms of properties of maps, not in terms of things intrinsic (like injections and surjections).

What's important here is not the theorem, but the *definition of an isomorphism.*

Lecture 22
10/15

Someone asked where to learn about categories. There is a standard reference by Saunders Maclane, called *Categories for the working mathematician.* It's a nice book—thin. But going in and learning about category theory is a slippery slope. It's kind of interesting, and you can spend a lot of time doing it.

A lot of it for this course, though, is just getting used to the language and the definitions.

Last time, we introduced the idea of a category, and showed that a functor takes isomorphisms to isomorphisms. This was an amazing result with a trivial proof. Today, we will characterize objects in terms of maps.

§1 Initial and terminal objects

22.1 Definition. Let $\mathcal{C}$ be a category. An **initial object** in a category is an object $X \in \mathcal{C}$ with the property that $\mathcal{C}(X, Y)$ has one element for all $Y \in \mathcal{C}$.

So there is a unique map out of X into each $Y \in \mathcal{C}$.

22.2 Example. If $\mathcal{C}$ is **Sets**, then the empty set $\emptyset$ is an initial object. The empty set is the set for indecisive people. To map out of the indecisive set, you never have to decide where anything goes—it just goes. There is a unique map from the empty set into any other set.

It seems too abstract to be useful. But it is.

There is a dual notion, called a **terminal object**, where every object can map into it in precisely one way.

22.3 Definition. A **terminal object** in a category $\mathcal{C}$ is an object $Y \in \mathcal{C}$ such that $\mathcal{C}(X, Y) = *$ for each $X \in \mathcal{C}$.

22.4 Example. The one point set is a terminal object in **Sets**.

The important thing about the next "theorems" is the conceptual framework.

22.5 Theorem. *Any two initial (resp. terminal) objects in $\mathcal{C}$ are isomorphic by a unique isomorphism.*

Proof. The proof is really easy. We do it for terminal objects. Say Y, Y' are terminal objects. Then $\mathcal{C}(Y, Y')$ and $\mathcal{C}(Y', Y)$ are one point sets. So there are unique maps $Y \to Y', Y' \to Y$, whose composites must be the identities: we know that $\mathcal{C}(Y, Y), \mathcal{C}(Y', Y')$ are one-point sets. This means that the maps $Y \to Y', Y' \to Y$ are isomorphisms. ▲

There is a philosophical point to be made here. We have characterized an object uniquely in terms of mapping properties. We have characterized it *uniquely up to unique isomorphism,* which is really the best you can do in mathematics. Two sets aren't generally the "same," but they may be isomorphism up to unique isomorphism. Like the sets of your father and Darth Vader: they're different (unless you're Luke), but the sets are isomorphic up to unique isomorphism.

Now we're going to talk about a bunch of other examples, which can all be phrased via initial or terminal objects in some weird category. This, therefore, is the proof for *everything* we will do today.

Say we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array} .$$

We can say what it means for this to be a **push-out**.

22.6 Definition. A square like this,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array} .$$

is a **pushout square** (and X is called the **push-out**) if, given a diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & & \downarrow \\ & \searrow & \downarrow \\ & & Y \end{array}$$

there is a unique map $X \to Y$ making the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \\ & \searrow & \downarrow \\ & & Y \end{array} .$$

22.7 Example. The following is a pushout square in the category of abelian groups:

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/4 \\ \downarrow & & \downarrow \\ \mathbb{Z}/6 & \longrightarrow & \mathbb{Z}/12 \end{array} .$$

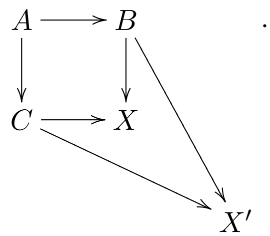
In the category of groups, the push-out is actually $\mathrm{SL}_2(\mathbb{Z})$—this is a cool theorem. The point is that being a push-out is actually dependent on the category.

22.8 Proposition. *If the push-out of*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

exists, it is unique up to unique isomorphism.

Proof. We can prove this in two ways. One is that suppose I had two pushout squares



Then there are unique maps $X \to X', X' \to X$ from the universal property, which have to be isomorphisms.

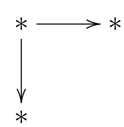
Alternatively, we can phrase push-outs in terms of initial objects. We could consider the category of all cartesian diagrams as above with A, B, C and mapping into something else; then the initial object in this category is the push-out. ▲

Now we abstract on this idea further.

§2 Colimits

We now want to generalize the push-out. Instead of a shape with A, B, C, we do something more general.

Start with a **small** category I: this is not meant in a pejorative sense, but that the objects of I form a set. What you're supposed to picture is that I is something like the category



or the category

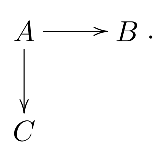
$$* \rightrightarrows *.$$

We will formulate the notion of a **colimit** which will specialize to the push-out when I is the first case. I is to be called the **indexing category**.

So we will look at functors

$$F : I \to \mathcal{C},$$

which in the case of the three-element category, will just correspond to diagrams



We will call a **cone** on F (this is an ambiguous term) an object $X \in \mathcal{C}$ equip'd[6] with maps $F_i \to X, \forall i \in I$ such that for all maps $i \to i' \in I$, the diagram below commutes:

$$\begin{array}{ccc} F_i & \longrightarrow & X \\ \downarrow & \nearrow & \\ F_{i'} & & \end{array} .$$

An example would be a cone on the three-element category above: then this is just a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} .$$

22.9 Definition. The **colimit** of the diagram $F : I \to \mathcal{C}$, written as colim$F$ or colim$_I F$ or $\varinjlim_I F$, if it exists, is a cone $F \to X$ with the property that if $F \to Y$ is any other cone, then there is a unique map $X \to Y$ making the diagram

$$\begin{array}{ccc} F & \longrightarrow & X \\ & \searrow & \downarrow \\ & & Y \end{array}$$

commute. (This means that the corresponding diagram with F_i replacing F commutes for each $i \in I$.)

We could think of some weird category where cones are objects and the colimit is initial. In any case, we see:

22.10 Proposition. colim$F$*, if it exists, is unique up to unique isomorphism.*

Let us go through some examples. We already looked at push-outs.

22.11 Example. Consider the category I described by

$$*, *, *, *.$$

A functor $F : I \to$ **Sets** is just a list of four sets A, B, C, D. The colimit is just the disjoint union $A \sqcup B \sqcup C \sqcup D$. This is the universal property of the disjoint union. To hom out of the disjoint union is the same thing as homming out of each piece.

22.12 Example. Suppose we had the same category I but we went into abelian groups. Then F corresponds, again, to a list of four abelian groups. The colimit is the direct sum. Again, the direct sum is characterized by the same universal property.

22.13 Example. Suppose we had the same I $(*, *, *, *)$ but the category of groups was $\mathcal{C}$. Then the colimit is the free product of the four groups.

[6]Imagine I wrote that with an English accent.

22.14 Example. Suppose we had the same I and the category $\mathcal{C}$ was of commutative rings with unit. Then the colimit is the tensor product.

So the idea unifies a whole bunch of constructions.
Now let us take a different example.

22.15 Example. Take

$$I = * \rightrightarrows *.$$

So a functor $I \to$ **Sets** is a diagram

$$A \rightrightarrows B.$$

Call the two maps $f, g : A \to B$. To get the colimit, we take B and mod out by the equivalence relation generated by $f(a) \sim g(a)$. To hom out of this is the same thing as homming out of B such that the pullbacks to A are the same.

This is the relation **generated** as above, not just as above. It can get tricky.

22.16 Definition. When I is just a bunch of points $*, *, *, \dots$ with no nonidentity morphisms, then the colimit over I is called the **coproduct**.

We use the coproduct to mean things like direct sums, disjoint unions, and tensor products.

22.17 Definition. When I is $* \rightrightarrows *$, the colimit is called the **coequalizer**.

22.18 Theorem. *If $\mathcal{C}$ has all coproducts and coequalizers, then it has all colimits.*

Proof. Exercise. It's not too hard, but it is—I don't know, I'll talk about it on Monday. It's worth racking your brain over. ▲

One of the reasons I talked about colimits is that we can talk about it in class and use the language. Also, there are a lot of examples we haven't done in class as we haven't studied filtered colimits.

§3 Filtered colimits

These are really useful, especially in algebraic topology. These are colimits over special I.

22.19 Definition. An indexing category is **filtered** if the following hold:

1. Given $i_0, i_1 \in I$, there is a third object $i \in I$ such that both i_0, i_1 map into i.
2. Given any two maps $i_0 \rightrightarrows i_1$, there exists i and $i_1 \to i$ such that the two maps $i_0 \rightrightarrows i$ are equal. Any two ways of pushing an object into another can be made into the same eventually.

22.20 Example. If I is the category

$$* \to * \to * \to \dots,$$

i.e. the category generated by the poset $\mathbb{Z}_{\geq 0}$, then that is filtered.

22.21 Example. If G is a torsion-free abelian group, the category I of finitely generated subgroups of G and inclusion maps is filtered. We don't actually need the lack of torsion.

22.22 Definition. Colimts over a filtered category are called **filtered colimits**.

22.23 Example. Any torsion-free abelian group is the filtered colimit of its finitely generated subgroups, which are free abelian groups.

This gives a simple approach for showing that a torsion-free abelian group is flat.

22.24 Proposition. *If I is filtered*[7] *and $\mathcal{C} = \mathbf{Sets}, \mathbf{Abgrp}, \mathbf{Grps}$, etc., and $F : I \to \mathcal{C}$ is a functor, then $\mathrm{colim}_I F$ exists and is given by the disjoint union of $F_i, i \in I$ modulo the relation $x \in F_i$ is equivalent to $x' \in F_{i'}$ if x maps to x' under $F_i \to F_{i'}$. This is already an equivalence relation.*

The fact that the relation given above is transitive uses the filtering of the indexing set. Otherwise, we would need to use the relation generated by it.

22.25 Example. Take $\mathbb{Q}$. This is the filtered colimit of the free submodules $\mathbb{Z}(1/n)$.

Alternatively, choose a sequence of numbers $m_1, m_2, \dots$, such that for all p, n, we have $p^n \mid m_i$ for $i \gg 0$. Then we have a sequence of maps

$$\mathbb{Z} \stackrel{m_1}{\to} \mathbb{Z} \stackrel{m_2}{\to} \mathbb{Z} \to \dots.$$

The colimit of this is $\mathbb{Q}$. There is a quick way of seeing this, which is left to the reader.

Lecture 23
10/18

§1 Filtered colimits

Last time, we talked about something called *filtered colimits.* In this, we had a special property of the indexing category I. It had the property that given any two $i_0, i_1 \in I$, there was a third one into which they mapped; moreover, given any two maps $i_0 \rightrightarrows i_1$, there was a third one which coequalized them. Filtered colimits were defined as colimits over a filtered category.

When we have a functor $F : I \to \mathbf{Sets}, \mathbf{Grps}, \mathbf{Modules}$ taking values in a "nice" category (e.g. the category of sets, modules, etc.), you can construct the colimit by taking the union of the $F_i, i \in I$ and quotienting by the equivalence relation $x \in F_i \sim x' \in F_{i'}$ if $f : i \to i'$ sends x into x'. This is already an equivalence relation, as one can check.

Another way of saying this is that we have the disjoint union of the F_i modulo the relation that $a \in F_i$ and $b \in F_{i'}$ are equivalent if and only if there is a later i'' with maps $i \to i'', i' \to i''$ such that a, b both map to the same thing in $F_{i''}$.

Suppose $F : I \to \mathbf{Ch}$ is a functor from a filtered category I to the category of chain complexes. For instance, I could be the category $* \to * \to * \to \dots$, leading to a

[7] Some people say filtering.

sequence of chain complexes $C_*^{(0)} \to C_*^{(1)} \to \dots$. This is the standard example you're supposed to keep in mind.

Then:

23.1 Proposition. *The homology of the colimit* $\varinjlim_I F$ *is the colimit of the homologies* $H(F_i)_{i\in F}$.

Proof. This is easy to prove. The deep idea is the formulation, not the proof.

We will first prove that the natural map

$$\mathrm{colim}_I(H_*F) \to H_*\mathrm{colim}_I F$$

is onto. Suppose we have something in $H_*(\mathrm{colim} F)$. Then this element x is represented by a n-cycle z in $(\mathrm{colim}_I F)_n$ for some n. The colimit $(\mathrm{colim}_I F)_n$ is just $\sqcup(F_i)_n$ modulo the equivalence relation. So z is represented by some $z' \in (F_i)_n$. We don't, a priori, know that z' is a cycle, i.e. that $dz' = 0$. If this were the case, then we would have a class in $\mathrm{colim}_I(H_*F)$ mapping onto x.

However, dz' does go to zero in the colimit $\mathrm{colim}_I F$ as z' is a cycle in this colimit. Because it is filtered, we know that there is a map $f : i \to i'$ such that dz' goes to zero in i'. In $F_{i'}$, z' becomes a cycle. So the homology class of x is in the image of $H_n(F_{i'})$, which maps into the colimit $\mathrm{colim}_I H_n(F_i)$, which in turn maps into the homology of the colimit. We have thus seen that

$$\mathrm{colim}_I H_n(F_i) \to H_n(\mathrm{colim}_I F_i)$$

is surjective.

Now let us prove that it is one-to-one. Suppose $x \in \mathrm{colim}_I H_n(F)$ goes to zero in the homology of the colimit $\mathrm{colim}_I F$. So x is represented by some cycle $z \in Z_n(F_i)$. In the colimit $\mathrm{colim}_I F$, x is a boundary $x = dy$. There is thus $\overline{y} \in F_{i'}$ representing y. By pushing forward into some mutually larger i'', we might as well suppose that $x = d\overline{y}$ in F_i itself. This means that x was zero in $H_n(F_i)$ itself. ▲

I hope that made sense. If it didn't, it's one of those things that's more complicated when you say it out loud than when you think it through for yourself. I can't remember whether this was in Hatcher or not. But then you'll just get what I said here in a less entertaining way. I find these kinds of things hard to digest when someone is standing there telling it to me.

But anyway, this is one of the main uses of filtered colimits—or directed colimits, as some people say.

§2 Colimits and the singular chain complex

There is a very serious reason for wanting these.

23.2 Proposition. *Suppose I is filtered and $F : I \to$* **HSpaces** *is a functor (to the category of Hausdorff topological spaces) with the property that every map $F_i \to F_{i'}$ is a closed inclusion.*

Then the obvious map

$$\mathrm{colim}_I C_* F_i \to C_*(\mathrm{colim}_I F_i)$$

is an isomorphism.

Just to be clear, the "obvious" map $\mathrm{colim}_I C_*(F_i) \to C_*(\mathrm{colim}_I F_i)$ can be explained as follows. Suppose we have functors

$$J \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{D}.$$

Then we have natural maps $F_j \to \mathrm{colim}_J F_j$ (assuming the colimit exists), so when we apply G, we get maps

$$G(F_j) \to G(\mathrm{colim}_J F_j).$$

Taking the colimit, we get a natural map

$$\mathrm{colim}_J G F_j \to G(\mathrm{colim}_J F_j).$$

23.3 Example. X might be a CW complex, and I might be the category with objects the finite subcomplexes and inclusion maps. This is obviously a filtered category, and there is an obvious functor $I \to$ **Spaces** since I was a subcategory of **Spaces**. X is the colimit of this functor.

We learn that

$$C_*(X) = \mathrm{colim}_I C_*(X_I)$$

which implies that **the homology of X is the colimit of the homology of the finite subcomplexes**. We will come back to giving applications of that.

Proof. We will give the proof while assuming that the indexing category is

$$1 \to 2 \to 3 \to \dots.$$

The general one is very similar, but I didn't get a chance to think through it before I came in today. We can leave the little tiny modifications to be made in general to the reader.

So we have a sequence of closed inclusions of topological spaces

$$X_1 \subset X_2 \subset \dots,$$

and X is the union $\bigcup X_i$ with the weak topology. We now have to show the following classic lemma in point-set topology, which will imply the result:

23.4 Lemma. *Any map $\Delta^k \to X$ factors through X_n for some n.*

This isn't special to Δ^k; we just need a compact Hausdorff space.

Proof. We have to do something that results in a ridiculous statement—a contradiction, not just something goofy.

Let $K = f(\Delta^k)$, which is a compact subset of X. Choose $s_n \in K - X_n$ for each n. We write $S = \{s_n\} \subset X$. I claim that S is discrete. For this we must show that any subset $S' \subset S$ is closed. But $S' \cap X_n$ is finite for each n, hence closed—since we are using the fact that each X_n is Hausdorff. Thus S' is closed. This is a property of the topology on the colimit.

We have now seen that S is discrete, which is a contradiction, as it is a subspace of the compact space K. ▲

Remark. There's a nice category-theoretic way of talking about this. Consider the **Sierpinski space** $D = \{0, 1\}$ with the topology that $\{0\}$ is closed but $\{1\}$ is not. The closure of $\{1\}$ is thus the whole space D. The important thing about the Sierpinski space is that the set of continuous maps $X \to D$ is in bijection with the set of closed subspaces of X. (If $F \subset X$ is closed, we send $X \to D$ by sending F to 0 and $X - F$ to 1.) You say that the functor associating a space to its set of closed subspaces is **represented** by the Sierpinski space.

Suppose $X = \mathrm{colim} X_i$, so to give a closed subspace of X is to give a continuous function $X \to D$. That's the same thing as giving a set of compatible maps $X_i \to D$. In other words, a collection of compatible closed subspaces of X_i. If you think through what this means, it says that a subspace of X is closed if and only if the intersection with each X_i is closed.

▲

Here is an example:

23.5 Proposition. *Suppose X is a CW complex. Let $X^{(n)}$ be the n-skeleton. Then the natural map*

$$H_n(X^{(n+1)}) \to H_n(X)$$

is an isomorphism.

Proof. For instance, use the long exact sequence

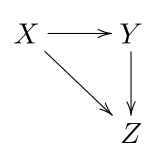
$$\dots \to H_{n+1}(X^{(n+2)}, X^{(n+1)}) \to H_n(X^{(n+1)}) \to H_n(X^{(n+2)}) \to H_n(X^{(n+2)}, X^{(n+1)}) \to \dots.$$

The two ends are zero because those are the homologies of $n+2$-spheres. More generally, we learn that the maps $H_n(X^{(n+1)}) \to H_n(X^{(n+2)}) \to H_n(X^{(n+3)}) \to \dots$ are all isomorphisms. The colimit is just therefore $H_n(X^{(n+1)})$. However, the previous arguments imply that the colimit is just $H_n(X)$. ▲

23.6 Corollary. *Cellular homology works for arbitrary CW complexes, not just finite-dimensional ones.*

Lecture 24
[Section] 10/18

The key observation made in class is that any diagram of the form



can be interpreted as a *functor* from a suitable *diagram category*. A *cone* on a functor $F : I \to \mathcal{C}$ can be defined as a collection of maps $Fi \to Z$. There is a category of cones one can define, and in this category, the initial object is the *colimit*.

Colimits don't have to exist.

Remark. Given a functor $F : I \to \mathcal{C}$ where $\mathcal{C}$ has a terminal object, you can always consider the trivial cone over the functor mapping each object $Fi, i \in I$ into the terminal object.

For limits, one reverses the arrows and defines a *co-cone* over a functor and considers the terminal object in the category of co-cones.

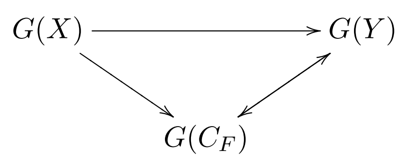
As we saw in class, given a functor G, we can always define a natural map

$$G(\mathrm{colim}_I F) \to \mathrm{colim}_I GF.$$

Here is an example. Given the category $J : * \to *$, a functor $J \to \textbf{Top}$ is just a morphism $X \to Y$. A colimit of this $X \to Y$ is just a space (the cone) C_F with maps

$$X \to C_F, Y \to C_F.$$

If G is a functor from **Top** to some other category, we have a commutative diagram



From this, we get a map from this cone into the universal cone C_{GF} over $CG(X) \to CG(Y)$. In particular, we get a map

$$G(C_F) \to C_{GF}.$$

Lecture 25
10/20

First, an announcement. There won't be a problem set for next week. If you feel like you're being cheated, I'll give you a dollar back from your tuition.

Today, we will talk about the Eilenberg-Steenrod axioms and start cohomology. However, we won't have gotten far enough to have a problem set.

§1 Eilenberg-Steenrod axioms

There are a lot of ways of formulating this. But let us start by writing out a bunch of properties of homology.

Suppose we have a collection of functors $h_* = \{h_n, n \in \mathbb{Z}\}$ which go from CW complexes to abelian groups. The motivating example will be **reduced homology** $\widetilde{H}_*$. Here are the properties we care about:

1. h_* is a functor (or a collection of functors) on the category of CW complexes and continuous maps.

2. h_* is homotopy invariant. Given two homotopic maps $f, g : X \rightrightarrows Y$, the induced maps $h_*(X) \rightrightarrows h_*(Y)$ are the same. In particular, h_* is a functor on the homotopy category.

3. If $A \subset X$ is a subcomplex, there is a long exact sequence

$$h_n(A) \to h_n(X) \to h_n(X/A) \to h_{n-1}(A) \to \dots$$

in the functors h_n. We want this long exact sequence to be *natural*, so the connecting map $h_n(X/A) \to h_{n-1}(A)$ is a natural transformation. In particular, if $(X, A) \to (Y, B)$, the diagram

$$\begin{array}{ccc} h_n(X/A) & \longrightarrow & h_{n-1}(A) \\ \downarrow & & \downarrow \\ h_n(Y/B) & \longrightarrow & h_{n-1}(B) \end{array}$$

commutes.

4. If Y is an infinite wedge[8] $\vee Y_\alpha$ of pointed spaces Y_α, then $h_*(Y) \simeq \bigoplus h_*(Y_\alpha)$. This is only of interest for an infinite wedge.

5. $h_*(S^0) = \mathbb{Z}$ when $n = 0$ and zero otherwise.

The big theorem is:

25.1 Theorem. *If h satisfies these properties, then h is naturally isomorphic to reduced homology $\widetilde{H}_*$.*

I.e., the properties we've studied actually determine homology. Everything you need to calculate about homology can be done using these properties, not going back to the definition of singular chains and whatnot.

We will give a partial proof of this.

Remark. We could modify this by requiring that $h_n(S^0) = M$ for $n = 0$ and zero otherwise, where M is an abelian group. We will then find that h is naturally isomorphic to singular homology with coefficients in M.

The last property is perhaps the weirdest. It isn't a general property. People eventually began to study functors satisfying the first four conditions but not the fifth.

25.2 Definition. A functor satisfying the above properties but the last one is called a **generalized cohomology theory**.

[8]The wedge is what happens when you glue two pointed spaces on the basepoints. It is the coproduct in the category of pointed spaces.

§2 Sketch of proof

The idea is that there is just enough structure to show that $h_*(X)$ can be calculated by the cellular chain complex.

Axiom 3 and axiom 4 imply that the homology commutes with filtered colimits. This is simple, but requires some setting up. I suggest you accept this for now.

We now mention:

25.3 Lemma. $h_*(*) = 0$. *So any contractible space has trivial* h_*.

Proof. Use the long exact sequence of

$$* \to * \to *.$$

This is kind of silly. But anyway. ▲

Let us check the proof. Consider the sequence

$$S^{n-1} \to D^n \to D^n/S^{n-1} \simeq S^n;$$

the associated long exact sequence (and the contractibility of D^n). Induction on n now implies that

$$h_*(S^n) = \mathbb{Z} \text{ or } 0$$

in the appropriate definitions. The spheres have the appropriate h.

Anyway, this now implies that for a CW complex X,

$$h_n(X^{(n+1)}) \to h_n(X)$$

is an isomorphism, because you attach cells of higher dimension when you go up to $X^{(n+2)}$ and so on. Similarly it implies that $h_k(X^{(n)}) = 0$ for $k < n$. Also, the map from

$$h_n(X^{(n+1)}) \to h_n(X^{(n+1)}/X^{(n-2)})$$

is an isomorphism, as one can check. This implies that

$$h_n(X) \simeq h_n(X^{(n+1)}) = \frac{\ker(h_n(X^{(n)}/X^{(n-1)}) \to h_{n-1}(X^{(n-1)}/X^{(n-2)}))}{\mathrm{Im}(h_{n+1}(X^{(n+1)}/X^{(n)}) \to h_n(X^{(n}/X^{(n-1)}))}.$$

as in the proof of cellular homology.

We now see that $h_n(X)$ is the n-th homology of the chain complex

$$\dots \to h_n(X^{(n)}/X^{(n-1)}) \to h_{n-1}(X^{(n-1)}/X^{(n-2)}) \to \dots$$

where each term is free on the n-cells. This looks very much like the cellular complex, except that the cellular boundary map might potentially be different from the usual one. This is really not very hard.

The main point is to show that a map $S^n \to S^n$ of degree k induces multiplication by k on $h_n(S^n) \to h_n(S^n)$. This can be checked if you use something we haven't proved. First, one can easily construct a map of degree k by mapping $S^n \to \vee_k S^n$ by crushing the complement of neighborhoods of k points. Then, we map $\vee_k S^n \to S^n$. This map has degree k. One can check that in any homology theory as above, this map induces multiplication by k.

Right now we don't know:

25.4 Theorem (Hurewicz)**.** *Any two maps $S^n \to S^n$ of the same degree are homotopic.*

But once we use this fact, and the fact that any map $S^n \to S^n$ is homotopic to one of the nice maps constructed above, that any map of degree k induces multiplication by k in h.

Remark. Excision was sort of built in here, because when you excise something from a pair, you don't change a quotient. Excision doesn't have as good a philosophical interpretation, and it's not necessary for this argument.

§3 A variation

Here is a variation:

25.5 Theorem. *Suppose h, h' are two generalized homology theories. Suppose we have a natural map*

$$\tau : h \to h'$$

which is an isomorphism on the zero-sphere. Then τ is a natural isomorphism.

This is "almost" a generalization of the other theorem. In some sense, it tells you that homology is determined by the homology of a point. The difference is that here you have a natural transformation between h, h' given a priori.

Proof. This is actually much easier to prove than the previous theorem. We'll sketch the proof and leave it as an exercise.

1. Show that $t : h_*(S^n) \to h'_*(S^n)$ is an isomorphism for all n.
2. By induction on n, show that $t : h_*(X) \to h'_*(X)$ if X has dimension $\leq k$.
3. Use filtered colimits to get all X.

▲

It makes a point that comes up all the time in homotopy theory. There's a big difference between knowing two things are isomorphic and having an actual map.

§4 Examples of generalized homology

This is just for fun.

We start with a dumb example.

25.6 Example. Fix a CW complex Y. Consider the functor $X \to H_*(X \times Y / * \times Y)$ where $* \in X$ is the basepoint. This actually satisfies all the axioms.

You get a weird homology theory here. It is one that fails to satisfy the last axiom: the "homology" of Y is the singular homology of Y. The last ("dimension") axiom is not satisfied. It works out, however, that this is just

$$h_*(X) = \widetilde{H}_*(X, H_*(Y)).$$

So this is homology of X with "coefficients in Y." I believe this is the Kunneth theorem. More precisely,

$$h_n(X) = \bigoplus_{k+l=n} H_k(X, H_l(Y)).$$

We don't really know it means for homology to have coefficients in a graded group, though.

Let us do another more interesting example.

25.7 Example. I suppose this is a rationale for the word "singular" in singular homology. We define something we call $MO_k(X)$. Given X, this is the k-th homology of the following chain complex:

$$C_k^{MO}(X) = \text{free abelian group on maps } N \to X$$

where N is a smooth k-manifold with boundary. The boundary $d : C_k^{MO}(X) \to C_{k-1}^{MO}(X)$ sends $N \to X$ to its restriction to its boundary. Since the boundary of a manifold-with-boundary is a closed manifold, we have $d^2 = 0$.

One can check that this is a homology theory. Even the homology of a point is interesting. In fact, this is just the group of equivalence classes of n-manifolds where two manifolds M, N are isomorphic if $M \sqcup N$ is the boundary of a bigger manifold.

These kinds of homology theories are very important. There is a very deep theorem of Thom:

25.8 Theorem (Thom)**.** *$MO_k(*)$ is a polynomial ring over $\mathbb{Z}/2$ on variables x_i where $i+1$ is not a power of 2. Moreover $MO_k(X)$ is homology of X with coefficients in the ring above, so the homology theory is actually ordinary.*

We'll have the tools for this next semester.

Lecture 26
10/22

We want to start talking about cohomology now.

§1 Singular cochains

In a way, I think it would be more natural to teach a course like this starting with cohomology than with homology. You'll see that it is formally the same, but it seems to have some advantages.

We start with some motivation. Remember when you study calculus, you might have a path $\gamma : [a, b] \to \mathbb{R}^2$, and a 1-form $pdx + qdy$. You study the integral $\int_\gamma pdx + qdy$ and prove various things about that. This 1-form can be thought of a map from paths γ to real numbers $\int_\gamma pdx + qdy$. This is what an n-cochain is.

26.1 Definition. An n**-cochain on** X **with values in** A (A an abelian group) is a function $c : \mathrm{Sing}_n(X) \to A$. Here $\mathrm{Sing}_n(X)$ is the set of all continuous maps $\Delta^n \to X$.

So a 1-form gives a 1-cochain (at least if you restrict to smooth simplicies).

The cochains can obviously be added, thanks to the group law in A, and they form a group.

26.2 Definition. We denote $C^n(X, A)$ for the group of n-cochains on X with values in A. We define the **coboundary** $\delta : C^n(X, A) \to C^{n+1}(X, A)$ as follows. Let $c \in C^n(X, A)$ and $s : \Delta^{n+1} \to X$. Then we define

$$(\delta c)(s) = c(\partial s) = \sum (-1)^i c(s_i),$$

where s_i is the i-th face of s. This defines δc as a function on $\mathrm{Sing}_{n+1}(X)$.

Let's go back and look at these calculus-style examples. A function $f : \mathbb{R}^n \to \mathbb{R}$ gives a zero-cochain. Namely, it assigns to each zero-simplex (i.e. $x \in \mathbb{R}^n$), we output $f(x)$. Then δf evaluated on a path γ (if it is smooth) is just $f(\gamma(1)) - f(\gamma(0))$. By Green's theorem, or whatever you want to call it, this is also the line integral $\int_\gamma f_x dx + f_y dy$ or the integral of the 1-form df.

So the cochain complex, while it is supposed to remind you of homology, actually has connections to what you learn about in calculus.

I have a lot to stay, and while this calculus formulation has a beautiful theory (the de Rham cohomology), we'll say more about this another time.

And now, we have made what we might call a *cochain complex* $C^*(X, A)$ with coboundary δ.

26.3 Definition. The **singular cohomology groups** of X with coefficients in A, denoted $Hj(X, A)$, are the cohomology groups of $C^*(X, A)$. Namely,

$$H^n(X, A) = \frac{\ker \delta : C^n \to C^{n+1}}{\mathrm{Im} \delta : C^{n-1} \to C^n}.$$

Your innocence is now over, and we're now going to have differentials that increase the degree instead of reduce them. Welcome to the real world. (OK, no, you'll find it easy.)

§2 Properties of cohomology

Fix an abelian group A. Cohomology has the following properties.

1. It's contravariant. If we have a map $f : X \to Y$, this induces a map

   $$f^* : H^*(Y, A) \to H^*(X, A).$$

   The identity induces the identity in cohomology, and $(f \circ g)^* = g^* \circ f^*$. So cohomology sends commutative diagrams to commutative diagrams. This should be obvious, but let's spell it out on the level of cochains. Let $c \in C^n(Y, A)$; then f^*c can be defined as follows. If $s : \Delta^n \to X$ is a simplex, we define

   $$(f^*c)(s) = c(f \circ s),$$

   because $f \circ s$ is the map from a simplex into X.

When f is an open immersion, we can think of this as a restriction map on cohomology.

Basically, for the same reason that homology is covariant, cohomology is contravariant.

2. Homotopy invariance. Homotopic maps give the same map in cohomology. We can't prove this now, but we'll give the idea later.

3. Mayer-Vietoris. Suppose $X = U \cup V$ for U, V having interiors covering X. Then there is a natural long exact sequence

   $$\dots \to H^n(X, A) \to H^n(U, A) \oplus H^n(V, A) \to H^n(U \cap V, A) \to H^{n+1}(X, A) \to \dots$$

   There are two ways this is different from homology. The arrows go in the opposite direction, and the connecting homomorphism raises, not lowers, degrees. This is just like how the differential raises degrees.

   This will follow from a tiny algebraic lemma that should fit into a single lecture.

4. Relative cohomology. Given a pair (X, B), the **relative cochain complex** can be defined as follows. $C^n(X, B, A)$ is defined as the set of maps $\mathrm{Sing}_n(X) \to A$ which are zero on $\mathrm{Sing}_n(B)$. Thus, we can define the **relative cohomology** $H^n(X, B, A)$ (or $H^n(X, B)$ when A is obvious).

5. Excision. If $Z \subset B \subset X$ and $\overline{Z} \subset Int(B)$, then the obvious map of relative cohomology:

   $$H^*(X, B) \to H^*(X - Z, B - Z)$$

   is an isomorphism. Here, of course, cohomology is to be interpreted as cohomology with coefficients in A.

So anyway, we find that cohomology satisfies the same Eilenberg-Steenrod-ish axioms in the other direction. To prove all this, we can do some algebra.

§3 The algebraic story

Let S be a set and A an abelian group. Then the set of all maps $f : S \to A$ is the same thing as $\mathrm{Hom}(\mathbb{Z}[S], A)$. We thus see that, for example,

$$C^n(X, A) = \mathrm{Hom}(C_n(X), A).$$

The differential is just the transpose of the differential $C_n \to C_{n-1}$. In particular,

$$C^*(X, A) = \mathrm{Hom}(C_*(X), A)$$

where Hom becomes a functor from complexes to complexes.

26.4 Proposition. *Suppose $C^1_* \xrightarrow{t} C^2_*$ is a map of chain complexes with the property that C^1_n, C^2_n are free abelian groups.*[9] *Suppose the map*

$$t_* : H_*(C^1) \to H_*(C^2)$$

[9] Like the singular chain complex.

is an isomorphism. Then the induced map in cohomology *with coefficients in* A,

$$H^*(C^2, A) \to H^*(C^1, A)$$

is an isomorphism. Here we have defined

$$H^*(C, A) = \text{cohomology of } \mathrm{Hom}(C_*, A).$$

We'll leave this for another lecture. Anyway, this implies everything we wanted. For instance, the Mayer-Vietoris sequence.

26.5 Example. The M-V sequence came as follows. Let $X = U_1 \cup U_2$. Then we defined

$$C_*^{\mathfrak{A}}(X) = \mathbb{Z}\left\{\mathrm{Sing}^{\mathfrak{A}}(X)\right\}$$

where $\mathrm{Sing}_n^{\mathfrak{A}}(X)$ consists of simplices whose images are contained in U_1 or U_2. These are "small" simplices. Mayer-Vietoris came from the sequence

$$0 \to C_*(U \cap V) \to C_*(U) \oplus C_*(V) \to C_*^{\mathfrak{A}}(X) \to 0$$

which gave a long exact sequence in homology. The key fact we used was that the map

$$C_*^{\mathfrak{A}}(X) \to C_*(X)$$

was an isomorphism in homology.

Now, the proposition tells us that the map $C^*(X, A) \to C^{\mathfrak{A}}(X, A)$, where we define

$$C^*(X, A) = \mathrm{Hom}(C_*(X), A), \quad C^{*\mathfrak{A}}(X, A) = \mathrm{Hom}(C_*^{\mathfrak{A}}(X), A),$$

is an isomorphism in homology. This is a good sign, but we need more.

We use:

26.6 Lemma. *Let* $0 \to L \to M \to V \to 0$ *be a short exact sequence of free abelian groups; then the sequence*

$$0 \to \mathrm{Hom}(V, A) \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(L, A) \to 0$$

is an exact sequence for any A.

Proof. This is because N is free and the sequence splits, $N \simeq L \oplus M$ in a nice way. ▲

OK. Let's now put all this together. We find that there is an exact sequence

$$0 \to C^{*\mathfrak{A}}(X) \to C^*(U) \oplus C^*(V) \to C^*(U \cap V) \to 0$$

by the lemma, by dualizing. Taking the long exact sequence of this, and using the fact that $C^{\mathfrak{A}}(X)$ has the cohomology of $C^*(X)$, now implies the Mayer-Vietoris sequence.

26.7 Example. You can continue in this manner, and define when X is a CW complex, the **cellular cohomology** of X. Here this is the cohomology of the complex $H^*(X^{(*)}, X^{(*-1)})$. You can define the **cellular differential** by sticking together the connecting homomorphisms. One can check, similarly, that the cohomology of X with coefficients in A can be computed via the cellular cochain complex.

26.8 Example. We could look at the cellular cochains on $\mathbb{RP}^n$. We would get a complex which is $\mathbb{Z}$ in degrees zero up to n; the maps would alternate between being zero and multiplication by two.

We find:

$$H^k(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } k \text{ even and } \leq n \\ 0 & \text{if } k \text{ odd and } < n \\ 0 & \text{if } k \text{ odd and } k = n \end{cases}$$

But remembering the cochain complex is better.

The point of this is that everything we did in homology can be done for cohomology. But what else is there? There is a special structure that can be exploited more easily for cohomology. We won't prove any of this today; it'll probably be done on Wednesday.

§4 Some remarks

Cohomology makes it really easy to exploit the fact that each space X has a diagonal map

$$X \to X \times X, \quad x \to (x, x).$$

This is a natural transformation between the identity functor and the functor $X \to X \times X$. If $f : X \to Y$, we get a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \times Y \end{array}.$$

There is something called the **Kunneth formula** for homology. Suppose R is a field. Then

$$H_*(X \times Y, R) \simeq H_*(X, R) \otimes H_*(Y, R).$$

The same is true for cohomology. This one requires one of X, Y to be finite type, though—it's a CW complex with only finitely many cells, for instance. This is something we haven't really talked about.

Remark. When I tensor graded groups together, I mean the following:

$$H_n(X \times Y, R) = \bigoplus_{i+j=n} H_i(X, R) \otimes_R H_j(Y, R).$$

The reason that cohomology seems to be more powerful is the following. Let $V = H_*(X)$ and $V^* = H^*(X)$ (with coefficients in some field R). Then there is some extra structure on homology. The diagonal maps imply that there are maps

$$V \to V \otimes V$$

and a map

$$V^* \otimes V^* \to V^*.$$

We're a lot better at thinking about the latter, though.

26.9 Theorem. $V^* = H^*(X)$ *is a ring with the above structure.*

V is a co-algebra, which is a little more complicated and harder to understand. Cohomology is a more powerful tool for us, because it is a ring.

Lecture 27
10/25

Jacob Lurie taught the next three lectures.

Remark. I have decided to reduce the extent of note-taking that I do.

So, recall:

Let X be a space, A abelian grp. You can define $H^*(X, A)$ as the cohomology of a chain complex. Namely, consider $C_*(X, \mathbb{Z})$, the complex of singular chains which assigns to each n, the free abelian group on the n-simplices.

27.1 Definition. $H^*(X, A)$ is the cohomology of the complex of singular cochains

$$C^*(X, A) = \mathrm{Hom}(C_*(X, \mathbb{Z}), A).$$

Question: What does this imply about $H^*(X, A)$?

Remark. If A is injective, meaning that $\mathrm{Hom}(-, A)$ is exact as a functor, and thus commutes with homology, then

$$H^*(X, A) = \mathrm{Hom}(H_n(X), A).$$

This is not true in general: not all A are injective. However, it is close to true.

Fix an exact sequence $0 \to M' \to M \to M'' \to 0$; consider the complex

$$0 \to \mathrm{Hom}(M'', A) \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M', A)$$

which is exact. Exactness on the right may fail.

Suppose $M' \subset M$, and $M' \to A$. Can we extend to $M \to A$? Sometimes. Well, we can form the push-out

$$\begin{CD} M' @>>> M \\ @VVV @VVV \\ A @>>> M \sqcup_{M'} A \end{CD}$$

which is a cocartesian, commutative diagram. We get a commutative exact diagram

$$\begin{CD} 0 @>>> M' @>>> M @>>> M'' @>>> 0 \\ @. @VVV @VVV @VVV @. \\ 0 @>>> A @>>> M \sqcup_{M'} A @>>> M'' @>>> 0 \end{CD}$$

which shows that what we need is a map $M \sqcup_{M'} A \to A$.

Upshot: Extending $M' \to A$ to $M \to A$ is to make the sequence $A \to M \sqcup_{M'} A$ split.

27.2 Definition. A, C ab. grps. An **extension** of C by A is an exact sequence

$$0 \to A \to B \to C \to 0.$$

Two extensions are *isomorphic* if they fit into a comm. diagram of isomorphisms. The set of isomorphism classes is denoted $\mathrm{Ext}(C, A)$. This has a canonical basepoint $A \oplus C$.

The above construction gives a map $\mathrm{Hom}(M', A) \to \mathrm{Ext}(M'', A)$ by sending $M' \to A$ to the sequence $0 \to A \to M \sqcup_{M'} A \to M'' \to 0$. Then the hom extends to M iff it the extension is trivial.

27.3 Proposition. *Let F free abelian. Any subgroup of F is free.*

Proof. Omitted. ▲

Next goal: understand $\mathrm{Ext}(M, A)$.

Lecture 28
11/3

Michael Hopkins is back. (Some lectures have been missed.)

We talked about the cup product and the Kunneth theorem last time. The cup product in relative cohomology also exists. Given $A \subset X$ a subspace, we can define $H^*(X, A, R)$ for R a ring.

28.1 Definition. $H^*(X, A, R)$ is the cohomology of the cochain complex $\mathrm{Hom}(C_*X/C_*A, R)$.

Henceforth we drop R and just assume it is there.

We can define the **cup product** for relative cohomology. This will give maps

$$H^*(X, A) \otimes H^*(X, B) \to H^*(X, A \cup B).$$

Let $c_1 \in H^n(X, A), c_2 \in H^m(X, B)$. Then we define $c_1 \cup c_2$ to be the cochain whose value on a simplex s is $c_1(s')c_2(s'')$ where s', s'' are the leading (resp. trailing) part of the simplex. So if $s = [a_0, \dots, a_{n+m}]$, then $s' = [a_0, \dots, a_m], s'' = [a_m, \dots, a_{n+m}]$. Then $c_1 \cup c_2$ is automatically zero on chains that lie either entirely in A or entirely in B. So $c_1 \cup c_2$ is zero on $C_*(A) + C_*(B)$. If we let $\mathfrak{A} = \{A, B\}$, then there is an inclusion $C_*^{\mathfrak{A}}(A \cup B) = C_*A + C_*B \to C_*(A \cup B)$. Dualizing gives a map

$$C^*(A \cup B) \to C^*_{\mathfrak{A}}(A \cup B)$$

which must also be an isomorphism in cohomology. Why?

We have an exact sequence

$$0 \to C^*_{\mathfrak{A}}(X, A \cup B) \to C_*(X) \to C^*_{\mathfrak{A}}(A \cup B) \to 0$$

We can draw another

$$0 \to C^*(X, A \cup B) \to C_*(X) \to C^*(A \cup B) \to 0$$

which fits into an exact diagram. The five lemma implies that $C^*(A\cup B) \to C^*_{\mathfrak{A}}(A\cup B)$ is an iso in cohomology.

So we have also an external relative cup product

$$H^*(X,A)\otimes H^*(Y,B) \to H^*(X\times Y, X\times B\cup A\times Y).$$

The Kunneth formula also holds at this level of generality.

28.2 Theorem. *If R is a field, and X, A, Y, B are finite CW complexes, then the map $H^*(X,A)\otimes H^*(Y,B) \to H^*(X\times Y, X\cup B, A\times Y)$ is an isomorphism in cohomology.*

28.3 Example. Take $(X,A) = (\mathbb{R}^n, \mathbb{R}^n - \{0\})$. We know that the cohomology of (X,A) with R-coefficients is R in dimension n and zero otherwise. The reason is that one can use the long exact sequence of the pair. Take $(Y,B) = (\mathbb{R}^m, \mathbb{R}^m - \{0\})$. Then $H^*(Y,B) = R$ if $* = m$ and 0 else. If we tensor these cohomologies together, there's an R at $n+m$ and zero in other dimensions. So this is the cohomology of $(X\times Y, X\times B\cup A\times Y)$. That is,

$$(\mathbb{R}^{n+m}, \mathbb{R}^{n+m} - 0)$$

as one can easily see. So the Kunneth formula is clearly verified.

28.4 Example. Note that

$$H^*(S^1\times S^1\times S^1) = H^*(S^1)\otimes H^*(S^1)\otimes H^*(S^1).$$

Thus the cohomology ring of the 3-torus can be computed:

$$R[\epsilon]/\epsilon^2 \otimes R[\epsilon]/\epsilon^2 \otimes R[\epsilon]/\epsilon^2$$

In particular, it is an exterior algebra.

28.5 Example. The cohomology ring of projective space. The answer is

$$H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[x]/x^{n+1},$$

where $x \in H^2(\mathbb{CP}^n)$ is the generator. This works in any ring (thanks to the universal coefficient theorem).

28.6 Example. $H^*(\mathbb{RP}^n, \mathbb{Z}/2) = \mathbb{Z}/2[w]/w^{n+1}$ where w generates $H^2(\mathbb{RP}^n)$. Here $\mathbb{Z}/2$ coefficients make things nice.

Lecture 29
11/19

A whole bunch of days have been skipped. Sorry. We have been calculating the homology groups of $SO(n)$ with $\mathbb{Z}/2$ coefficients. What we did was to construct a map

$$SO(n) \to S^{n-1} = SO(n)/SO(n-1).$$

We argued that if there was a *section* to this projection, then we would have

$$SO(n) \simeq SO(n-1) \times SO(n)/SO(n-1) = SO(n-1) \times S^{n-1},$$

and by the Kunneth formula, we could compute the homology. Nonetheless, by looking at some long exact sequences of pairs, we showed that if there is only a section in homology:

$$\begin{array}{ccc} & & H_*(SO(n)) \\ & \nearrow & \downarrow \\ H_*(S^{n-1}) & \longrightarrow & H_*(SO(n-1)) \end{array}$$

we can still conclude that

$$H_*(SO(n)) \sim H_*(SO(n-1)) \otimes H_*(S^{n-1})$$

as with the Kunneth formula. For this, we must find a space X mapping to the $n-1$-sphere and a diagram

$$\begin{array}{ccc} & & SO(n-1) \\ & \nearrow & \downarrow \\ X & \longrightarrow & S^{n-1} \end{array}$$

such that $H_*(X) \to H_*(S^{n-1})$ is surjective. Then we can lift this to get a homology section (since by projectivity $H_*(X) \to H_*(S^{n-1})$ splits). We will take $X = \mathbb{RP}^{n-1}$. Well, we write

$$S^{n-1} = \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2}.$$

Given a random line through the origin in n-space, we look at where it intersects the unit sphere in the upper half-plane. This is how we *could* get the isomorphism displayed above.

But this is not the *best way.*

On the other hand, we give a map to $SO(n)$. Namely given a line ℓ, at angle θ from the e_n-axis, we define the rotation sending the last e_n to the line at angle 2θ from it, and the identity on the orthogonal complement from the plane spanned by e_n and this vector. More generally if ℓ_1, ℓ_2 are lines ϕ, spanning a plane P, then reflecting through $\ell_2^\perp$ and then reflecting through $\ell_1^\perp$ is going to be a rotation in the plane P by angle 2ϕ.

We're going to map

$$\mathbb{RP}^{n-1} \to S^{n-1}$$

by sending a line ℓ to the following two things. First, reflect through $e_n^\perp$; then reflect through $\ell^\perp$. This composite is a map T_ℓ that moves the e_n vector that moves through twice the angle between θ. This is a very nice map from $\mathbb{RP}^n/\mathbb{RP}^{n-1}$ to S^{n-1} (anything in $\mathbb{RP}^{n-1}$ gets sent to the south pole). This obviously lifts to $SO(n)$, $\ell \to T_\ell$. It is a homeomorphism modulo $\mathbb{RP}^{n-2}$, so clearly onto in homology with $\mathbb{Z}/2$ coefficients. It also induces a map $\ell \to T_\ell$. So we have a diagram

$$\begin{array}{ccc} & & SO(n) \\ & \nearrow & \downarrow \\ \mathbb{RP}^{n-1} & \longrightarrow & S^{n-1} \end{array}$$

This is a **really important map**

$$\boxed{\mathbb{RP}^{n-1} \to SO(n)}$$

Since the map $\mathbb{RP}^{n-1} \to S^{n-1}$ is surjective, we find

29.1 Theorem. *$H_*(SO(n), \mathbb{Z}/2)$ (with the Pontryagin product) is the exterior algebra $\bigwedge[b_1, \ldots, b_n]$ where b_i has degree i.*

In fact, something much better happens. There is a **cell decomposition** of $SO(n)$. We can get that in this way. But that's worth coming back to later. This is anyway in Hatcher, where he writes this all up in great detail.

Let's build on this answer a bit more. Now I want to ask: we know the homology of $SO(n)$; what's the *cohomology ring*? That's a little bit trickier. We are going to be able to do this, but purely by algebraic manipulation.

Let's remember. The ring structure in cohomology comes from the diagonal map

$$SO(n) \to SO(n) \times SO(n).$$

In cohomology this leads to

$$H^*(SO(n)) \otimes H^*(SO(n)) \to H^*(SO(n)).$$

So what does the diagonal map do in cohomology? Well, first, what does it do in homology? We can then take dual vector spaces.

We will not do everything in full generality (i.e. restrict to $SO(3)$). In homology, we have the map

$$H_*(SO(n)) \to H_*(SO(n)) \otimes H_*(SO(n)).$$

This is a *coproduct,* and it goes in the wrong direction as an algebra structure. However, it is also a ring homomorphism (where both homologies are rings via the Pontryagin product) since the diagonal homomorphism is a group homomorphism. However, we have the super-duper-important map

$$\mathbb{RP}^{n-1} \to SO(n),$$

leading to a commutative diagram

$$\begin{array}{ccc} \mathbb{RP}^{n-1} & \longrightarrow & \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \\ \downarrow & & \downarrow \\ SO(n) & \longrightarrow & SO(n) \times SO(n) \end{array}$$

Since these are ring homomorphsims, we just need to figure out where the b_i go in $H_*(SO(n)) \otimes H_*(SO(n))$. And we can find the top row via the cup-product structure in projective space. Well, let $H^*(\mathbb{RP}^{n-1})$ have the basis $1, x, \ldots, x^{n-1}$; we can get $e_0, \ldots, e_{n-1} \in H_*(\mathbb{RP}^{n-1})$. The fact that $x^i \cup x^j = x^{i+j}$ implies that the map in homology sends e_m to the sum $\sum_{i+j=m} e_i \otimes e_j$. That describes the map in the homology of $\mathbb{RP}^{n-1}$ induced by the diagonal map. Anyway, this lets us compute the cohomology ring of $SO(n)$.

29.2 Example. $SO(3)$. We want to figure out the map

$$H_*(SO(3)) \to H_*(SO(3)) \otimes H_*(SO(3)).$$

The first on e has a basis consisting of $1, b_1, b_2, b_1b_2$. Under this $1 \to 1 \otimes 1$. $b_1 \to b_1 \otimes 1 + 1 \otimes b_1$. $b_2 \to b_2 \otimes 1 + 1 \otimes b_2 + b_1 \otimes b_1$. The product goes to the product. You can multiply those out. Now we can pass to the dual vector space to think of the cup product.

If we write $c_1 = b_1, c_2 = b_2, c_3 = b_1b_2$, and $c_0 = 1$, we find that under these maps

$$c_1 \to c_1 \otimes 1 + 1 \otimes c_1 \tag{1}$$
$$c_2 \to c_2 \otimes 1 + c_1 \otimes c_1 + 1 \otimes c_2 \tag{2}$$
$$c_3 \to c_3 \otimes 1 + c_2 \otimes c_1 + c_1 \otimes c_2 + c_3 \otimes c_1 \tag{3}$$

This is precisely of the form $a_m \to \sum_{i+j=m} a_i \otimes a_j$. In particular, if we take $\gamma \in H^1(SO(3))$ to be dual to c_1, then $H^*(SO(3))$ is $\mathbb{Z}/2[y]/(y^4)$. In fact

$$SO(3) \simeq \mathbb{RP}^3.$$

Remark. Let V be a vector space with an algebra structure, i.e. a map $V \otimes V \to V$. Let ϵ_i be a basis and suppose

$$\epsilon_i \otimes \epsilon_j \to \sum a_{ij}^k \epsilon_k.$$

Now V^* will be a dual basis ϵ^i. There is a map

$$V^* \to V^* \otimes V^*.$$

Its matrix form can be worked out explicitly. There will be a map

$$\epsilon^i \to \sum d_{jk}^i \epsilon^j \otimes \epsilon^k$$

Then the d_{jk}^i will be like the a_{ij}^k with some permutation. This is a bit of formalism with linear algebra.

Lecture 30
11/24

Let $p : E \to B$ be a fiber bundle with fiber F, with F compact and Hausdorff.

Last time:

30.1 Proposition. *p is trivial iff there is a map $E \to F$ such that $p^{-1}b \to E \to F$ is a homeomorphism for all $b \in B$.*

30.2 Proposition. *Work with coefficients in a field. If the fiber bundle looks trivial in cohomology, i.e. there is a section*

$$H^*(F) \to H^*(E)$$

which becomes an iso on each fiber, then the map

$$H^*(B) \otimes H^*(F) \to H^*(E)$$

is an iso.

Proof. Suppose $E \to B$ is trivial and the map is projection $B \times F \to B$. Then we have a map

$$H^*(B) \otimes H^*(F) \simeq H^*(B \times F) \leftarrow H^*(F)$$

such that for each $b \in B$,

$$H^*(\{b\} \times F) \leftarrow H^*(F)$$

is an iso. Using some algebra (**omitted!**), the result is not hard to show.

The general case is proved as follows. Suppose $B = U_1 \cup U_2$ where E is trivial on U_1, U_2. Then use Mayer-Vietoris. Namely, we get

$$H^{*-1}(p^{-1}(U_1 \cap U_2)) \to H^*(E) \to H^*(p^{-1}U_1) \oplus H^*(p^{-1}U_1) \to \dots$$

and we know the theorem for U_1, U_2. From this the five lemma gives the result. In general, use induction on the number of open sets in a cover. Then that does it for a base space covered by finitely many compact sets. ▲

30.3 Example. Consider a fiber bundle

$$\mathbb{RP}^{n-1} \to E \to B,$$

for instance a *projective space bundle.* Suppose there is an element $x \in H^1(E, \mathbb{Z}/2)$ that restricts to a generator of every fiber $\mathbb{RP}^{n-1}$, then using cup products of x we can get a map

$$H^*(E) \leftarrow H^*(\mathbb{RP}^{n-1})$$

Then

$$H^*(E) \simeq H^*(B) \otimes \left\{1, \alpha, \dots, \alpha^{n-1}\right\}$$

30.4 Proposition. *Consider a principal G-bundle*

$$G \to E \xrightarrow{p} B.$$

If there is a homology section, then we have

$$H_*(E) \simeq H_*(B) \otimes H_*(G)$$

Proof. Again, step 1: check for trivial bundles: step 2: use Mayer-Vietoris; step 3: use filtered colimits.

▲

This will enable us to compute the homology of $SO(n)$ as before. We find that

$$H * (SO(n)) \simeq H_*(S^1) \otimes H_*(S^2) \otimes \cdots \otimes H_*(S^{n-1}).$$

Next semester, we will learn another relationship between the homologies of fiber bundles, called a *spectral sequence.*

We would like to end with a *puzzle.* Consider the Hopf bundle (a principal S^1-bundle)

$$S^1 \to S^{2n-1} \to \mathbb{CP}^{n-1}.$$

It is easy to check that this is locally trivial. Namely, we can find a section over

$$U_i = \{[z_0, \dots, z_{n-1}] : z_i \neq 0\}$$

by sending $[z_0, \dots, z_{n-1}]$ to $(z_0/z_i, \dots, z_i/z_i, \dots, z_n/z_i)$ and scaling by the norm. So on the open sets U_i, this bundle has a section.

Question: Can we do this trivialization with fewer than n open sets?

The theorems that we have studied don't apply, as $H^*(S^{2n-1})$ is clearly not $H^*(\mathbb{CP}^{n-1}) \otimes H^*(S^1)$. Let $x \in H^2(\mathbb{CP}^{n-1}, \mathbb{Z})$ be the generator. Suppose there is $a \in H^*(B)$ that goes to zero in $H^*(E)$. Then if U is an open subset on which the bundle becomes trivial, a goes to zero in $H^*(U)$, because being trivial means that $H^*(U) \to H^*(p^{-1}(U))$ is injective.

Suppose $B = U \cup V$ and the bundle was trivial on each. Then a goes to zero in both U and V, so it comes from $H^*(B, U)$ as well as in $H^*(B, V)$. It follows that $a^2 = 0$ as it is in $H^*(B, U \cup V) = 0$.

In particular, we see that if B can be covered by two open sets, on which p is trivial, then any such element a has square zero. There is an obvious generalization of this. If B can be covered by k open sets as above, then $a^k = 0$.

So suppose $x \neq 0$ in $\mathbb{CP}^{n-1}$. Then x goes to zero in S^{2n-1}. So if $\mathbb{CP}^{n-1}$ can be covered by fewer than n open sets, we'd have $x^k = 0$ which is silly.

Lecture 31
11/29

§1 Grassmannians

We are now going to talk about the cohomology of Grassmannians. There are two different ways of calculating this, and we look at this down.

31.1 Definition. $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ is the space of k-planes through the origin in $\mathbb{R}^{n+k}$.

31.2 Example. $\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$.

If we have a k-plane $V \subset \mathbb{R}^{n+1}$, we can choose an orthonormal basis $\{v_1, \dots, v_k\} \subset V$. Here $v_1, \dots, v_k$ and $v_1', \dots, v_k'$ determine the same plane if there is an orthogonal matrix $T \in O(k)$ such that $T(v_i) = v_i', \forall i$. In other words,

$$\mathrm{Gr}_k(\mathbb{R}^{n+k}) = \{\text{orthonormal } k-\text{frames modulo }\}\, O(k)$$

Here the set V_k of orthonormal frames is a subset of a product of spheres, so it has a topology. Thus the Grassmannian gets the quotient topology.

31.3 Proposition. $\mathrm{Gr}_k(\mathbb{R}^{n+k})$ *is a manifold of dimension* nk.

Proof. Pick $V \subset \mathbb{R}^{n+k}$. Let $W = V^\perp$. Let U be the set of all $V' \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ such that $V' \cap W = \{0\}$. That is, the projection from any such V' down to V is an isomorphism, so by the **vertical line test** this subspace V' is a graph of a map $T : V \to W$. This identifies U with the set of maps $\mathrm{Hom}(V, W)$, which is a linear space of dimension nk. ▲

We will now decompose the Grassmannian into a cell decomposition, into a bunch of things called **Schubert cells.** Let's remember what went on in $\mathbb{RP}^n$. We filtered $\mathbb{R}^{n+1}$ by various $\mathbb{R}^i, i \le n+1$ and looked at the set $e_i = \left\{\ell : \ell \subset \mathbb{R}^{i+1}, \ell \not\subset \mathbb{R}^i\right\}$. If we intersected the set e_i with the upper half of the i-sphere, then we got a homeomorphism. So e_i is an open cell.

If there is a plane $V \subset \mathbb{R}^{n+k}$, we can look at the various numbers

$$\dim V \cap \mathbb{R}^1, \dots, \dim V \cap \mathbb{R}^{n+k}.$$

This can be various sequences of numbers.

31.4 Example. Let $V \in \mathrm{Gr}_3(\mathbb{R}^5)$ with basis $(1,1,0,0,0), (0,-1,1,1,0), (2,-3,0,0,1)$. Then the sequence goes $0,1,1,2,3$.

What is interesting is where the dimension jumps. Let $V \in \mathrm{Gr}_k(\mathbb{R}^{n+k})$ and let $j_1, \dots, j_k$ be the places where the dimension $\mathbb{R}^i \cap V$ jumps by one. This is a sequence of k numbers because V is k-dimensional.

Given a sequence $\mathbf{j} = (j_1, \dots, j_k) \in [1, n+k]$, we define the object $H_i = \mathbb{R}^{j_i}$ so that there is a sequence

$$\mathbb{R}^1 \subset H_1 \subset H_2 \subset \cdots \subset H_k \subset \mathbb{R}^{n+k}.$$

31.5 Definition. Fix the above notations.

The **Schubert variety** S is the set of all planes $V \subset \mathbb{R}^{n+k}$ such that $\dim V \cap H_i \ge i$.

Given these notations, we let $\mathbf{a} = (a_1, \dots, a_k)$ where each $a_i = j_i - i$. We shall write $S_{\mathbf{a}}$ for the Schubert variety.

31.6 Example. $\mathrm{Gr}_1(\mathbb{R}^{n+1})$. Here the possible $\mathbf{j}$'s are just $1, 2, \dots, n+1$. The corresponding element $\mathbf{a}$ is just $0, 1, \dots, n$ and the Schubert cell S_a is just the set of lines which is contained in $\mathbb{R}^{a+1}$, so alternatively $\mathbb{RP}^a$. The Schubert varieties are just the lower projective spaces as a result.

(denoted Σ_a in $\Sigma\epsilon c\tau\iota\nu$)

In general, $S_{\mathbf{a}}$ is **not** a manifold. It's an algebraic variety with singularities. As we shall see,

$$\dim S_{\mathbf{a}} = \sum a_i.$$

31.7 Lemma. *$S_{\mathbf{a}}$ is contained in $S_{\mathbf{a}'}$ iff $a_i \le a_i'$ for all i.*

Note that:

$$0 \le a_1 \le \cdots \le a_k \le n.$$

31.8 Example. $\mathrm{Gr}_2(\mathbb{R}^5)$. Then $\mathbf{a}$ can be $(0,0), (0,1), (0,2), (1,2), (2,2)$ etc. and we can draw a lattice of what these can look like.

As we will see, the Schubert varieties are actually cells, and they actually give a basis for the homology of the Grassmannian.

31.9 Proposition. *$e_{\mathbf{a}} = S_{\mathbf{a}} - \bigcup_{\mathbf{a}' < \mathbf{a}} S_{\mathbf{a}'}$ is a euclidean space of dimension $\mathbb{R}^{a_1 + \cdots + a_k}$.*

So $\mathbf{e_a}$ is the set of all $V \subset \mathbb{R}^{n+k}$ such that $\dim V \cap H_i = i$ for all i, while $\dim V \cap H_{i-1} = i - 1$. So the jumps occur exactly at these spots. This is like what happened earlier.

Proof. This corresponds to a bunch of lines in $\mathbb{R}^{j_i} - \mathbb{R}^{j_i - 1}$. More details? ▲

31.10 Proposition. *The $\mathbf{e_a}$ are the interiors of a cell decomposition of the Grassmannian. In the cellular chain complex with $\mathbb{Z}/2$ coefficients, the differential is zero.*