GEOMETRY, STEINBERG REPRESENTATIONS AND COMPLEXITY

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Group representation theory often relates quite different areas of mathematics and we shall give yet another example of this phenomenon. A construction from finite geometries will lead us to a new concept in representation theory which we shall then apply to the representation theory of Lie type groups. This, in turn, will involve ideas from the homological approach to modular representations. We shall, therefore, cover a spectrum of ideas.

One construction of finite projection planes involves the use of spreads. Suppose that V is a $2n$-dimensional vector space over a finite field k of characteristic p. A spread $\mathcal{S}$ is a collection of n-dimensional subspaces whose (set-theoretic) union is all of V but where the intersection of any two members of the collection is zero. A group of linear transformations of V preserves the spread $\mathcal{S}$ if its elements permute the members of $\mathcal{S}$.

Proposition 1. *If E is an elementary abelian 2-group of linear transformations of V which preserve $\mathcal{S}$ and $p = 2$ then there is a subgroup F of E with the following two properties:*

i) V is free as a kF-module;

ii) The space of fixed-points V^F, of V under F, equals V^E.

This is the key idea and we shall now formulate it in more generality. If E is an elementary abelian p-group and k is any field of characteristic p then the kE-module M is said to be subfree if there is a subgroup F of E with the two properties of the proposition. Before proceeding, let us note that this gives us some interesting parameters. The dimension of the subspace $V^F = V^E$ is called the breadth of M. Since M is a free kF-module we have that $\dim M = |F| \dim M^F$ so $|F|$ is independent of the choice of F. If $|F| = p^d$ then we say that d is the depth of M.

Returning to the geometry, we shall see that this property arises in a "layered" manner.

Proposition 2. *Under the same hypotheses as Proposition 1, let Q be a 2-group of linear transformations of V which preserve $\mathcal{S}$. Assume, moreover,*

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that V^Q is not contained in any single member of $\mathcal{S}$. It follows that whenever N is a normal subgroup of the subgroup P of Q and P/N is elementary abelian then V^N is a subfree $k[P/N]$-module.

We now turn to the study of how these ideas arise in the representation theory of groups of Lie type. Assume that k is an algebraically closed field of characteristic p and let $G = SL(2,q)$, the special linear group over the q-element subfield k_q of k, where $q = p^e$. Let X be the subgroup of upper uni-triangular elements of G so X is a Sylow p-subgroup of order q. Let $X(\lambda)$ be the element of X which has λ as the first row second column entry. We can now state a surprising result [2].

Theorem 1. *If V is a kG-module, and $e > 1$, then the following are equivalent:*

i) V is simple of dimension a power of p;

ii) V is a subfree kX-module of breadth 1.

It is quite unexpected that simplicity can be described in these terms. The motivation is again geometric. If $p = 2$ then a non-simple module with property ii), written over a finite subfield of k, would be an interesting candidate for a spread left invariant by G. Geometric motivation again suggests the following question in the case $p = 2$: Which simple kG-modules are subfree on restriction to every 2-subgroup?

This theorem is based, in part, on a determination of the non-identity subgroups of X which act freely on particular simple modules. Such modules are necessarily of dimension a power of p so let us describe all the simple kG-modules of such a dimension. The basic Steinberg module St_1 for kG is the $(p-1)st$ symmetric power of the standard two-dimensional module so St_1 is p-dimensional and it is simple as well. Let $\sigma_1, \ldots, \sigma_d$ be d distinct automorphisms of k_q so $d \leq e$. Then Galois conjugate modules $\sigma_i(St_1), 1 \leq i \leq d$, are distinct and

$$S = \sigma_1(St_1) \otimes \ldots \otimes \sigma_d(St_1)$$

is of dimention p^d and simple. Such modules, called partial Steinberg modules, give all the simple kG-modules of dimension a power of p. The tensor product of all e conjugates is of dimension $p^e = q$ and is the Steinberg module.

This situation generalizes considerably so we shall give the adjunct to the theorem in this broader context. Now set $G = SL(n,q)$ and let X be the root subgroup which consists of matrices with ones on the main diagonal, zeros elsewhere, except perhaps in the first row second column entry. Let $X(\lambda)$ be the corresponding element of X. There is a basic Steinberg module St_1 for kG of dimension $p^{n(n-1)/2}$ and partial Steinberg modules

$$S = \sigma_1(St_1) \otimes \ldots \otimes \sigma_d(St_1)$$

(as well as the Steinberg module). The adjunct result is as follows [3].

Theorem 2. *The subgroup generated by* $X(\lambda_1), \ldots, X(\lambda_d)$ *is of order* p^d *and free on* S *if, and only if,* $\det(\sigma_i(\lambda_j)) \neq 0$.

This is only a special case of the general result: we can deal with p-subgroups of all orders, all simple kG-modules as well as some other groups of Lie type. It would be also nice to have a generalisation of Theorem 1 to the case of $SL(n,q)$ but there is a significant obstacle: a basic Steinberg module St_1 need *not* be subfree on restriction to the root subgroup X. However, the ideas in Proposition 2 are the way round this problem and we shall illustrate this first by looking at another Lie type group.

We let $p = 2$ and let $H = Sz(2^{2f+1})$ be the Suzuki group, $f \geq 1$. Here there is a basic Steinberg module St_1 of dimension four and partial Steinberg modules

$$T = \tau_1(St_1) \otimes \ldots \otimes \tau_c(St_1), \ \ c \leq 2f+1,$$

which give all the simple kH-modules in fact (and the Steinberg module has dimension 4^{2f+1}). Let Y be a Sylow 2-subgroup of H, so $|Y| = 4^{2f+1}$, the center Z of Y is elementary abelian of order 2^{2f+1} as is Y/Z.

Theorem 3. *As a* kZ*-module,* T *is subfree of breadth* 2^{2c+1} *while* T^Z *is a subfree* $k[Y/Z]$*-module of breadth one.*

Presumably these conditions are also sufficient to characterize simple modules and give a result analogous to Theorem 1.

We have not indicated the ideas of the proofs. There are direct methods [2,3] but Carlson has shown that complexity theory and homological ideas are important [5]. In order to return to the special linear groups we require a first instalment of these techniques. Let E be an elementary abelian p-group and let $\operatorname{aug}(kE)$ be the augmentation ideal. If $g_1, \ldots, g_d$ is a minimal generating set of E then $V_E = \operatorname{aug}(kE)/\operatorname{aug}(kE)^2$ is a d-dimensional vector space with a basis consisting of the cosets of the elements $g_1 - 1, \ldots, g_d - 1$. If $x_1, \ldots, x_t$ are elements of $\operatorname{aug}(kE)$ whose cosets modulo $\operatorname{aug}(kE)^2$ are linearly independent elements of V_E then the units $1 + x_1, \ldots, 1 + x_t$ are of order p and generate an elementary abelian p-group of order p^t in the unit group of kE. Such a group is called a shifted subgroup (e.g. see [6]) of E. If M is a kE-module then M is called shifted subfree if there is a shifted subgroup F such that $M^E = M^F$ and M is a free kF-module. Again we can speak of the breadth and depth of M.

We now return to the group $G = SL(n,q)$. Let L be the subgroup stabilizing a fixed m-dimensional subspace, $0 < m < n$, of the standard n-dimensional space on which G acts. Let Q be the normal subgroup of L of all elements which induce the identity on the m-dimensional subspace and in its $(n-m)$-dimensional quotient so Q is an elementary abelian p-group of order $q^{m(n-m)} = p^{em(n-m)}$. We are going to state a result about the action of Q on the partial Steinberg module S (as above). Since L/Q contains a subgroup isomorphic with $SL(m,q) \times SL(n-m,q)$ and S^Q is a tensor product of partial Steinberg modules for this direct product, it becomes possible to use our result to give, inductively, a "layered" result about the action of a Sylow-p-subgroup of G on S analogous to the previous theorem. However, we shall restrict ourselves to Q.

Theorem 4. *The kQ-module S is shifted subfree of depth $dm(n-m)$.*

We also believe that when $q > p$ the partial Steinberg modules can be characterized in terms of subfree "layers" and the parameters involved and already have made considerable progress towards such a goal. The proof involves complexity theory, to which we turn for out last theorem.

Let G be an arbitary finite group and let M be a kG-module. The complexity $C_G(M)$ of M is a non-negative integer, a homological invariant of M. It is zero exactly when M is projective and it is one, when it is not zero, but there is a projective resolution

$$\to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$$

of M such that $\dim P_n$ is bounded, independently of n, (i.e. $\dim P_n$ is bounded by a degree zero polynomial in n). Similarly, it is two, if it is not less than two and there is such a resolution where $\dim P_n$ is bounded bu a degree one polynomial in n, and so on for higher complexities.

Theorem 5. *The complexity of the basic Steinberg module St_1 for $SL(n,q)$ is $(e-1)[n^2/4]$.*

The way this result is connected with our previous results is as follows. The complexity $C_G(M)$ equals the maximum of the complexities $C_E(M_E)$ as E runs over all elementary abelian p-subgroups of G [1]. In turn, $C_E(M_E)$ can be calculated in two complementary fashions, using Carlson's rank variety [4] or Kroll's shifted subgroup approach [7]. The rank variety $V_E(M)$ is defined as follows, as a subset of V_E (notation as above). If X is in $\mathrm{aug}(kE)$ and not in $\mathrm{aug}(kE)^2$ then the coset $x + \mathrm{aug}(kE)^2$ lies in $V_E(M)$ if, and only if, M is not a free module for $< 1 + x >$; this does not depend on the choice of coset representative. The variety $V_E(M)$ consists of these elements together with the zero vector, it is a homogeneous affine variety of dimension $C_E(M_E)$. If $|E| = p^s$ then there are subspaces of V_E of dimension $s - C_E(M_E)$ which intersect $V_E(M)$ in zero, by the homogeneity of $V_E(M)$, and no such subspaces of any larger dimension. This implies that there are shifted subgroups of E of order $p^{s-C_E(M_E)}$ for which M is a free module and none of any greater order with this property.

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