A Construction of Endo-permutation Modules

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1 Introduction

Endo-permutation modules for finite p-groups appear in the representation theory of finite groups in several ways, for example as sources of simple modules and in connection with equivalences between blocks [6, 7, 11]. The main problem is their complete description and classification. We shall give a simple new construction of such modules, study their properties and apply the results to the group of endo-trivial modules, determining its rank.

Let P be a fixed finite p-group and k an algebraically closed field of characteristic p. All modules for kP will be assumed finite-dimensional and all P-sets will be finite. If X is a P-set let $\Delta(X)$ be the kernel of the map from the kP-module kX to the trivial modules k sending each element of X to 1 (the augmentation map). Recall that a kP-module U is an endo-permutation module if the kP-module $\mathrm{Hom}_k(U,U) = \mathrm{End}_k(U)$ is a permutation module and U is an endo-trivial module if $\mathrm{Hom}_k(U,U)$ is the direct sum of k and a free kP-module.

Theorem 1. *The module $\Delta(X)$ is an endo-permutation module.*

Theorem 2. *The module $\Delta(X)$ is indecomposable if, and only if, no orbit of P on X is a homomorphic image, as P-set, of another orbit of P on X.*

Theorem 3. *If $\Delta(X)$ is indecomposable and Q is a subgroup of P then the class $[Br_Q^P(\Delta(X))]$ in the Dade group of $N(Q)/Q$ equals $[\Delta(X^Q)]$.*

Here X^Q are the fixed points of Q on X and the Brauer map notation is the usual one for endo-permutation modules (called the slash operation in [6].) We shall review the Dade definition in the course of the proof. Recall that if Y is a P-set then $(kY)^Q$ is the direct sum of kY^Q and the sum of traces from proper subgroups of Q and the quotient of $(kY)^Q$ by these traces is $\mathrm{Br}_Q^P(kY)$.

Theorem 4. *The rank of the group of endo-trivial kP-modules is as follows:*

a. If the p-rank of P is one it is zero;

b. If the p-rank of P is two then it is the number of conjugacy classes of maximal elementary abelian subgroups;

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c. *If the p-rank of P is three or more then it is one plus the number of conjugacy classes of maximal elementary abelian subgroups which are of order p^2.*

As usual, the p-rank of a group is the rank of the elementary abelian p-subgroups of maximal order.

The remainder of this paper is devoted to the proofs of the results, in order. All the arguments are direct and elementary. In a sequel we shall examine this topic from the point of view of relative homological algebra; the work actually began there following the suggestion of Lluis Puig that this topic was relevant to the study of endo-permutation modules.

2 Proof of Theorem 1

Let C be the two term complex $kX \to k$ where the map is the augmentation ϵ and let C^* be the complex $k \to kX$ where the map is σ which takes 1 to the sum of the elements of X, so C^* is isomorphic with the dual of C. The homology of C is just $\Delta(X)$ and that of C^* is the dual $\Delta(X)^*$. We now get a double complex

$$\begin{array}{ccc} k[X \times X] & \xrightarrow{\epsilon \times 1} & kX \\ {\scriptstyle 1\times\sigma}\uparrow & & \uparrow{\scriptstyle \sigma} \\ kX & \xrightarrow[-\epsilon]{} & k \end{array}$$

by taking the tensor product and the homology of the associated total complex is just $\Delta(X) \otimes \Delta(X)^*$ in the middle dimension. We shall prove that this complex is split so that $\Delta(X) \otimes \Delta(X)^*$, which is isomorphic with $\mathrm{Hom}_k(\Delta(X), \Delta(X))$, is then a summand of a permutation module, since the four modules in the diagram are permutation modules. But P is a p-group so the result will then be proved.

Hence, we need to give a homomorphism of KX to $K[X \times X]$ and of kX to k such that the sum of these two maps, in the total complex, followed by the differential, is the identity on kX. And we need to do the analogous thing at the other "corner" kX. The map of kX to k will be σ and the map of kX to $k[X \times X]$ takes each $x \in X$ to minus the sum of all pairs (x, y) where y runs over the elements of X other than x. Turning to the other corner, the map of k to kX is just σ again, and the map of $k[X \times X]$ to kX takes a pair (x, y) to y if x and y are distinct and to $2y$ otherwise. The verifications are immediate.

3 Proof of Theorem 2

First, we need some preliminary results.

Lemma 1. *If X and Y are transitive P-sets then there is an epimorphism from kX to kY if, and only if, there is an epimorphism of P-sets of X to Y.*

Proof. Suppose ϕ is an epimorphism of kX to kY. Choose $x \in X$ so that $\phi(x)$ is not in the radical of kY. Let H be the stabilizer of x so H fixes $\phi(x)$ and therefore $\phi(x)$ is a linear combination of orbit sums of H on Y. Since the radical of kY is the unique maximal submodule, by transitivity, and is the kernel of the augmentation map, there is such a sum with a non-zero augmentation. That is, the cardinality of the orbit of $y \in Y$ under H is not divisible by p. If K is the stablizer of y then we have $|H : H \cap K|$ is prime to p so it is one and H is contained in K, as desired. □

Lemma 2. *If B and C are bases of a finite-dimensional vector space over k and $B = B_1 \cup B_2$ is a partition of B then there is a partition $C = C_1 \cup C_2$ of C such that $B_1 \cup C_2$ and $B_2 \cup C_1$ are bases.*

Proof. We may as well assume that the vector space is the standard row space of length n, that the second basis is the standard one with elements $e_1, \dots, e_n$ while the first one consists of $v_1, \dots, v_n$ partitioned into the first m vectors and the last $n - m$ vectors. The matrix, indexed by the v_i on the rows and the e_j on the columns, giving the coefficients of the expressions of the v_i as linear combinations of the e_j, is non-singular. Hence, by the Laplace expansion of the determinant, for the expansion using the first m rows of the matrix (rather than one row), after renumbering the e_j if needed, we can assume the matrix has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A and B have m rows and m and $n - m$ columns, respectively, while C and D have $n - m$ rows and m and $n - m$ columns, respectively, and where A and D are non-singular. Hence, the matrices

$$\begin{pmatrix} A & B \\ 0 & I \end{pmatrix}, \begin{pmatrix} I & 0 \\ C & D \end{pmatrix}$$

are also non-singular and the result is proved.

□

Turning to the theorem, suppose that X is the disjoint union of the P-sets Y and Z and that ϕ is a P-set epimorphism of Y onto Z. It is then easy to see that $\Delta(X)$ is the direct sum of the module spanned by all elements $y - \phi(y)$ and $\Delta(Z)$ so one half of the theorem is clear.

On the other hand suppose that $\Delta(X) = U \oplus V$ is a non-trivial direct sum of kP-modules. Let X be the disjoint union of the orbits $X_1, \dots X_n$. We wish to show that one of these orbits is, as a P-set, an epimorphic image of another. Hence, we may assume no orbit has cardinality one. Thus, the socle of $\Delta(X)$ equals the socle of kX and is n-dimensional. Therefore, by Lemma 2, there is a partition of $N = \{1, 2, \dots, n\} = I \cup J$ so that we have two direct sum decompositions,

$$\mathrm{soc}(kX) = \mathrm{soc}(kX_I) \oplus \mathrm{soc}(V)$$

$$= \mathrm{soc}(U) \oplus \mathrm{soc}(KX_J)$$

where X_I and X_J are the obvious unions of orbits. In particular, $U \cap kX_J = 0$ and $V \cap kX_I = 0$.

We will now use Goursat's Theorem ([2], p. 25) on the structure of submodules of a direct sum of two modules in terms of projections and intersections with the two summands. We have now already, with the obvious notation, that $U \cong P_I(U)$ (as the intersection with kX_J is zero) and, similarly, that $V \cong P_J(V)$. But $U \oplus V$ is of codimension one in KX so it must be that one of these projections is also surjective; without loss we assume that $P_J(V) = kX_J$.

But V is contained in $\Delta(X)$ and $P_J(V)$ is not so there must be some $i \in I$ such that $P_i(V)$ is not contained in $\Delta(X_i)$. For the image of V under the augmentation map for X is the sum of the images under the augmentation maps for each X_i, i running over i, plus the image of the augmentation map for X_J. Thus, the projection of V on X_i, in terms of the decomposition of kX into the direct sum of all kX_t, t running over X, is not contained in the unique maximal submodule $\Delta(X_i)$ of kX_i, so this projection is all of kX_i.

Now let $J' = J \cup i$ and we shall study the projection $P'_J(V)$ as a submodule of kX'_J (using the decomposition of N into J' and $I - i$). First, note that $kX_i \cap P'_J(V) = 0$. For otherwise, there is an element of V whose "coordinates" in J are all zero but whose "i-th coordinate" is not zero, that is, a non-zero element of $kX_I \cap V$, a contradiction. Hence, by Goursat's Theorem, applied to the submodule $P'_J(V)$ of $kX_i \oplus KX_J$, $k(X_i)$ is an epimorphic image of kX_J. But since kX_i has a unique maximal submodule that implies that there is j in J with kX_i an epimorphic of kX_J. But Lemma 1 shows that X_i is an epimorphic image of the P-set X_j. Hence the result is proved.

4 Proof of Theorem 3

If $\Delta(X)$ is indecomposable then it is of dimension not divisible by p, by Theorem 2, so, as P is a p-group, it is not induced from a proper subgroup of P and therefore has vertex P. Hence, it defines an element of the Dade group of P. The module for $N_P(Q)$ corresponding with it is constructed as follows [6]. Let $A = \mathrm{End}_k(\Delta(X))$ and suppose that Y is a permutation basis of A. Let B= A^Q so $B = kY^Q + I$, a direct sum of vector spaces where I is the ideal of traces and $B/I = Br_Q^P(A)$. We have [6] that B/I is also a matrix algebra over k and its simple module, as module for $N_P(Q)$, is the corresponding module. Hence, we need only show that $\Delta(X^Q)$ is a simple module for B and that this respects the action of $N(Q)$. We shall do this by showing, in turn, it has the right dimension, is a module for B/I and the action is respected.

Suppose that exactly e of the elements of X have stablizers, in P, which contain Q. Then $|X^Q| = e$ so $\dim\Delta(X^Q) = e - 1$. We claim that B has dimension $(e-1)^2$. That is, the number of elements in a permutation basis of A which have stabilizers containing Q is exactly $(e-1)^2$. However, the proof of Theorem 1 gives the following isomorphism:

$$k \oplus k[X \times X] \cong kX \oplus kX \oplus A.$$

Thus, the number of stablizers we are after is the number for $X \times X$ plus one less $2e$. Hence, we need only see the number for $X \times X$ is exactly e^2. But the stabilizer of the pair

(x, y), for x and y in X, contains Q if, and only if, the stablizers of x and y each do, so the number is correct. Hence, $\Delta(X^Q)$ is of dimension equal to that of the simple B/I-module.

Since $\Delta(X)$ is an A-module (as A consists of all linear transformations on $\Delta(X)$) it follows that $(\Delta(X))^Q$ is a $B = A^Q$-module. But we have a vector space direct sum $(kX)^Q = kX^Q \oplus J$, where J consists of the traces. Moreover, J is contained in $(\Delta(X))^Q$ since every orbit sum making up a trace is a sum of a power of p number of terms. Thus, it follows that we have

$$(\Delta(X))^Q = \Delta(X^Q) \oplus J$$

since the intersection of $\Delta(X)^Q$ with $k(X^Q)$ is $\Delta(X^Q)$.

$$\begin{tikzcd}
(kX)^Q \arrow[r, dash] & (\Delta(X))^Q \arrow[r, dash] & J \\
kX^Q \arrow[u, dash] \arrow[r, dash] & \Delta(X^Q) \arrow[u, dash] \arrow[r, dash] & 0 \arrow[u, dash]
\end{tikzcd}$$

Next, we have that $I.\Delta((X))^Q$ contained in J. For suppose that R is a proper subgroup of Q, a is in A^R and d is in $(\Delta(X))^Q$ and we compute the sum of terms $gag^{-1}.d$ where g runs over a set of coset representatives of R in Q. Then this equals a sum of terms $g.ad$ and ad now lies in $(kX)^R$. Thus, $(\Delta(X)^Q/J$ is a B/I-module and as such B/I induces all the linear transformations on this space. Hence, we have the structure of B/I module for $\Delta(X^Q)$, by the First Isomorphism Theorem. It is therefore enough to verify compatibility with the action of $N_P(Q)$. But this is immediate: if b is in B, d in $\Delta(X)^Q$ and n in $N_P(Q)$ then $nbn^{-1}.nd = n.bn$.

5 Proof of Theorem 4

The key step is the following result.

Proposition 1. *If F is a maximal elementary abelian subgroup of P and F has order p^2 then there is an endo-trivial kP-module whose class, in the group of endo-trivial modules, has a non-trivial restriction to F and a trivial restriction to every other maximal elementary abelian subgroup of P not conjugate with F.*

First, we state and then prove a preliminary result.

Lemma 3. *If F is a maximal elementary abelian subgroup of P of order p^2 and F is contained in the Frattini subgroup of P then the following conditions are satisfied:*

a. The p-rank of P is two;

b. F is normal in P;

c. Every other maximal elementary abelian subgroup of P is not contained in the Frattini subgroup.

Proof. First, suppose that the normal p-rank is one, so every normal abelian subgroup of P is cyclic. Then, by a result of P. Hall ([8], p.198) we have that $p = 2$ and P is dihedral or quasi-dihedral and the condition that F is contained in the Frattini subgroup is violated, by inspection. Hence, we may let N be a normal elementary abelian subgroup of order p^2 in P. Since F is contained in the Frattini subgroup it follows that F centralizes N so, by maximality, $F = N$. Maximality also implies that $C = C_P(F)$ is either P or of index p in P and that F contains all the elements of order p in C. If $C = P$ then everything is proved so assume that C is of index p in P. Now, if E is any maximal elementary abelian subgroup then either E is contained in C, so $E = F$, or $E \cap C = Z(P) \cap F$ is of order p so E has order at most p^2 and E is not contained in the Frattini subgroup (as E has a non-trivial image in a quotient of P of order p, namely P/C). Thus, we have proved the lemma. □

Let us use the Proposition to see that the theorem follows before establishing the Proposition. For any p-group H let $T(H)$ denote the group of endo-trivial kH-modules. The restriction map for endo-trivial modules gives a map

$$T(P) \to \Pi T(E)$$

where E runs over representatives of the conjugacy classes of elementary abelian subgroups of P. The kernel is a finite torsion group by a theorem of Puig [10]. Now if P has p-rank one then the right-hand product has one term and is of order one or two so the Theorem is valid in this case. If the p-rank of P is two then every subgroup E in the product is of order p^2 and the Proposition applies immediately. Finally, suppose that the p-rank of P is at least three so the normal p-rank is at least two, by Hall's theorem, so there is a normal elementary abelian subgroup N of order p^2. Each of the groups $T(E)$ is now infinite cyclic and generated by $\Delta(E)$, for the E-set E [7] and the restriction of $\Delta(P)$ to the product gives the element whose coordinates are these generators. Each element in the product is therefore a product of powers of the generators $\Delta(E)$. Hence, in view of the Proposition, we need only show that any element in the product which is in the image has all the powers the same for every E which is of order at least p^3. However, if E_1 and E_2 are two such subgroups with an intersection of order p^2 then the powers of $\Delta(E_1)$ and $\Delta(E_2)$ must coincide since restrictions to the intersection of E_1 and E_2 from E_1 and E_2 must agree: the restriction map from $T(E)$ to $T(E')$ where E and E' are elementary abelian of order greater than p is always an isomorphism. We now finish by employing N. If E is elementary abelian of order at least p^3 then either E contains N or $C_E(N)$ is of index p in E. In either case, E intersects another elementary abelian subgroup of the same order in a group of order p^2 and the latter group contains N. Thus, the transitive relation on maximal elementary abelian subgroups given by having intersections of order at least p^2 makes all these subgroups equivalent and the proof is complete.

Next, we need an easy result before proving the Proposition.

Lemma 4. *If Y and Z are P-sets then there is an exact sequence of kP-modules*

$$0 \to \Delta(Y) \otimes \Delta(Z) \to k[Y \times Z] \to \Delta(Y \cup Z) \to 0$$

The map which sends (y, z), for y in Y and z in Z, to $y - z$ is easily seen to be an epimorphism. The obvious map of $\Delta(Y) \otimes \Delta(Z)$ to $k[Y \times Z] \cong k[Y] \otimes k[Z]$ has image in its kernel and dimension counting confirms the exactness.

Let us now turn to the proof of the Proposition.

Proof. First, assume that F is not contained in the Frattini subgroup of P so there is a maximal subgroup M of P with F not contained in M. Thus, there is a subgroup Q of order p in F such that $P = Q.M$ and so $C_M(Q)$ is of p-rank one (or else F intersect M is not a maximal elementary abelian subgroup of M). But $C_M(Q) \cong N_P(Q)/Q$ so this quotient is of p-rank one as well. Hence, the $kN_P(Q)/Q$-module $\Delta(N_P(Q)/Q)$ is endo-trivial and its class in the group of endo-trivial modules has order dividing four (by periodicity of projective resolutions).

Let $X = P/Q$ be a P-set so, using Theorem 2, we have that $[\mathrm{Br}_Q^P(X)] = [\Delta(N_P(Q)/Q)]$ and, since Q has order p, $\mathrm{Br}_R^P(\Delta(X) = 1$ for any subgroup R of P not conjugate with Q. Hence, by Puig's criterion [[10], p. 518] the fourth power of $\Delta(X)$ is an endo-trivial module.

Let E be a maximal elementary abelian subgroup of P not conjugate with F. Thus, no conjugate of Q lies in E, or else $F =< F \cap Z(P), Q >$ would be conjugate with a subgroup of E. Hence, the kX is a free kE-module so $\mathrm{Res}_E^P[(\Delta(X)] = [\Delta(E)]$.

Notice that if Q is normal in P then $P = Q \times M$ and F is the direct product of Q and the unique central subgroup of order p in M so F is the unique maximal elementary abelian subgroup of P and the Proposition holds. Hence, we assume that Q is now not normal in Q.

We have from above that restriction to E of $[\Delta(X)][\Delta(P)]^{-1}$ is trivial so, in order to complete the proof in this case, we need only prove that the restriction to F of $[\Delta(X)]^4$ does not equal $[\Delta(F)]^4$. As Q is not normal then P induces non-trivial automorphisms on F so there are exactly p subgroups of F of order p, $Q = Q_1, Q_2, \ldots, Q_p$, which are conjugate to Q in P. Now let $X_i = F/Q_i$ be an F-set, $i = 1, \ldots, p$. Since all these subgroups are conjugate we get that the restriction of $\Delta(X)$ to F is a positive power of $\Delta(X_1 \cup \ldots \cup X_p)$. Lemma 4, applied to the group F, now gives us an exact sequence

$$0 \to \Delta(X_1) \otimes \Delta(X_2) \to kX_1 \otimes kX_2 \to \Delta(X_1 \cup X_2) \to 0$$

But the middle term is free so we get that

$$[\Delta(X_1)][\Delta(X_2)] = [\Delta(F)][\Delta(X_1 \cup X_2)]$$

and iterating this argument gives

$$[\Delta(X_1)] \ldots [\Delta(X_p)] = [\Delta(F)]^{p-1}[\Delta(X_1) \cup \ldots \cup \Delta(X_p)]$$

Since each element $[\Delta(X_i)]$ has order two, taking the fourth power of this relation gives us more than we need - we get $[\Delta(X)]^4$ a negative power of $[\Delta(F)]$, and the proof is complete.

Now we examine the remaining case, that F is contained in the Frattini subgroup of P. Let $E_1, \ldots, E_n$ be representatives of the conjugacy classes of other maximal elementary abelian subgroups of P so, by Lemma 3, none of these n subgroups is contained in the Frattini subgroup of P. Applying what we have just done to E_i, we have that there is an endo-trivial module U_i such that the restriction of $[U_i]$ to E_i is non-trivial but it has trivial restrictions to E_j, for every j not i, and to F. Since the group of endotrivial modules of an abelian group A is cyclic with generator $[\Delta(A)]$, it follows that we can take a tensor product of powers of the $[U_i]$, $i = 1, \ldots, n$ with a power of $[\Delta(P)]$ to get the desired result. Hence, the Proposition and, therefore, Theorem 4, are proved. $\square$

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