Math 207 Final Examination Solutions

December 6, 2006

Note: the wording of some of the problems has been changed to make them clearer, as a few were misinterpreted.

1. (20 points) Let (X, d) be a metric space, and let F be either $\mathbb{R}$ or $\mathbb{C}$ in the usual metric. Let $(\mathrm{BC(X,F)}, \mathrm{d}')$ be the space of bounded and continuous functions from X to F with metric $\mathrm{d}'(\mathrm{f},\mathrm{g}) = \sup_{\mathrm{x}\in\mathrm{X}} |\mathrm{f(x)} - \mathrm{g(x)}|$. Show that $(\mathrm{BC(X,F)}, \mathrm{d}')$ is complete.

   Proof. Let f_n be a Cauchy sequence. Let $\epsilon > 0$. Then, there exists some $N \in \mathbb{N}$ such that $\mathrm{d}'(\mathrm{f_n}, \mathrm{f_n}) < \epsilon$ for all $m, n > N$. Then for any $x_0 \in X$, we have $|f_n(x_0) - f_m(x_0)| \leq \sup_{x\in X} |f_n(x) - f_m(x)|$, so $(f_n(x_0))$ is a Cauchy sequence. Since F is complete, $(f_n(x))$ converges for every $x \in X$. For each $x \in X$, let $f(x) = \lim_n f_n(x)$.

   Let $\epsilon > 0$. Then there exist $N \in \mathbb{N}$ such that $\mathrm{d}'(\mathrm{f_n}, \mathrm{f_m}) < \epsilon/2$ for all $m, n > N$. Let $x \in X$. Then for sufficiently large m, $|f(x) - f_m(x)| < \epsilon/2$. Therefore $|f(x) - f_n(x)| < |f(x) - f_m(x)| + |f_m(x) - f_n(x)| < \epsilon$. Since x was arbitrary, we have that $|f(x) - f_n(x)| < \epsilon$ for all $x \in X$ and $n > N$.

   We have that for some $n \in \mathbb{N}$, $|f_n(x) - f(x)| < 1$ for all $x \in X$. Therefore $|f(x)| < |f_n(x)| + 1$. Since f_n is bounded, so is f.

   Let $\epsilon > 0$, and let $x_0 \in X$. From above, there is some $n \in N$ such that for all $x \in X$, $|f(x) - f_n(x)| < \epsilon/3$. Since f_n is continuous, there is some $\delta > 0$ such that if $d(x, x_0) < \delta$, then $|f_n(x) - f_n(x_0)| < \epsilon/3$. Then, if $d(x, x_0) < \delta$, $|f(x) - f(x_0)| < |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| = \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Thus f is continuous.

   Therefore, $f \in BC(X, F)$ and $\lim f_n = f$, hence every Cauchy sequence has a limit, so $BC(X, F)$ is complete.

   □

2. (20 points) Let p be a prime, and let $\mathbb{Q}_p$ be the p-adic numbers. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}_p$. Let $b_n = \sum_{k=1}^{n} a_k$. Show that $\{b_n\}$ converges if and only if $\lim a_i = 0$.

Proof. If $\lim a_n = 0$, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n|_p < \epsilon$ whenever $n > N$. Then if $n > m > N$, $|b_n - b_m|_p = \left|\sum_{i=m+1}^{n} a_i\right|_p \leq \max(\{|a_{m+1}|_p, \dots |a_n|_p\}) < \epsilon$. Thus (b_n) is Cauchy and thus converges.

Conversely, if (b_n) converges, $\lim a_n = \lim(b_{n+1} - b_n) = \lim b_{n+1} - \lim b_n = 0$. □

3. (20 points) Let p be a prime and r a rational number. Prove that the p-adic expansion of r is repeating.

Proof. By multiplying r by a power of p (which will not change periodicity of the expansion of r, we may assume that $r \in R_p$.

Let $r = a/b$ in lowest terms with $b > 0$. We wish to find (a_i) such that

$$\frac{a}{b} = \sum_{i=0}^{\infty} a_i p^i.$$

To begin, we choose $a_0 \in \{0, 1, \dots, p-1\}$ such that $\left|\frac{a}{b} - a_0\right|_p = |a - ba_0|_p < 1$, and at stage n we choose $a_n \in \{0, 1, \dots, p-1\}$ such that $|a - b\left(\sum_{i=0}^{n} a_i\right)|_p < p^{-n}$. We have that $p^n \mid a - b\left(\sum_{i=0}^{n-1} a_i\right)$, and hence $c_n = \left(a - b\left(\sum_{i=0}^{n-1} a_i\right)\right)/p^n$ is an integer for all n.

By construction, a_n is chosen so that $|a - b\left(\sum_{i=0}^{n} a_i\right)|_p = |p^n c_n - ba_n p^n|_p = p^{-n}|c_n - ba_n|_p < p^{-n}$, that is $|c_n - ba_n|_p < 1$. Thus, c_n uniquely determines a_n, and since c_n and a_n determine c_{n+1}, if (c_n) ever repeats a value, then (c_n) is periodic, and thus (a_n) is periodic.

Since $c_{n+1} = (c_n p^n - ba_n p^n)/p^{n+1} = (c_n - ba_n)/p$, we have that $|c_{n+1}| < b + |c_n|/p \leq \max(c_n, \frac{bp}{p-1})$, and thus that (c_n) takes on only finitely many values. Therefore, some value must be repeated. Therefore, (a_n) is periodic. □

Proof. Here is a sketch of a second solution. There are only finitely many congruence classes in $\mathbb{Z}/(b)$, so there exist $m, n \geq 0$ such that $p^m \equiv p^n \pmod{b}$. Since GCD(p, b) = 1, p is a unit in $\mathbb{Z}/(b)$ and thus $p^{n-m} \cong 1 \pmod{b}$, and hence we can find $c, k \in \mathbb{N}$ with $bc = p^k - 1$.

First, assume that r is positive. Write $r = \frac{a}{b} = \frac{ac}{p^k-1} = m - \frac{d}{p^k-1}$ where $m \in \mathbb{N} \cup \{0\}$ and $0 \leq d < p^k - 1$. Since $\frac{-d}{p^k-1} = \sum dp^{kn}$, we have that $\frac{-d}{p^k-1}$ has a periodic p-adic expansion on length k. Since m is positive, it clearly has a finite expansion (just it's base p expression). The proof (from the homework) that the sum of two periodic decimal numbers is periodic then shows that r is periodic.

If r is negative, let $\sum a_i p^i$ be the expansion of $-r$. Then $1 + \sum((p-1) - a_i)p^i$ is an expansion of of r. □

4. (20 points) Let p be a prime. Prove that in $\mathbb{Q}_p$, every bounded infinite sequence has a convergent subsequence (i.e., $\mathbb{Q}_p$ satisfies the Bolzano-Weirstrass property).

Proof. If either an infinite number of a_i are zero or if for every n, there exists a nonzero a_i with $|a_i|_p < p^{-n}$, then there exists a subsequence converging to 0. Assume otherwise. Then there is some n with $|a_i| > p^{-n}$ for all $i \in \mathbb{N}$. Since (a_i) is bounded, there is also an m with $|a_i| < p^m$ for all $\imath \in \mathbb{N}$. Thus, $|a_i|$ takes on only finitely many values, and so we can find some $n \in \mathbb{Z}$ such that $|a_i| = p^{-n}$ infinitely often. Let b_1 be one of those elements.

Since $p^n U_n$ can be written as p disjoint cosets of $p^n \wp$, an infinite number of a_i must occur in one of these cosets. Let b_2 be one of these elements occurring after b_1. In general, b_i will lay in some coset of $p^{n+i-1}\wp$, which in turn can be broken into p cosets of $p^{n+1}\wp$, one of which will contain an infinite number of a_i. Let b_{i+1} be an sequence member in this coset occurring after b_i. Since $|b_i - b_{i+1}|_p < p^- i + n$, (b_n) is Cauchy, and thus a convergent subsequence. □

5. (a) (10 points) Find the 5-adic expansion of 3/7.

   Proof. The algorithm from problem 3 can be used to show that $3/7 = 4+5*(0+2*5+1*5^2+4*5^3+2*5^4+3*5^5)+5^7*(0+2*5+1*5^2+4*5^3+2*5^4+3*5^5)+\ldots$. Alternately, $3/7 = 1-4/7 = 1-8928/(5^6-1)$ and 8928 has expansion $3+0*5+2*5^2+1*5^3+4*5^4+2*5^5$, and the expansion follows. □

   (b) (10 points) In $\mathbb{Q}_5$, what rational number has the 5-adic expansion $1+3*5+1*5^2+3*5^3+\ldots$?

   Proof. Let $x = 1+3*5+1*5^2+3*5^3+\ldots$. Then $5^2 x = x-(1+3*5)$, so $24x = -16$ or $x = -2/3$ □

6. (a) (10 points) Let $n \in \mathbb{N}$ and $1 \leq p, q \leq \infty$. Show that there exist $c, C > 0$ such that $c\mathrm{d_p(x,y)} \leq \mathrm{d_q(x,y)} \leq \mathrm{Cd_p(x,y)}$ for all $x, y \in \mathbb{R}^n$.

   Proof. First note that $d_\infty(x,y) \leq d_p(x,y) \leq d_1(x,y)$ for all $1 \leq p \leq \infty$. Also, $n * \sup|x_i - y_i| \geq \sum |x_i - y_i|$, and hence

$$\begin{aligned} \frac{1}{n} d_p(x,y) &\leq \frac{1}{n} d_1(x,y) \leq d_\infty(x,y) \\ &\leq d_q(x,y) \leq d_1(x,y) \\ &\leq n d_\infty(x,y) \leq n d_p(x,y) \end{aligned}$$

   Thus, we can take $c = 1/n$ and $C = n$. □

(b) (10 points) Prove that $U \subset \mathbb{R}^n$ is open in ℓ_n^p if an only if U is open in ℓ_n^q.

Proof. Let $B_{\epsilon,p}(x)$ denote the ϵ-ball around x in ℓ_n^p. If U is open in ℓ_n^p, then for each $x \in U$, we can find ϵ such that $B_{\epsilon,p}(x) \subset U$. Then $B_{c\epsilon,q}(x) \subset B_{\epsilon,p}(x) \subset U$, and thus B is open in ℓ_n^q. The other direction is similar. $\square$

7. Let (X, d) be a metric space, and define $\mathrm{d}' : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$ by

$$\mathrm{d}'(\mathrm{x}, \mathrm{y}) = \frac{\mathrm{d}(\mathrm{x}, \mathrm{y})}{1 + \mathrm{d}(\mathrm{x}, \mathrm{y})}$$

(a) (10 points) Prove that d' is a metric on X.

Proof. i. Since $\mathrm{d}(\mathrm{x}, \mathrm{y}) \geq 0$, we have $\mathrm{d}'(\mathrm{x}, \mathrm{y}) \geq 0$. Furthermore, since $1 + \mathrm{d}(\mathrm{x}, \mathrm{y})$ is always positive, $\mathrm{d}(\mathrm{x}, \mathrm{y})/(1 + \mathrm{d}(\mathrm{x}, \mathrm{y})) = 0 \Leftrightarrow \mathrm{d}(\mathrm{x}, \mathrm{y}) = 0 \Leftrightarrow \mathrm{x} = \mathrm{y}$. Thus d' is positive definite.

ii. $\mathrm{d}'(\mathrm{x}, \mathrm{y}) = \mathrm{d}(\mathrm{x}, \mathrm{y})/(1 + \mathrm{d}(\mathrm{x}, \mathrm{y})) = \mathrm{d}(\mathrm{y}, \mathrm{x})/(1 + \mathrm{d}(\mathrm{y}, \mathrm{x})) = \mathrm{d}'(\mathrm{y}, \mathrm{x})$.

iii. Since d is a metric, $\mathrm{d}(\mathrm{x}, \mathrm{z}) \leq \mathrm{d}(\mathrm{x}, \mathrm{y}) + \mathrm{d}(\mathrm{y}, \mathrm{z})$, and since $t/(1+t) = 1 - 1/(t+1)$ is increasing when $t \geq 0$, we have

$$\begin{aligned}
\mathrm{d}'(\mathrm{x}, \mathrm{z}) &= \frac{\mathrm{d}(\mathrm{x}, \mathrm{z})}{1 + \mathrm{d}(\mathrm{x}, \mathrm{z})} \\
&\leq \frac{\mathrm{d}(\mathrm{x}, \mathrm{y}) + \mathrm{d}(\mathrm{y}, \mathrm{z})}{1 + \mathrm{d}(\mathrm{x}, \mathrm{y}) + \mathrm{d}(\mathrm{y}, \mathrm{z})} \\
&= \frac{\mathrm{d}(\mathrm{x}, \mathrm{y})}{1 + \mathrm{d}(\mathrm{x}, \mathrm{y}) + \mathrm{d}(\mathrm{y}, \mathrm{z})} \\
&\quad + \frac{\mathrm{d}(\mathrm{y}, \mathrm{z})}{1 + \mathrm{d}(\mathrm{x}, \mathrm{y}) + \mathrm{d}(\mathrm{y}, \mathrm{z})} \\
&\leq \frac{\mathrm{d}(\mathrm{x}, \mathrm{y})}{1 + \mathrm{d}(\mathrm{x}, \mathrm{y})} + \frac{\mathrm{d}(\mathrm{y}, \mathrm{z})}{1 + \mathrm{d}(\mathrm{y}, \mathrm{z})} \\
&= \mathrm{d}'(\mathrm{x}, \mathrm{y}) + \mathrm{d}'(\mathrm{y}, \mathrm{z}).
\end{aligned}$$

$\square$

(b) (10 points) Prove that $U \subset X$ is open in (X, d) if and only if U is open in (X, d').

Proof. It is easy to verify that for all $x_0 \in X$ and $\epsilon > 0$ we have $B_\epsilon(x_0) \subset B'_\epsilon(x_0)$ and $B'_{\epsilon/(1-\epsilon)}(x_0) \subset B_\epsilon(x_0)$. The result follows.

$\square$

8. (20 points) Let (X, d) and (X', d') be metric spaces, and define $\widetilde{\mathrm{d}} : (X \times X') \times (X \times X') \to \mathbb{R}$ by $\widetilde{\mathrm{d}}((x_1, y_1), (x_2, y_2)) = \sup(\mathrm{d}(\mathrm{x}_1, \mathrm{x}_2), \mathrm{d}'(\mathrm{y}_1, \mathrm{y}_2))$. Show that $(X \times X', \widetilde{\mathrm{d}})$ is a metric space.

Proof. Positive definiteness and symmetry are easy to check and similar to the previous problem. For the triangle inequality, we have that $\mathrm{d}(\mathrm{x}_1, \mathrm{z}_1) \leq \mathrm{d}(\mathrm{x}_1, \mathrm{y}_1) + \mathrm{d}(\mathrm{y}_1, \mathrm{z}_1)$ and $\mathrm{d}'(\mathrm{x}_2, \mathrm{z}_2) \leq \mathrm{d}'(\mathrm{x}_2, \mathrm{y}_2) + \mathrm{d}'(\mathrm{y}_2, \mathrm{z}_2)$, and hence

$$\begin{aligned}
\widetilde{\mathrm{d}}((x_1, x_2), (z_1, z_2)) &= \sup(\mathrm{d}(\mathrm{x}_1, \mathrm{z}_1), \mathrm{d}'(\mathrm{x}_2, \mathrm{z}_2)) \\
&\leq \sup(\mathrm{d}(\mathrm{x}_1, \mathrm{y}_1) + \mathrm{d}(\mathrm{y}_1, \mathrm{z}_1), \mathrm{d}'(\mathrm{x}_2, \mathrm{y}_2) + \mathrm{d}'(\mathrm{y}_2, \mathrm{z}_2)) \\
&\leq \sup(\mathrm{d}(\mathrm{x}_1, \mathrm{y}_1), \mathrm{d}'(\mathrm{x}_2, \mathrm{y}_2)) \\
&\quad + \sup(\mathrm{d}(\mathrm{y}_1, \mathrm{z}_1), \mathrm{d}'(\mathrm{y}_2, \mathrm{z}_2)) \\
&= \widetilde{\mathrm{d}}((x_1, x_2), (y_1, y_2)) + \widetilde{\mathrm{d}}((y_1, y_2), (z_1, z_2))
\end{aligned}$$

□

9. Let (X, d) be a metric space. A subset A is said to be *disconnected* if there exist open sets $U, V \subset X$ such that

   (a) $(A \cap U) \neq \emptyset, (A \cap V) \neq \emptyset$
   (b) $(A \cap U) \cup (A \cap V) = A$
   (c) $(A \cap U) \cap (A \cap V) = \emptyset$

   A is said to be *connected* if it is not disconnected.

   (a) (10 points) Prove that if $A \subset X$ is connected and $f : X \to Y$ is a continuous function, then $f(A)$ is connected.

   Proof. Assume U and V disconnect $f(A)$, that is

   i. $(f(A) \cap U) \neq \emptyset, (f(A) \cap V) \neq \emptyset$
   ii. $(f(A) \cap U) \cup (f(A) \cap V) = f(A)$
   iii. $(f(A) \cap U) \cap (f(A) \cap V) = \emptyset$

   Then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets such that

   i. $(A \cap f^{-1}(U)) \neq \emptyset, (A \cap f^{-1}(V)) \neq \emptyset$
   ii. $(A \cap f^{-1}(U)) \cup (A \cap f^{-1}(V)) = A$
   iii. $(A \cap f^{-1}(U)) \cap (A \cap f^{-1}(V)) = \emptyset$

   Thus, if A is connected, $f(A)$ is connected.

   □

   (b) (10 points) A subset $I \subset \mathbb{R}$ is said to be an interval if whenever $a, b \in I$ and $a < x < b$, then $x \in I$. Prove that $A \subset \mathbb{R}$ is connected if and only if A is an interval.

   Proof. If A is not an interval, then we can find, $a < x < b$ with $a, b \in A$, $x \notin A$. Let $U = (-\infty, x), V = (x, \infty)$. Then U and V disconnect A.

   Conversely, suppose that A is an interval, and that there exist U and V disconnecting A. Without loss of generality, we can find $a \in$

$A \cap U$, $b \in A \cap V$ with $a < b$. Let $c = \sup(\{x \in [a,b] \mid U \cap [a,x]$ is an interval$\})$. Since A is an interval, $c \in A$. Note that if $a < x < c$, then $x \in U$. We cannot have $c \in V$ (e.g., $b = c$) since otherwise we could find some ϵ-ball around c contained in V, and then $c - \epsilon/2 \in (A \cap U) \cap (A \cap V)$. If $c \in U$, then we can find some ϵ-ball around c contained in U, and then $[a, c + \epsilon/2]$ is an interval, contradicting maximality of c. Thus, $c \notin (A \cap U) \cup (A \cap V) = A$, contradicting that A is an interval. □

10. (20 points) Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Prove that f attains some maximum value on X, i.e., there exists some $x \in X$ such that if $y \in X$, $f(y) \leq f(x)$.

    Proof. Since X is compact and f is continuous, $f(X)$ is compact, and hence closed and bounded. Since $\mathbb{R}$ has the least upper bound property, $f(X)$ a least upper bound ℓ, and since $f(X)$ is closed $\ell \in f(X)$. Thus, for some $x \in X$, $f(x) = \ell \geq f(y)$ for all $y \in Y$. □

11. Let (X, d) be a metric spaces, and let $A, B \subset X$. We define the distance between A and B as $\mathrm{d(A,B)} = \inf\{\mathrm{d(a,b)} \mid \mathrm{a \in A, b \in B}\}$. We say the distance is realized if there exist $a \in A, b \in B$ such that $\mathrm{d(A,B)} = \mathrm{d(a,b)}$. For each of the following, give a proof or a counterexample.

    (a) (10 points) If A and B are closed, must $\mathrm{d(A,B)}$ be realized?

    (b) (10 points) If A is closed and B is compact, must $\mathrm{d(A,B)}$ be realized?

    In both these cases, $\mathrm{d(A,B)}$ need not be realized. Consider $\mathbb{Q}$ in the usual metric, and let $A = \{x \mid x^2 < 2\}$ and $B = \{2\}$. A is closed, B is compact, and $\mathrm{d(A,B)} = 2 - \sqrt{2}$ is not realized.

    For an example with X complete, let $X = B(\mathbb{N}, \mathbb{R})$. Let e_n denote the function which is 1 on n and 0 elsewhere. Let $A = \{(1 + 1/n)e_n \mid n \in \mathbb{N}\}$ and $B = \{0\}$. All the points of A are isolated, so A is closed, and $\mathrm{d(A,B)} = 1$ is not realized.

    (c) (10 points) If A and B are compact, must $\mathrm{d(A,B)}$ be realized?

    Let $\ell = \mathrm{d(A,B)}$. Then for all $n \in \mathbb{N}$, we can find $a_n \in A, b_n \in B$ such that $\mathrm{d(a_n, b_n)} < \ell + \mathrm{1/n}$. Since A is compact, (a_n) has some convergent subsequence (a_{i_n}). Since B is compact, (b_{i_n}) has some convergent subsequence (b_{j_n}). Let $a = \lim a_{j_n}$, $b = \lim b_{j_n}$. Then for any $n \in \mathbb{N}$, we have $\mathrm{d(a,b)} \leq \mathrm{d(a,a_n)} + \mathrm{d(a_n,b_n)} + \mathrm{d(b,b_n)}$. Therefore, $\mathrm{d(a,b)} \leq \ell$. By the definition of ℓ, we must have that a and b realize the distance.