Automorphisms of Quotients of Symmetric Groups

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Abstract

We describe the automorphism group of the quotient of the symmetric group on a countably infinite set by the group of finitary permutations. We show that the outer automorphism group is infinite cyclic. In the proof, we examine the structure of centralizers in this quotient.

1 Introduction

In this paper, S will denote $Sym(\Omega)$, the full symmetric group on a countably infinite set Ω. For any $g \in S$, define the *support* of g to be $supp(g) = \{\alpha \in \Omega\colon \alpha g \neq \alpha\}$ and the *fixed point set* to be $fix\, g = \{\alpha \in \Omega\colon \alpha g = \alpha\}$. It is well-known (see Scott [2] or originally Baer [1]) that S has precisely two proper non-trivial normal subgroups. These are $FS(\Omega) := \{g \in S\colon |\, supp(g)| < \aleph_0\}$ and $\mathrm{Alt}(\Omega)$, the subgroup of index two in $FS(\Omega)$ consisting of even permutations. Throughout this paper, we let $G := S/FS(\Omega)$. This group is simple. The main goal of this paper is to describe $Aut(G)$.

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Theorem 1.1 *Let $G = Sym(\Omega)/FS(\Omega)$, where Ω is a countably infinite set. Then the outer automorphism group $Out(G)$ is infinite cyclic.*

To see why G admits an infinite cyclic group of outer automorphisms, observe that if Ω is identified with N, then the shift $n \mapsto n+1$, acting on G 'by conjugation up to finitely many mistakes', induces an outer automorphism of infinite order.

We give in Section 2 a more formal description of the infinite cyclic group of outer automorphisms. In Section 3 we describe in Proposition 3.2 and Theorem 3.3 the structure of centralizers in G, and thereby show that certain conjugacy classes of G are invariant under $Aut(G)$. We believe that these results are of independent interest, as the centralizers do not have as simple a structure as might first be imagined. In Section 4, we show that there are no outer automorphisms other than those described in Section 2, and thereby prove Theorem 1.1.

We frequently refer to elements of S by their cycle type. The set of *cycle lengths* is the set of cardinals $\{n: 1 \leq n \leq \omega\}$, and a *nontrivial cycle* has length $n > 1$. If T is a set of cycle lengths, we say that g has cycle type $\prod(n^{a_n}: n \in T)$ if g has exactly a_n cycles of length n for each $n \in T$, $(0 \leq a_n \leq \omega)$, and no others. Observe that if $g, h \in S$ have cycle types $1^\omega . \prod(n^{a_n}: 1 < n \leq \omega)$ and $1^\omega . \prod(n^{b_n}: 1 < n \leq \omega)$ respectively, then $FS(\Omega)g, FS(\Omega)h$ are conjugate in G if and only if their cycle types differ finitely: that is, if

(a) $a_n = b_n$ for all but finitely many finite cycle lengths n;

(b) $a_\omega = b_\omega$;

(c) $a_n = \omega$ if and only if $b_n = \omega$, for $1 \leq n < \omega$.

It is important that here g and h have infinitely many fixed points. As a small warning, we remark that G has three conjugacy classes of involutions, corresponding to elements in S of the cycle types $1^\omega \cdot 2^\omega$ and 2^ω and $1^1 \cdot 2^\omega$.

If $g \in S$ then $\bar{g}$ will be the corresponding element $FS(\Omega)g$ of G, and similarly, given $\bar{g} \in G$, we use g to denote any preimage of it under the natural surjection from S to G. We shall use $\triangle$ to denote symmetric difference, and write $U \equiv V$ if $U \triangle V$ is finite. Also, we extend the notion of support to elements $\bar{g} \in G$. If $g \in S$ has support $\Gamma \subseteq \Omega$, then $\bar{g}$ has support

$$supp(\bar{g}) = \{\Delta \subseteq \Omega: |\Gamma \triangle \Delta| < \omega\}.$$

Thus $supp(\bar{g})$ lies in the Boolean Algebra $\mathcal{B} = \mathrm{Pow}(\Omega)/\mathcal{I}$, where $\mathcal{I}$ is the ideal of finite subsets of Ω. Observe the distinction between $supp(g)$, which is a subset of Ω, and $supp(\bar{g})$, which is an element of $\mathcal{B}$. All wreath products whose top group is a permutation group of infinite degree will be unrestricted, and similarly all direct products will be unrestricted (that is, will be the full Cartesian product). Finally, recall that a moiety Γ of a set Ω is a subset satisfying $|\Gamma| = |\Omega \setminus \Gamma|$; for a countably infinite set Ω this is an infinite coinfinite subset.

The paper by J.K. Truss in the present volume sketches an alternative proof of Theorem 1.1, due to Truss and Shelah, in which the Boolean algebra $\mathcal{B}$ is shown to be first-order interpretable in G. Their approach seems to be more powerful,

since they obtain some generalisations of Theorem 1.1 for symmetric groups of uncountable cardinality. The strongest results concerning the interpretability of $\mathcal{B}$ in G were obtained by M. Rubin in Theorem 4.2 of [8]. There, he showed the following. Let $\mu < \kappa \leq \lambda^+$ be infinite cardinals and $G_{\lambda\kappa\mu}$ denote the group of permutations (of support of size less than κ) of λ, modulo the normal subgroup of permutations of support of size less than μ, and $B_{\lambda\kappa\mu}$ be the corresponding Boolean algebra. Then $B_{\lambda\kappa\mu}$, its boolean algebra structure, and the action of $G_{\lambda\kappa\mu}$ on it, is first-order interpretable in the abstract group $G_{\lambda\kappa\mu}$, and furthermore this interpretation is uniform, that is, the formulas in the interpretation are independent of the choice of λ, κ, μ. It seems, though, that the step from such results to a description of $Aut(()G_{\lambda\kappa\mu}$ is never immediate. Our approach yields en route the group-theoretic information about centralizers.

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2 The group of near symmetries

In this section we describe an infinite cyclic group of outer automorphisms of $G = Sym(\Omega)/FS(\Omega)$, where Ω is a countably infinite set. In the rest of the paper, we will show that this group is the whole outer automorphism group.

Define the set of *near bijections* of Ω to be

$$NB(\Omega) = \{f\colon f \text{ is a bijection } \Omega_1 \to \Omega_2, \text{ where } \Omega_1, \Omega_2 \text{ are cofinite subsets of } \Omega\}.$$

Clearly, $S \subseteq NB(\Omega)$. Extending our notation for sets, we define an equivalence relation $\equiv$ on $NB(\Omega)$ by the rule

$$f_1 \equiv f_2 \text{ if there is a cofinite } \Omega' \subseteq \Omega \text{ such that } f_1 |_{\Omega'} = f_2 |_{\Omega'} .$$

It is easily seen that $\equiv$ is indeed an equivalence relation. We define $NS(\Omega)$, the set of *near symmetries* of Ω, to be the set of equivalence classes of this relation. This set naturally forms a group induced from composition of maps.

The natural projection from S to $NS(\Omega)$ is a group homomorphism with kernel $FS(\Omega)$. Hence we have an exact sequence

$$FS(\Omega) \longrightarrow S \longrightarrow NS(\Omega).$$

Thus $G \leq NS(\Omega)$. The elements of G are the equivalence classes of permutations.

For $f \in NB(\Omega)$ with Ω_1, Ω_2 as above, we define the *index* of f to be

$$\operatorname{ind} f = |\Omega - \Omega_2| - |\Omega - \Omega_1|.$$

Lemma 2.1 *The map* $ind: NB(\Omega) \to Z$ *induces a mapping* $ind: NS(\Omega) \to Z$ *which is a homomorphism onto* $(Z, +)$ *with kernel* G.

Proof. First note that if Ω' is a cofinite subset of Ω then for any $f \in NB(\Omega)$ we have $\operatorname{ind} f = \operatorname{ind}(f\mid_{\Omega'})$. Take $f, g \in NB(\Omega)$ with $f \equiv g$. We want to show that $\operatorname{ind} f = \operatorname{ind} g$. There is a cofinite subset Ω' of Ω such that $f\mid_{\Omega'} = g\mid_{\Omega'}$. But then clearly

$$\operatorname{ind} f = \operatorname{ind}(f\mid_{\Omega'}) = \operatorname{ind}(g\mid_{\Omega'}) = \operatorname{ind} g.$$

Thus ind is a well-defined function on $NS(\Omega)$.

We now show that $\operatorname{ind}(fg) = \operatorname{ind} f + \operatorname{ind} g$. Restricting the domains of f and g if necessary, we may assume that $\operatorname{dom} g$ is equal to $\operatorname{ran} f$, the range of f, that is, we have $f: \Omega_1 \to \Omega_2$ and $g: \Omega_2 \to \Omega_3$. Then

$$\begin{aligned} \operatorname{ind}(fg) &= |\Omega - \Omega_3| - |\Omega - \Omega_1| \\ &= |\Omega - \Omega_3| - |\Omega - \Omega_2| + |\Omega - \Omega_2| - |\Omega - \Omega_1| \\ &= \operatorname{ind} g + \operatorname{ind} f. \end{aligned}$$

Now the kernel of this homomorphism consists of all bijections $f: \Omega_1 \to \Omega_2$ such that $|\Omega - \Omega_1| = |\Omega - \Omega_2|$. Any such map is $\equiv$-equivalent to a permutation. Thus $\ker(\operatorname{ind}) = G$.

From this lemma it follows that G is a normal subgroup of $NS(\Omega)$ and we have an exact sequence

$$1 \longrightarrow FS(\Omega) \longrightarrow S \longrightarrow NS(\Omega) \longrightarrow Z \longrightarrow 1.$$

Lemma 2.2 $C_{NS(\Omega)}(G)$ *is trivial.*

Proof. Let h be a near-bijection representing an element of $C_{NS(\Omega)}(G)$. Then for every permutation $g \in S$, we have $h^{-1}gh \equiv g$. Now, for any $g \in S$, the sets $supp(h^{-1}gh)$ and $(supp(g))h$ have finite symmetric difference, where $supp(h^{-1}gh)$ is the the set of $\alpha \in \Omega$ such that $\alpha h^{-1}gh$ is defined and not equal to α. It follows that $supp(g) \equiv supp(g)h$ for any $g \in S$. Since any set $\Phi \subset \Omega$ is the support of some permutation, we have $\Phi h \equiv \Phi$ for all $\Phi \subseteq \Omega$. But this is possible only if h has finite support, that is, $h \equiv id$.

Proposition 2.3 $NS(\Omega)$ *induces an infinite cyclic group of outer automorphisms of* G.

Proof. Since $GNS(\Omega)$, we see that $NS(\Omega)$ induces a group of automorphisms of G by conjugation. Since $C_{NS(\Omega)}(G)$ is

trivial, the outer automorphism group induced by $NS(\Omega)$ is $NS(\Omega)/G$, which is isomorphic to Z.

Remark. Let $\Omega = \{\alpha_i : i < \omega\}$ be any enumeration of Ω. A typical generator of $NS(\Omega)/G$ is the equivalence class of the near-bijection $\sigma : \Omega \to \Omega$ defined by $\alpha_i \mapsto \alpha_{i+1}$ (for $i < \omega$).

3 Structure of centralizers in G

In this section we give partial information on the structure of centralizers in G. From this, it will follow that $Aut(G)$ induces an automorphism of $\mathcal{B}$, and hence that $Aut(G)$ fixes certain conjugacy classes of G.

For the first four lemmas we denote by $\bar{g}$ a fixed element of G, and choose a fixed $g \in S$ such that $\bar{g} = gFS(\Omega)$. We suppose that g has cycle type $\prod(i^{n_i} : i \in T)$, where $T \subseteq N \cup \{\omega\}$. We define an equivalence relation $\mathcal{F}$ on Ω whose equivalence classes are listed below:

- The union of all the infinite cycles of g is an $\mathcal{F}$-class F_ω.
- If $1 \leq i < \omega$ and g has infinitely many i-cycles, then the union of the i-cycles of g is an $\mathcal{F}$-class F_i.
- The remaining points of Ω form a single $\mathcal{F}$-class F_0, called the *dregs*.

We define the *strong centralizer* of $\bar{g}$ in G, to be

$$SC_G(\bar{g}) := \{\bar{h} \in C_G(\bar{g}) : h \text{ fixes each } \mathcal{F}\text{-class for some representative } h \text{ of } \bar{h}\}.$$

Note that, even though $\mathcal{F}$ depends on g, this definition does not depend on the choice of g. For, let g' be another preimage of $\bar{g}$, and let $\mathcal{F}'$ be the corresponding equivalence relation. Take $h \in S$ such that $\bar{h} \in SC_G(\bar{g})$. Now, $\mathcal{F}'$ and $\mathcal{F}$ differ only on a finite set Δ of points. There is a finitary permutation k fixing each $\mathcal{F}$-class such that $hk \in S_{(\Delta)}$. Then hk fixes each $\mathcal{F}'$-class.

Lemma 3.1 *Let $g, \bar{g}$ be as above. Then the group $C_G(\bar{g})/SC_G(\bar{g})$ is free abelian (where the trivial group is regarded as free abelian).*

Proof. Let $\{F_i : i \in I\}$ be the set of $\mathcal{F}$-classes, and let $K := \bigoplus(Zf_i : i \in I)$ be free abelian with $\{f_i : i \in I\}$ as basis. We define a homomorphism $\chi : C_G(\bar{g}) \to K$ as follows: if $\bar{k} \in C_G(\bar{g})$ then for each $i \in I$, we define

$$\chi_i(\bar{k}) = |F_i k \setminus F_i| - |F_i \setminus F_i k|.$$

Then $\chi_i(\bar{k})$ is well-defined, that is, it does not depend on the choice of the preimages g and k. Also put

$$\chi(\bar{k}) := \sum_{i \in I} \chi_i(\bar{k}) f_i.$$

Now g and $k^{-1}gk$ agree on all but finitely many points, so k fixes all but finitely many $\mathcal{F}$-classes and $\chi_i(\bar{k}) = 0$ for all but finitely many $i \in I$, so $\chi(\bar{k})$ does indeed lie in the direct sum K. It can be checked that χ is a homomorphism, its image is a subgroup of a free abelian group so is free abelian, and its kernel is $SC_G(\bar{g})$.

Problem 3.1 *Find an expression for the rank of $C_G(\bar{g})/SC_G(\bar{g})$, in terms of the cycle type of g.*

We will need a considerable amount of information about all centralizers in G. To find $C_G(\bar{g})$, we must describe all elements $h \in S$ such that $[g,h] \in FS(\Omega)$. We list five types of such elements, and then show that the images in G of elements of such types generate $C_G(\bar{g})$.

1. **Obvious Centralizer Type** These are elements of $C_S(g)$. Conjugation by these elements fixes g. Note that $C_S(g) \cong \prod_{i \in T}(C_i \operatorname{Wr} S_{n_i})$, where C_i denotes the cyclic group of order i.

2. **Finite/Finite Type** These elements convert one finite set of finite cycles of g into another. Let A, B be the unions of supports of finitely many finite cycles of g, where

   - $|A| = |B| > 0$;
   - No cycle in B has the same length as one in A;
   - A or B contains the support of a g-cycle of length i only if $n_i = \omega$, that is, g has infinitely many cycles of length i.

   Let $\chi\colon A \to B$ be a bijection and define a permutation h as follows:

   - h agrees with χ on A;
   - If A contains an i-cycle, then h maps $F_i \setminus A$ to F_i, respecting the cycle structure;
   - If B contains an i-cycle, then h maps F_i to $F_i \setminus B$, respecting the cycle structure;
   - h fixes all elements of all other $\mathcal{F}$-classes.

   Then g^h agrees with g except on the finite set B, so $[g,h] \in FS(\Omega)$.

3. **Finite/Infinite Transfers** Conjugation by these elements moves a finite cycle of g into an infinite cycle. Let $(\ldots a_{-2}a_{-1}a_0a_1a_2\ldots)$ be an infinite cycle of g. Let $(b_1b_2\ldots b_k)$ be a finite cycle of g from the infinite $\mathcal{F}$-class F_k. Define a permutation h as follows:

   - h fixes a_i for $i \leq 0$;
   - $a_ih = b_i$ for $1 \leq i \leq k$;

- $a_i h = a_{i-k}$ for $i > k$;
- h maps F_k to $F_k \setminus \{b_1, b_2, \ldots, b_k\}$, respecting the cycle structure;
- h fixes all elements of all other cycles.

Then g^h agrees with g except on $\{a_0, b_k\}$, and so $[g, h] \in FS(\Omega)$.

We refer to the inverses of such elements as infinite/finite transfers.

4. Infinite/Infinite Transfer These are elements which move a single point from one infinite cycle of g into another. Let

$$(\ldots a_{-2}a_{-1}a_0a_1a_2\ldots) \text{ and } (\ldots b_{-2}b_{-1}b_0b_1b_2\ldots)$$

be two infinite cycles of g, and let h be the permutation of Ω whose only non-trivial cycle is

$$(\ldots b_{-2}b_{-1}b_0a_0a_1a_2\ldots).$$

Then the conjugate g^h differs from g only in that a_0 lies in the b-cycle of g^h. Hence g^h agrees with g except on the set $\{a_{-1}, a_0, b_0\}$, so $[g, h] \in FS(\Omega)$.

5. Infinite/Infinite Recombination These are elements which cut two infinite cycles of g in half, then reattach the left side of one to the right side of the other. Let g have two infinite cycles as above, and let

$$h := (a_1b_1)(a_2b_2)\cdots.$$

Then g^h agrees with g except on $\{a_0, b_0\}$, so $[g, h] \in FS(\Omega)$.

Remark. Elements of types 1,4 and 5 lie in $SC_G(\bar{g})$, but elements of types 2 and 3 do not.

Proposition 3.2 *(a) The group $SC_G(\bar{g})$ is generated by elements of G represented by elements of S of types 1,4 and 5.*

(b) The group $C_G(\bar{g})$ is generated by elements of G represented by the above types.

Proof. (a) Let $k \in S$ be such that $[k, g] \in FS(\Omega)$, and assume that k fixes setwise each $\mathcal{F}$-class. We say that two infinite g-cycles

$$(\ldots a_{-2}a_{-1}a_0a_1a_2\ldots)$$

and

$$(\ldots b_{-2}b_{-1}b_0b_1b_2\ldots)$$

are *initially equal* if there are integers M and m such that $a_i = b_{i-m}$ for all $i < M$. *Finally equal* is defined similarly.

For all but finitely many of the infinite g-cycles, its conjugate under k is also a g-cycle. Let $(\ldots a_{-2}a_{-1}a_0a_1a_2\ldots)$ be any infinite g-cycle. Then

$$(\ldots a_{-2}k, a_{-1}k, a_0k, a_1k, a_2k, \ldots)$$

is initially equal to some g-cycle, and finally equal to some, perhaps different, g-cycle. Then we can use infinite/infinite recombinations to reduce to the case where (a_ik) is initially and finally equal to a particular infinite g-cycle. We then use a series of infinite/infinite transfers, together with finitary permutations to reduce to the case where k preserves the structure of all the infinite cycles. Using further finitary permutations, we may suppose that k preserves all finite g-cycles, and it follows that $\bar{k} \in C_G(\bar{g})$.

(b) Let $k \in S$ with $[k, g] \in FS(\Omega)$. Observe that $Fk \triangle F$ is finite for every $\mathcal{F}$-class F, and that all but finitely many of the $\mathcal{F}$-classes are fixed setwise by k. By adjusting k by a finitary permutation, we can reduce to the case where, for each $\mathcal{F}$-class F, either $F \subseteq Fk$ or $Fk \subseteq F$. Note that since the dregs class F_0 contains only finite cycles and has just finitely many cycles of any length, we must have $F_0k = F_0$.

Now consider the classes F_i where $1 \leq i < \omega$. We wish to adjust k to ensure that $F_ik = F_i$ for each such i, for then also $F_\omega k = F_\omega$. So fix i with $1 \leq i < \omega$, and suppose that $F_i \subset F_ik$. (The case when $F_i \supset F_ik$ is similar.) If g has an infinite cycle then we can use a series of infinite/finite transfers to move cycles of $F_ik \setminus F_i$ into F_ω, then a combination of finitary permutations on F_ω and finite/infinite transfers to move appropriate elements of F_ω back into F_i. If g has no infinite cycle, we multiply k by a single element of finite/finite type to reduce to the case where k fixes all $\mathcal{F}$-classes setwise. We can then use a finitary permutation to ensure that k respects the cycle structure of finite cycles. After these reductions we have $k \in SC_G(\bar{g})$, so can apply part (a).

We now give a partial structure theorem for centralizers in G. We first define some normal subgroups of $C_G(\bar{g})$. Let $g \in S$ be as above. Let

$$OC(\bar{g}) := C_S(g) \cdot FS(\Omega)/FS(\Omega) = \{\bar{h} : h \in S, [g, h] = 1\}$$

the group of images of elements of obvious centralizer type. Also, define

$$N_1(\bar{g}) := \{\bar{h} \in OC(\bar{g}) : h \text{ fixes setwise the support of each infinite } g\text{-cycle}\},$$

$$N_2(\bar{g}) := \{\bar{h} \in SC_G(\bar{g}) : |Dh \triangle D| < \aleph_0, \text{ if } D \text{ is the support of an infinite } g\text{-cycle}\},$$

and

$$N_3(\bar{g}) := \langle N_2(\bar{g}), OC(\bar{g})\rangle.$$

These definitions are independent of the choice of the representative g.

Theorem 3.3 *(a) The groups*

$$N_1(\bar{g}), OC(\bar{g}), N_2(\bar{g}), N_3(\bar{g}) \quad \text{and } SC_G(\bar{g})$$

are each normal subgroups of $C_G(\bar{g})$, with

$$N_3(\bar{g}) \leq SC_G(\bar{g}) \leq C_G(\bar{g}).$$

(b) $N_2(\bar{g}) \cap OC(\bar{g}) = N_1(\bar{g})$. Hence we have the isomorphisms $N_3(\bar{g})/OC(\bar{g}) \cong N_2(\bar{g})/N_1(\bar{g})$ and $N_3(\bar{g})/N_2(\bar{g}) \cong OC(\bar{g})/N_1(\bar{g})$.

(c) (i) The groups $C_G(\bar{g})/SC_G(\bar{g})$ and $N_2(\bar{g})/N_1(\bar{g})$ are free abelian, and the latter has rank n if g has $n+1$ infinite cycles (n finite) and $\aleph_0$ if g has infinitely many infinite cycles.

(ii) The quotient $OC(\bar{g})/N_1(\bar{g})$ is isomorphic to the symmetric group on the set of infinite cycles of $\bar{g}$.

(iii) The quotient $SC_G(\bar{g})/N_3(\bar{g})$ is isomorphic to the finitary symmetric group on the set of infinite g-cycles.

(d) If g has more than one infinite cycle then the five normal subgroups listed in (a) are distinct, and if g has at most one infinite cycle then they are all equal.

Proof. All parts of this are easy, so we omit details. To show that $N_2(\bar{g})/N_1(\bar{g})$ is free abelian, we argue much as in the proof of Lemma 3.1. This quotient is generated by infinite/infinite transfers, modulo $N_1(\bar{g})$. By Proposition 3.2(a), the finitary symmetric group $SC_G(\bar{g})/N_3(\bar{g})$ is generated by infinite/infinite recombinations (modulo $N_3(\bar{g})$).

We now define, for any $h \in S$, the group

$$G(\bar{h}) := \{\bar{k} : supp(k) \subseteq fix(h)\}.$$

Observe that this definition is independent of the choice of the element $h \in S$ such that $\bar{h} = h \cdot FS(\Omega)$. Also, clearly $G(\bar{g})C_G(\bar{g})$.

Lemma 3.4 *(a) Suppose that $g \in S$ has infinitely many fixed points. Then $G(\bar{g})$ is isomorphic to G, and is the only normal subgroup of $C_G(\bar{g})$ which is isomorphic to G.*

(b) Suppose that $g \in S$ has just finitely many fixed points. Then there is no normal subgroup of $C_G(\bar{g})$ isomorphic to G.

(c) Let $g, h \in S$ and $\varphi \in Aut(G)$, and suppose $\bar{g}\varphi = \bar{h}$. Then g has infinitely fixed points if and only if h has infinitely many fixed points, and if they do, then $G(\bar{g})\varphi = G(\bar{h})$.

Proof. (a) Let $\Gamma := fix(g)$. Clearly

$$G(\bar{g}) = Sym(\Gamma) \cdot FS(\Omega)/FS(\Omega) \cong Sym(\Gamma)/FS(\Gamma) \cong G.$$

Now let $\bar{N}$ be any normal subgroup of $C_G(\bar{g})$ isomorphic to G. We shall show that $\bar{N} = G(\bar{g})$. First observe that $G(\bar{g})$ is simple, so for any normal subgroup M of $C_G(\bar{g})$, either $G(\bar{g}) \leq M$ or $G(\bar{g}) \cap M = 1$. In particular, by Lemma 3.1, as $\bar{N}$ is non-abelian it must lie in $SC_G(\bar{g})$. Also, by Theorem 3.3, as G is not embeddable in a free abelian group or a finite or finitary symmetric group, we must have $\bar{N} \leq OC(\bar{g})$.

Define $N := \{h \in S : \bar{h} \in \bar{N}\}$. Observe that

$$C_S(g) = \prod(C_i \operatorname{Wr} S_{n_i} : i \in T),$$

and that $1 \in T$ and n_1 is infinite, and that $OC(\bar{g}) = C_S(g) \cdot FS(\Omega)/FS(\Omega)$. Put

$$K := C_1 \operatorname{Wr} S_{n_1} \times \prod(C_i^{n_i} : i \in T, i > 1),$$

and

$$L := \{k \in S : supp(k) \subseteq fix(g)\} \cong S_{n_1} = S.$$

Then $L \leq K$. Put $\bar{K} := K \cdot FS(\Omega)/FS(\Omega)$ and $\bar{L} := L \cdot FS(\Omega)/FS(\Omega)$. Then $\bar{L}$ and $\bar{K}$ are normal subgroups of $OC(\bar{g})$ and $\bar{K}/\bar{L}$ is abelian. The proof will be complete if we can show that $\bar{N} \leq \bar{L}$ (since $\bar{L}$ is simple), so we shall suppose that this is false. It follows that $\bar{N} \cap \bar{K} = 1$; for otherwise $\bar{N} \leq \bar{K}$ which is impossible as $\bar{K}/\bar{L}$ is abelian and $\bar{N}$ is non-abelian and simple.

Now pick $h \in N \setminus L \cdot FS(\Omega)$. Then $h \notin K \cdot FS(\Omega)$, so, up to a finitary permutation, h induces a permutation of infinite support on the set of non-trivial cycles of g. It is now easy to choose $k \in K$ so that $[h,k] \notin L \cdot FS(\Omega)$. Now $[h,k] \in N \cap (K \cdot FS(\Omega) \setminus L \cdot FS(\Omega))$, which contradicts the last paragraph.

Part (b) follows from the proof of uniqueness in (a), and (c) follows from (a) and (b).

We will adopt the notation $G(\bar{g})$, $G(\bar{h})$ later in the paper. Also, from now on φ will denote an element of $Aut(G)$.

Lemma 3.5 *Let $\varphi \in Aut(G)$, let $\bar{g}, \bar{h}, \bar{k} \in G$, and assume that g, h, k each have infinitely many fixed points. Then*

(a) $supp(\bar{g}) = supp(\bar{h})$ if and only if $supp(\bar{g}\varphi) = supp(\bar{h}\varphi)$,

(b) $supp(\bar{g}) < supp(\bar{h})$ if and only if $supp(\bar{g}\varphi) < supp(\bar{h}\varphi)$,

(c) $supp(\bar{k}) = supp(\bar{g}) \wedge supp(\bar{h})$ if and only if $supp(\bar{k}\varphi) = supp(\bar{g}\varphi) \wedge supp(\bar{h}\varphi)$.

Proof. To see (a), observe that $supp(\bar{g}) \leq supp(\bar{h})$ if and only if $G(\bar{h}) \leq G(\bar{g})$, and by Lemma 3.4(c), this holds if and only if $G(\bar{h}\varphi) \leq G(\bar{g}\varphi)$. But, again by Lemma 3.4(c), we have $G(\bar{g}\varphi) = G(\bar{g})\varphi$ and $G(\bar{h}\varphi) = G(\bar{h})\varphi$.

Part (b) follows directly from (a). Part (c) then follows since

$$supp(\bar{k}) = supp(\bar{g}) \wedge supp(\bar{h})$$

if and only if $supp(\bar{k}) \leq supp(\bar{g}) \wedge supp(\bar{h})$ and $supp(\bar{\ell}) \leq supp(\bar{k})$ for all $\bar{\ell} \in G$ such that $supp(\bar{\ell}) \leq supp(\bar{g}) \wedge supp(\bar{h})$.

It follows from this lemma that $Aut(G)$ embeds in $Aut(\mathcal{B})$ (via the action of each automorphism on the set of supports of elements with infinitely many fixed points). Observe that if $supp(\bar{g}) \wedge supp(\bar{h}) = 0$ then there are representatives g of $\bar{g}$ and h of $\bar{h}$ with disjoint supports.

Next, we define the conjugacy classes in which we will be especially interested. For any subset $T \subseteq N \setminus \{0\}$, let

$$C(T)_{inf} := \{\bar{g} : g \in S, g \text{ has cycle type } 1^{\omega} \cdot \prod(n^{\omega} : n \in T)\}.$$

Also, define

$$C(T) := \{\bar{g} : g \in S, g \text{ has cycle type } 1^{\omega} \cdot \prod(n^{1} : n \in T)\}.$$

Both $C(T)_{inf}$ and $C(T)$ are conjugacy classes of G. Our goal is to show that they are preserved by any automorphism of G.

For $T = \{2\}$ we shall write $C(2)_{inf}$ for $C(\{2\})_{inf}$.

Lemma 3.6 *Let $\varphi \in Aut(G)$, and let $T \subseteq N \setminus \{0\}$, and $\bar{g} \in C(T)_{inf}$. Then $\bar{g}\varphi \in C(T)_{inf}$.*

Proof. First, observe that for any i with $1 < i < \omega$ and $h \in S$ with infinitely many fixed points, h has infinitely many i-cycles if and only if

$$supp(\bar{h}^{i}) < \bigwedge(supp(\bar{h}^{j}) : 1 \leq j < i, j|i).$$

By Lemma 3.5, it follows that if $k \in S$ with $\bar{k} = \bar{g}\varphi$, then for each i with $1 < i < \omega$, the element k has infinitely many i-cycles if and only if $i \in T$.

Next, again assuming $h \in S$ has infinitely many fixed points, note that h has a single infinite cycle and just finitely many non-trivial finite cycles if and only if $supp(\bar{h}) \neq 0$ and we cannot write $\bar{h} = \bar{h}_1\bar{h}_2$ with the supports of $\bar{h}_1$ and $\bar{h}_2$ disjoint and not equal to 0 or $1 \in \mathcal{B}$. Hence, given such an element h, there is $k \in S$ with a single infinite cycle and no other non-trivial cycles such that $\bar{k} = \bar{h}\varphi$. Also, if $h \in S$ is not of this form, then h has at least one infinite cycle if and only if there are $h_1, h_2 \in S$ such that $\bar{h} = \bar{h}_1\bar{h}_2$, the $\bar{h}_i$ have disjoint supports not equal to 0 or 1 in

$\mathcal{B}$, and h_1 has a single infinite cycle and no other non-trivial cycles. It follows from Lemma 3.5 that if $k \in S$ with $\bar{k} = \bar{g}\varphi$, then k has no infinite cycles. In fact, this argument shows that if $k_1, k_2 \in S$ and $\bar{k}_1\varphi = \bar{k}_2$ and k_1 has infinitely many fixed points, then k_1 and k_2 have the same number of infinite cycles.

Finally, suppose that $h \in S$ has infinitely many fixed points and no infinite cycles. Then the dregs class F_0 of h is finite if and only if, whenever there are $h_1, h_2 \in S$ such that $\bar{h} = \bar{h}_1\bar{h}_2$ with $\bar{h}_1, \bar{h}_2$ having disjoint non-trivial supports and h_1 having finitely many i-cycles for all i $(1 < i < \omega)$, the elements $\bar{h}_2$ and $\bar{h}$ are conjugate in G. Again, this condition is φ-invariant, so the lemma follows from this and the previous two paragraphs.

Lemma 3.7 *Let $\varphi \in Aut(G)$, and let T be an infinite set of nontrivial finite cycle lengths. Then $C(T)$ is φ-invariant.*

Proof. We will show that $C(T)$ is the only conjugacy class of $G \setminus \{1\}$ whose elements $\bar{g}$ satisfy all the following properties.

(a) there is $\bar{k} \in C(T)_{inf}$ with $supp\,\bar{k} \wedge supp\,\bar{g} = 0$ such that $\bar{k}\bar{g} \in C(T)_{inf}$;

(b) for any moiety R of T, there is no $\bar{k} \in C(R)_{inf}$ with $supp\,\bar{k} \wedge supp\,\bar{g} = 0$ and $\bar{k}\bar{g} \in C(R)_{inf}$;

(c) if $\bar{g} = \bar{h}\bar{k}$ where $\bar{h}, \bar{k} \in G\setminus\{1\}$, $supp\,\bar{h} \wedge supp\,\bar{k} = 0$ and $supp\,\bar{h} \vee supp\,\bar{k} = supp\,\bar{g}$, then one of (a) or (b) fails when $\bar{g}$ is replaced by $\bar{h}$.

By Lemma 3.5, the conditions on supports are preserved by φ, and by Lemma 3.6, $C(T)_{inf}$ is φ-invariant, so the conditions (a)-(c) are preserved by φ. Hence, given the above, $C(T)$ is φ-invariant, as required.

Clearly if $\bar{g} \in C(T)$ then it satisfies (a)–(c). Conversely, assume that $\bar{g} \in G \setminus \{1\}$ satisfies (a)–(c), and choose a representative g. Property (a) says that g has infinitely many fixed points, that all but finitely many of the cycles of g have lengths lying in T, and that g has no infinite cycles. Property (b) says that all but finitely many elements of T arise as lengths of cycles of g. Thus (a) and (b) imply that the set of cycle lengths of g is almost exactly T. If the cycle type of g differs more than finitely from $1^\omega \cdot \prod(n^1 : n \in T)$, then (c) fails when we let h be the product of exactly one n-cycle of g for each $n \in T$, and let k be the product of the remaining g-cycles.

4 Proof of the Theorem

In this section, we prove Theorem 1.1, that is, that every outer automorphism of G is one of the outer automorphisms induced by a near symmetry as described in Section 2.

Let φ be any automorphism of G. We must find $f_\varphi \in NB(\Omega)$ such that the action of φ on G is that of conjugation by f_φ. In the introduction, we defined the *support* of an element of G, and in

Lemma 3.5 we showed that φ respects supports of elements of G and the 'containment' relation induced on supports by the subset relation on Ω. We also showed that certain conjugacy classes of G in which we are interested are respected by φ, that is, are not fused by φ. We now pick a particular conjugacy class $C(T)$ of G, a subset $\mathcal{D}$ of unordered pairs from $C(T)$, and an equivalence relation $\mathcal{E}$ on $\mathcal{D}$, and show that $C(T)$, $\mathcal{D}$ and $\mathcal{E}$ are invariant under φ. Elements of $\mathcal{D}/\mathcal{E}$ will be naturally identified with ordered moieties of Ω, and this enables us to define $f_\varphi \in NB(\Omega)$ corresponding to φ. We show that f_φ and φ agree in their action on one non-trivial conjugacy class of G. Then since G is a simple group, this conjugacy class generates G and so the actions of f_φ and φ agree on the whole of G, as required.

Fix a set $T = \{n_i : i < \omega\}$ of finite nontrivial cycle lengths such that for all $i < \omega$ we have $n_i > 2\sum_{j<i} n_j$. We note without proof the following easy lemma.

Lemma 4.1 *Let T be as above. Then*

(a) if $a, b, c \in T$ then $c \neq 2a + b - 2$;

(b) if $a, b, c \in T$ then $2a \neq 2b + c - 1$;

(c) if $a, b \in T$ then $a \neq 2b - 1$;

(d) if $a, b, c, d \in T$ with $a + b = c + d$ then $\{a, b\} = \{c, d\}$;

(e) if $a, b, c, d \in T$ with $2a + b = 2c + d$ then $a = c$ and $b = d$.

We say that $\{g, h\}$ is a *matched $C(T)$-pair* if:

(a) g, h have cycle types $1^\omega \cdot \prod(i^1 : i \in T)$;

(b) $supp(g) \cup supp(h)$ is a moiety of Ω;

(c) for each $n \in T$, the n-cycle of g meets $supp(h)$ in a singleton lying in the n-cycle of h.

Next, let $\mathcal{D}$ be the set of all unordered pairs $\{\bar{g}, \bar{h}\}$ for which there are corresponding elements $g, h \in S$ which form a matched $C(T)$-pair. We say that $\{g, h\}$ is a *matched cover* of $\{\bar{g}, \bar{h}\}$.

If $\{\bar{g}, \bar{h}\} \in \mathcal{D}$, we will write $supp(g, h)$ for $supp(g) \cup supp(h)$. If $\{g, h\}$ is a matched cover of $\{\bar{g}, \bar{h}\}$, let the *spine* $\sigma(g, h)$ of $\{g, h\}$ be the ω-sequence from Ω whose i^{th} entry is the point in which the n_i-cycles of g and h intersect. Then the *ribcage* of $\{g, h\}$ is the set $supp(g, h) \setminus (supp(g) \cap supp(h))$, that is the elements of the supports of g and h which are not in the spine.

Lemma 4.2 *The set $\mathcal{D}$ defined above is φ-invariant for any $\varphi \in Aut(G)$.*

Proof. We show that if $\bar{g}, \bar{h} \in G$ then $\{\bar{g}, \bar{h}\} \in \mathcal{D}$ if and only if the following conditions all hold:

(a) $\bar{g}, \bar{h} \in C(T)$;

(b) $supp(\bar{g}) \vee supp(\bar{h}) \neq 1$;

(c) for all $\bar{k} \in C(2)_{inf}$ such that $supp(\bar{k}) = supp(\bar{g}) \wedge supp(\bar{h})$ we have $\bar{g}\bar{k}, \bar{h}\bar{k} \in C(Q)$ for some subset Q of $\{n_i + n_j : n_i, n_j \in T\}$;

(d) $\bar{g}\bar{h} \in C(R)$, where $R = \{2n - 1 : n \in T\}$.

Here, part (b) says that $supp(g) \cup supp(h)$ is a moiety of Ω. Part (c) ensures that for some $g, h \in S$ representing $\bar{g}, \bar{h}$, every non-trivial cycle of g intersects $supp(h)$ in at most a singleton, and similarly with g, h interchanged. For, if the n_i-cycle of g intersects $supp(k)$ in a singleton, which k pairs with a singleton from the n_j-cycle of g, then gk contains an (n_i+n_j)-cycle. On the other hand, if, say, there are infinitely many cycles of g which each intersect $supp(h)$ in more than one point, then k can be chosen so that gk contains an infinite cycle. We leave the details to the reader.

If the m-cycle of g meets the k-cycle of h in a singleton, then they form an $(m+k-1)$-cycle in the product gh. By Lemma 4.1(d), if $m, k, n \in T$ then $m+k-1 = 2n-1$ if and only if $m = k = n$. Thus part (d) ensures that these cycles are matched up correctly, that is, the n-cycle of g meets the n-cycle of h.

By Lemmas 3.5, 3.6 and 3.7, the conditions in (a) to (d) are preserved by any automorphism. So $\mathcal{D}$ is φ-invariant.

Next, we define an equivalence relation $\mathcal{E}$ on $\mathcal{D}$. If $\{\bar{g}_1, \bar{h}_1\}, \{\bar{g}_2, \bar{h}_2\} \in \mathcal{D}$, then we say that they are equivalent under $\mathcal{E}$ if they have matched covers $\{g_1, h_1\}$ and $\{g_2, h_2\}$ respectively, such that $\sigma(g_1, h_1) = \sigma(g_2, h_2)$.

Lemma 4.3 *Let $\varphi \in Aut(G)$. Then the equivalence relation $\mathcal{E}$ is φ-invariant.*

Proof. We say that $g_1, g_2, g_3, g_4 \in S$ form a *matched quadruple* if

(a) $\{\bar{g}_i, \bar{g}_j\} \in \mathcal{D}$ and has $\{g_i, g_j\}$ as a matched cover for all $1 \leq i < j \leq 4$;

(b) all sequences $\sigma(g_i, g_j)$, $(1 \leq i < j \leq 4)$ are equal;

(c) $\bigcup\{supp(g_i) : 1 \leq i \leq 4\}$ is a moiety of Ω;

Claim: The set of matched quadruples in G is φ-invariant.

Proof of Claim: Note that $\{\bar{g}_1, \bar{g}_2, \bar{g}_3, \bar{g}_4\}$ is a matched quadruple if and only if $\{\bar{g}_i, \bar{g}_j\} \in \mathcal{D}$ for all $1 \leq i < j \leq 4$ and $supp(\bar{g}_i) \wedge supp(\bar{g}_j) = \bigwedge\{supp(\bar{g}_k) : 1 \leq k \leq 4\}$ and $\bigvee\{supp(\bar{g}_k) : 1 \leq k \leq 4\} \neq 1$. These conditions are clearly preserved by φ, which yields the claim.

Now take $\{\bar{g}_1, \bar{h}_1\}, \{\bar{g}_2, \bar{h}_2\} \in \mathcal{D}$ such that $\{\bar{g}_1, \bar{h}_1\}\mathcal{E}\{\bar{g}_2, \bar{h}_2\}$. Let $\{g_1, h_1\}$ and $\{g_2, h_2\}$ be matched covers. We can find $\{\bar{g}_3, \bar{h}_3\} \in \mathcal{D}$ and $\{\bar{g}_4, \bar{h}_4\} \in \mathcal{D}$ with matched covers $\{g_3, h_3\}$ and $\{g_4, h_4\}$ such that

$supp(g_1, h_1) \cup supp(g_3, h_3)$, $supp(g_3, h_3) \cup supp(g_4, h_4)$ and $supp(g_2, h_2) \cup supp(g_4, h_4)$

are moieties, the spines $\sigma(g_i, h_i)$ all agree and the ribcages of $supp(g_1, h_1)$ and $supp(g_3, h_3)$ are disjoint, as are the ribcages of $supp(g_3, h_3)$ and $supp(g_4, h_4)$, and the ribcages of $supp(g_4, h_4)$ and $supp(g_2, h_2)$. Then each of $\{\bar{g}_1, \bar{h}_1, \bar{g}_3, \bar{h}_3\}$, $\{\bar{g}_3, \bar{h}_3, \bar{g}_4, \bar{h}_4\}$ and $\{\bar{g}_4, \bar{h}_4, \bar{g}_2, \bar{h}_2\}$ is a matched quadruple of G. Hence, by the claim, their images under φ are all matched quadruples of G. Thus

$$\{\bar{g}_1\varphi, \bar{h}_1\varphi\}\mathcal{E}\{\bar{g}_3\varphi, \bar{h}_3\varphi\}\mathcal{E}\{\bar{g}_4\varphi, \bar{h}_4\varphi\}\mathcal{E}\{\bar{g}_2\varphi, \bar{h}_2\varphi\},$$

as required.

It follows from the last lemma that φ has a natural induced action on the set of almost-equality classes of ω-sequences which enumerate moieties of Ω. We are now in a position to construct a near-bijection f_φ corresponding to φ. Let $\Omega = \Omega_0 \cup \Omega_1$ be a partition of Ω into moieties, and let $\{\bar{g}, \bar{h}\}, \{\bar{k}, \bar{l}\} \in \mathcal{D}$ have matched covers $\{g, h\}$, $\{k, l\}$ respectively such that $\sigma(g, h)$ is an enumeration of Ω_0 and $\sigma(k, l)$ is an enumeration of Ω_1. Since φ preserves the Boolean algebra $\mathcal{B}$, we see that if $\{g', h'\}$ and $\{k', l'\}$ are matched covers of $\{\bar{g}\varphi, \bar{h}\varphi\}$ and $\{\bar{k}\varphi, \bar{l}\varphi\}$ respectively, then $\sigma(g', h')$ and $\sigma(k', l')$, regarded as sets, have finite intersection and their union is cofinite in Ω. There is a near-bijection f_φ of Ω which induces a map taking $\sigma(g, h)$ to $\sigma(g', h')$ and taking $\sigma(k, l)$ to $\sigma(k', l')$ (except on a finite set).

The next lemma shows essentially that this action preserves subsequences. (Note that these subsequences are not necessarily increasing subsequences; that is, they may involve permutations of the elements of the sequence, so may not induce the same ordering.)

Lemma 4.4 *Let $\varphi \in Aut(G)$. Suppose that $\{\bar{g}, \bar{h}\}, \{\bar{k}, \bar{l}\} \in \mathcal{D}$ have matched covers $\{g, h\}$, $\{k, l\}$ respectively, and that $\sigma(g, h)$ meets $supp(k) \cap supp(l)$ in a subsequence of $\sigma(k, l)$, that is, there is $A \subseteq \omega$ and an injection $\chi: A \to \omega$ such that if $i \in A$ then the i^{th} element of $\sigma(g, h)$ is the $\chi(i)^{th}$ element of $\sigma(k, l)$, and if $i \notin A$ then the i^{th} element of $\sigma(g, h)$ does not lie in $\sigma(k, l)$. Then there are matched covers $\{g', h'\}$ and $\{k', l'\}$ of $\{\bar{g}\varphi, \bar{h}\varphi\}$ and $\{\bar{k}\varphi, \bar{l}\varphi\}$ respectively such that if $i \in A$ then the i^{th} element of $\sigma(g', h')$ is the $\chi(i)^{th}$ element of $\sigma(k', l')$, and if $i \notin A$ then the i^{th} element of $\sigma(g', h')$ does not occur in $\sigma(k', l')$.*

Proof. By replacing $(\bar{k}, \bar{l})$ by a pair $\mathcal{E}$-equivalent to it, and using Lemma 4.3, we may suppose that

$$supp(k) \cap supp(l) \cap supp(g) \cap supp(h) = supp(g, h) \cap supp(k, l).$$

It follows that we may choose matched covers $\{g', h'\}$ of $\{\bar{g}\varphi, \bar{h}\varphi\}$ and $\{k', l'\}$ of $\{\bar{k}\varphi, \bar{l}\varphi\}$ so that

$$supp(k') \cap supp(l') \cap supp(g') \cap supp(h') = supp(g', h') \cap supp(k', l').$$

Hence there is a subset B of ω and an injection $\chi' : B \longrightarrow \omega$ such that if $i \in B$ then the i^{th} element of $\sigma(g', h')$ is the $\chi'(i)^{th}$ element of $\sigma(k', l')$, and if $i \notin B$ then the i^{th} element of $\sigma(g', h')$ is not in $\sigma(k', l')$, for almost all $i \in \omega$. We must show that $A = B$ and $\chi = \chi'$ almost everywhere (for then, by finite adjustments of g', h', k', l', we may arrange that equality holds everywhere).

It is clear that ghk has cycle type $C(S)$ where

$$S = \{2n_i - 1 : i \notin A\} \cup \{2n_i + n_{\chi(i)} - 2 : i \in A\} \cup \{n_i : i \notin \chi(A)\}.$$

Similarly, $g'h'k'$ almost has cycle type

$$S = \{2n_i - 1 : i \notin B\} \cup \{2n_i + n_{\chi'(i)} - 2 : i \in B\} \cup \{n_i : i \notin \chi'(B)\}.$$

It follows by Lemma 4.1(a),(b),(c) and (e) that $A \triangle B$ is finite and $\chi(i) = \chi'(i)$ for almost all $i \in \omega$, as required.

Lemma 4.5 *Let f_φ be as defined above. Then the $\equiv$-class of f_φ is independent of the choice of Ω_0, Ω_1 and of $\bar{g}, \bar{h}, \bar{k}, \bar{l}$.*

Proof. Pick another partition $\Omega = \Omega_0^* \cup \Omega_1^*$ into moieties, and let $\{\bar{g}^*, \bar{h}^*\}, \{\bar{k}^*, \bar{l}^*\} \in \mathcal{D}$ have matched covers $\{g^*, h^*\}, \{k^*, l^*\}$ so that $\sigma(g^*, h^*)$ is an enumeration of Ω_0^* and $\sigma(k^*, l^*)$ is an enumeration of Ω_1^*. Define a corresponding near-bijection f_φ^* of Ω. We must show $f_\varphi \equiv f_\varphi^*$. This, however, follows almost immediately from Lemma 4.4.

To complete the proof of the theorem, we must prove the following lemma.

Lemma 4.6 *Let f_φ be as above, and let $\bar{f}_\varphi$ be the element of $NS(\Omega)$ corresponding to f_φ. Then for all $\bar{g} \in C(2)_{inf}$ we have $\bar{f}_\varphi^{-1} \bar{g} \bar{f}_\varphi = \bar{g}\varphi$.*

Proof. Let $g := \prod_{i \in \omega}(a_{2i} a_{2i+1})$. Choose permutations $x, y \in S$ such that $\{\bar{x}, \bar{y}\} \in \mathcal{D}$, the set $\{x, y\}$ is a matched $C(T)$-pair, and such that $\sigma(x, y)$ is the sequence $(a_i : i \in \omega)$. Then $supp(\bar{g}) = supp(\bar{x}) \wedge supp(\bar{y})$, so $supp(\bar{g}\varphi) = supp(\bar{x}\varphi) \wedge supp(\bar{y}\varphi)$. Also, $\{\bar{x}\varphi, \bar{y}\varphi\} \in \mathcal{D}$, so we may choose a corresponding matched $C(T)$-pair $\{x', y'\}$. Let $\sigma(x', y') = (b_i : i < \omega)$. Then by Lemma 4.5, $a_i f_\varphi = b_i$ for all but finitely many $i \in \omega$. In particular, $f_\varphi^{-1} g f_\varphi$ and $\prod_{i<\omega}(b_{2i} b_{2i+1})$ are $\equiv$-equivalent. Also, by Lemma 3.6 (or a direct argument), $\bar{g}\varphi \in C(2)_{inf}$, and if $g' \in S$ with $\bar{g}' = \bar{g}\varphi$, then the symmetric difference $supp(g') \triangle \{b_i : i \in \omega\}$ is finite. It remains to show that g' and $\prod_{i \in \omega}(b_{2i} b_{2i+1})$ differ by a finitary permutation.

Let $u := gx$. Then u has cycle type $1^\omega \prod((n_{2i} + n_{2i+1})^1 : i < \omega)$. This says that, for each $i < \omega$, g pairs up the n_{2i}-cycle of x with the n_{2i+1}-cycle of x. By Lemma 3.7, $\bar{u}\varphi$ and $\bar{u}$ are conjugate in G, so have preimages in S with cycle types differing finitely. If g' induces the transposition $(b_i b_j)$ then $x'g'$ has an $(n_{2i}+n_{2i+1})$-cycle. It now follows from Lemma 4.1(d) that g' differs finitely from $\prod_{i\in\omega}(b_{2i}b_{2i+1})$, as required.

Proof of Theorem 1.1. The main theorem follows immediately from Lemma 4.6. For, since the conjugacy class $C(2)_{inf}$ generates G, conjugation by $\bar{f}_\varphi$ induces the action of φ on all elements of G.

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