Algebraic Cobordism

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Before We Start

Many parts of America are still reeling from the tragic murders of George Floyd and Breonna Taylor, among others. These unjust killings reflect deep corruption in society at large and within ourselves as individuals.

If you're angry and not sure what to do, a few ways to get started:

- Donate to an organization fighting police brutality and other structural racism in the criminal justice system, such as BLM Chicago, CAARPR, BYP100, or the CCBF.
- Read about the black experience in America: *The New Jim Crow, Between the World and Me, The South Side, The Fire Next Time, How to be an Anti-Racist, etc.* If you send me your address after this, I will buy you one of these books.
- Follow black authors and activists on social media and listen to what they have to say.

References

- *Algebraic Cobordism* slides by Marc Levine.
- Talks by Adeel Khan, Ben Knudsen, Lukas Brantner, and Brian Hwang at the 2014 Talbot Seminar. (Notes by Amelia Perry)
- *Algebraic Cobordism* by Marc Levine and Fabian Morel.
- *Motivic Stable Homotopy Groups* by Daniel C. Isaksen and Paul Arne Ostaver (in Haynes Miller's *Handbook of Homotopy Theory*.)
- Among others. You can google the names of people I've attributed results to.

Overview

Given a topological space X, we like to consider the cohomology ring $H^*(X)$. There are two ways to generalize this to algebraic geometry:

1. Given a smooth variety X, the Chow Ring $CH^*(X)$ is similar to cellular cohomology, using subvarieties of X in place of cells.
2. Given a motivic space X, we've previously discussed its motivic cohomology $H^{*,*}(X)$.

These two constructions are related. For smooth varieties X,

$$CH^n(X) = H^{2n,n}(X).$$

Overview

Given a topological space X, we like to consider the complex K-theory ring $KU^*(X)$ built from complex vector bundles. There are two ways to generalize this to algebraic geometry:

1. Given a (nice) smooth variety X, the algebraic K-theory $K^*(X)$ of X is built from the category of vector bundles over X.
2. Given a motivic space X, there is a cohomology theory $KGL^{*,*}(X)$ represented by a spectrum coming from the (motivic) space $\mathbb{Z} \times BGL$.

These two constructions are related. For smooth varieties X,

$$K^n(X) = KGL^{2n,n}(X).$$

Overview

Given a topological space X, we like to consider the complex cobordism ring $MU^*(X)$. There are two ways to generalize this to algebraic geometry:

1. Given a smooth variety X, the algebraic cobordism $\Omega^*(X)$ of X is built from ????
2. Given a motivic space X, there is a cohomology theory $MGL^{*,*}(X)$, represented by a spectrum coming from ????

Are these two constructions related? For smooth varieties X,

$$\Omega^n(X) \overset{???}{=} MGL^{2n,n}(X)$$

Overview

Outline of this talk:

1. Preliminaries: How to...
   - ...projectivize a bundle
   - ...build a motivic spectrum
   - ... orient a manifold
   - ...get a formal group law from a cohomology theory
2. Cobordism of Manifolds and MU
3. The spectrum MGL
4. Algebraic cobordism $\Omega^*(-)$

How to Projectivize a Bundle

- Recall that projective space $\mathbb{R}\,\mathrm{P}^n$ is built by quotienting $\mathbb{R}^{n+1} \setminus \{0\}$ by the relation $(x_0, \cdots, x_n) \simeq (cx_0, \cdots cx_n)$
- We can apply the same construction to vector bundles. The projectivization of a rank $n+1$ vector bundle $E \to X$ is the space $E \setminus X$ modulo the above relation on each fiber.
- This produces a bundle $\mathbb{P}(E)$ with fiber $\mathbb{R}\,\mathrm{P}^n$.
- In the language of (Noetherian, affine) schemes, a vector bundle over $X = \operatorname{Spec} R$ is equivalent data to a finitely-generated projective R-module M.
- The bundle corresponding to M is $\operatorname{Spec}(\operatorname{Sym}_R M) \to X$, and the projectivisation of this bundle is $\operatorname{Proj}(\operatorname{Sym}_R M) \to X$.

How to Build a Motivic Spectrum

- Marie defined the ∞-category of motivic spectra for us as the homotopy limit of the tower:

$$\cdots \to_{\Omega_{\mathbb{P}^1}} \mathrm{MSpaces}_* \to_{\Omega_{\mathbb{P}^1}} \mathrm{MSpaces}_* \to_{\Omega_{\mathbb{P}^1}} \mathrm{MSpaces}_* .$$

  In other words, a motivic spectrum is a collection of motivic spaces X_n for $n \geq 0$, along with homotopy equivalences $X_n \to \Omega_{\mathbb{P}^1} X_{n+1}$.
  (Quasi-categories exercise: check this. The objects of the limit correspond to maps from Δ^0, which you can identify using the universal property.)

- In practice, we can start with *prespectra*: collections of (motivic) spaces Y_n and maps $\mathbb{P}^1 \wedge Y_n \to Y_{n+1}$.
- The corresponding spectrum is given by

$$X_n = \mathrm{colim}_k \ \Sigma^\infty \Omega^k_{\mathbb{P}^1} Y_{n+k}.$$

How to Orient a Manifold

Definition

Let E be a ring spectrum, and X be a smooth n-manifold. An *E-orientation* of X is a class r in $E_n(X)$ such that for each x, the image of r under the composition

$$E_n(X) \to E_n(X, X - \{x\}) = E_n(D^n, S^{n-1}) = E_0$$

generates E_* as an E_*-module.

This gives us all sorts of nice things, like Poincare duality!

Example

A manifold is $H\mathbb{Z}$-orientable iff it is orientable in the usual sense.

Example (Atiyah-Bott-Shapiro)

A manifold is KO-orientable iff it is Spin.

How to Orient a Manifold

Question: which homology theories orient *all* manifolds?

Theorem (Thom)

If E is orients all manifolds, then E is a direct sum of shifts of $H\mathbb{F}_2$.

This is kind of dumb, so "all manifolds" was too much to ask for. Instead, let's try to orient a specific collection of manifolds.

Example

$H\mathbb{Z}$, MU, and KU orient all complex manifolds

If we generalize orientability a bit to include general vector bundles, we can define a complex-oriented (co)homology theory to be one that orients all complex vector bundles.

How to Get a Formal Group Law from a Cohomology Theory

Definition

A cohomology theory $E^*(X)$ is complex-oriented if it comes with a good theory of Chern classes for complex vector bundles.

Specifically, we ask that:

1. E^* has a commutative multiplication.
2. We have a chosen element $c_1 \in E^2(\mathbb{C}\,\mathrm{P}^\infty)$ which maps to 1 under the composition

$$E^2(\mathbb{C}\,\mathrm{P}^\infty) \to E^2(\mathbb{C}\,\mathrm{P}^1) = E^2(S^2) = E_0.$$

This definition lets us find elements $c_1, c_2, \cdots$ so that

$$E^*(BU(n)) = E_*[[c_1, c_2, \cdots, c_n]].$$

Since n-dimensional vector bundles V over X correspond to maps $f_V : X \to BU(n)$, we obtain characteristic classes $c_i(V) := f_V^*(c_i)$ living in $E^*(X)$.

How to Get a Formal Group Law from a Cohomology Theory

Given line bundles V and W over X, we can take their (fiberwise) tensor product to get a new line bundle $V \otimes W$.

Proposition

In ordinary cohomology, we have $c_1(V \otimes W) = c_1(V) + c_1(W)$.

Naive hope: maybe this is always true!

Counterexample

In complex K-theory, we have $c_1(V \otimes W) = c_1(V) + c_1(W) + \beta^{-1} c_1(V) c_1(W)$, where $\beta \in KU_2$ is the Bott element.

Question: what is true in general?

How to Get a Formal Group Law from a Cohomology Theory

Definition

A (one-dimensional, commutative) formal group law over a ring R is a power series $F(x, y)$ in two variables such that:

1. $F(x, 0) = F(0, x) = x.$
2. $F(x, y) = F(y, x)$
3. $F(x, F(y, z)) = F(F(x, y), z).$

(Exercise: there exists a power series $\iota(x)$ with $F(x, \iota(x)) = 0$.)

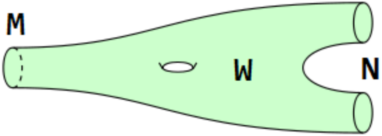
Theorem

For any complex-oriented cohomology theory $E^*(-)$, there exists a formal group law F over E_* such that for any line bundles V, W over X,

$$c_1(V \otimes W) = F(c_1(V), c_1(W)).$$

Cobordism of Manifolds, I

- A *cobordism* of n-dimensional (smooth) manifolds M and N is an $(n+1)$-dimensional manifold W and an identification of ∂W with $M \coprod N^{\mathrm{op}}$.



- We say two manifolds are cobordant if there is a cobordism between them.
- Hard to generalize because varieties don't have boundaries!
- Better for today: a cobordism $M \to N$ is an $(n+1)$ dimensional manifold W and a (smooth, proper) map $f : W \to \mathbb{R}$ such that $f^{-1}(0) = M$ and $f^{-1}(1) = N$.

Cobordism of Manifolds, II

- (Unoriented) closed manifolds up to cobordism form a graded ring MO_* under disjoint union and cartesian product.

Theorem (Dold, using work of Thom)

MO_* is isomorphic to $\mathbb{F}_2[c_i]_{i+1\neq 2^k}$, where c_i is represented by an explicit dimension i manifold.

- By requiring every manifold in sight to have additional structure, we obtain more interesting forms of cobordism.
- Complex cobordism MU_*: cobordism of manifolds with an "almost complex structure on their stable normal bundle."

Example

In MO_2, orientable surfaces are all cobordant to $\emptyset$. In MU_2, the Riemann sphere $\mathbb{C}P^1$ is nontrivial, and genus g Riemann surfaces are cobordant to $g-1$ copies of $-\mathbb{C}P^1$.

Cobordism of Manifolds, III

(Complex) cobordism gives us a cohomology theory $MU^n(X)$.

Definition (Non-standard, just for this slide)

A map $f : Y \to X$ is an n-map if X and Y are both manifolds, and if is smooth, proper, complex-oriented, and has codimension n.

Definition

For manifolds X, $MU^n(X)$ is the free group generated by n-maps with target X, modulo the relations $[f \coprod g] = [f] + [g]$ and $[h^{-1}(X \times \{0\})] = [h^{-1}(X \times \{1\})]$ for each $(n+1)$-map $h : Y \to X \times \mathbb{R}$ transverse to $X \times \{0, 1\}$.

Warning: the *homology* theory $MU_n(X)$ has a similar (and easy to mix up) definition!

Question: What is the spectrum MU representing this?

Aside: The Thom construction

Typically, we build Thom spaces by adding points to turn vector bundles into sphere bundles, then collapsing all the new points. We'll use a definition that works equally well motivically.

Definition (Atiyah)

The Thom Space of a vector bundle $E \to X$ is the quotient $\mathbb{P}(E \oplus \mathbb{R})/\mathbb{P}(E)$.

(Exercise: show that this is equivalent to the usual definition)

Examples

- *The Thom Space of the trivial bundle* $X \times \mathbb{R}^n \to X$ *is* $\Sigma^n X_+$.
- *The Thom Space of the Mobius Strip is* $\mathbb{R}P^2$.

Theorem (Thom)

For $k \geq 0$, *there is an isomorphism* $H^k(X; \mathbb{F}_2) \to \tilde{H}^{k+n}(T(E); \mathbb{F}_2)$

The Spectrum MU

- Let $MU(2n)$ be the Thom space of the real vector bundle underlying the universal bundle $EU(n) \to BU(n)$.
- The inclusion $U(n) \to U(n+1)$...

  ...induces a bundle map $EU(n) \oplus \mathbb{R}^2 \to EU(n+1)$...

  ...which induces a map $\Sigma^2 MU(2n) \to MU(2n+2)$.
- In other words, the $MU(2n)$'s assemble into a spectrum MU!
- The "direct sum" operation on vector bundles induces a "fully" commutative multiplication $MU \times MU \to MU$.

Going Backwards

Motivic cobordism developed a bit backwards, in that the spectrum *MGL* was studied before algebraic cobordism was defined. For the rest of this talk, we work over a field k of characteristic zero.

1. Let $\mathrm{Gr}(m, n)$ denote the Grassmanian: that is, the projective variety of m-dimensional subspaces of $\mathbb{A}^n$.
2. The colimit $BGL(m) := \mathrm{colim}_n \mathrm{Gr}(m, n)$ comes with a natural vector bundle $EGL(m)$, so we can take

$$\mathrm{Th}(EGL(m)) := \mathbb{P}(EGL(m) \oplus k^1)/\mathbb{P}(EGL(m))$$

3. The map $i : EGL(m) \to EGL(m+1)$ sending X to $X \oplus k^1$ induces a map $\mathbb{P}^1 \wedge \mathrm{Th}(EGL(m)) \to \mathrm{Th}(EGL(m+1))$.
4. We call the corresponding (commutative ring) spectrum *MGL*.

Universality

Theorem (Adams?)

MU is the universal complex-oriented spectrum: orientations on a ring spectrum E correspond to ring maps $MU \to E$.

Theorem (Panin-Pimenov-Rondigs)

For a natural motivic generalization of complex orientations, MGL is the universal oriented motivic spectrum.

The proof is very computational. One direction is not too hard: once you have an orientation on MGL, then rings maps $MGL \to E$ give you orientations for free! The corresponding Chern classes for E cohomology are just the images of those for MGL-cohomology.

Sketch of proof of Panin-Pimenov-Rondigs

To sketch reverse direction, we have a couple of intermediate steps:

1. Given an oriented spectrum E, show that $E^{*,*}(BGL(m)) = E_{**}[[c_1, \ldots, c_n]]$ with $|c_i| = (2i, i)$.
2. Use this to show that $E^{*,*}(MGL) = E_{**}[[c_1, c_2, \ldots]]$.
3. Show that an orientation produces an element of $E^{0,0}(MGL)$, and hence a map $\varphi : MGL \to E$.
4. Redo this whole story to understand $E^{*,*}(MGL \wedge MGL)$. All the diagrams you'd hope might commute do, so φ is a ring map.

The coefficient ring of MGL

This universality statement show up in the algebra of MU as well:

Theorem (Milnor, Quillen)

MU_* is a polynomial ring on generators x_i in degree $2i$. The corresponding formal group is the universal one.

In general MGL_{**} is hard, but we have partial information!

Theorem (Levine, Levine-Morel)

$MGL_{2*,*} = MU_{2*}$, carrying the universal formal group law.

Theorem (Spitzweck, others)

The slice spectral sequence for MGL has

$$E^1_{m,q,n} = H\mathbb{Z}_{**}[x_1, x_2, \cdots] \Rightarrow \pi_{m,n} MGL$$

with $|x_i| = (2i, i, i)$.

K-theory from cobordism

Theorem (Conner-Floyd)

We can recover complex K-theory from complex cobordism:

$$KU^*(X) = MU^*(X) \otimes_{MU_*} KU_*.$$

The same is true motivically!

Theorem (Panin-Pimenov-Rondigs)

For small motivic spaces X, we can recover algebraic K-theory from algebraic cobordism.

$$KGL^{*,*}(X) = MGL^{*,*}(X) \otimes_{MGL_{2*,*}} KGL_{2*,*}$$

There's actually a whole theory of Landweber exactness, much like in the topological case! I did not have time to learn the details of it and this talk is too long anyways.

"Higher Algebraic" Constructions

Theorem (Snaith)

Both KU and periodic MU can be constructed "algebraically": there exists $\beta \in \pi_2^s(BU)$ *such that:*

$$PMU = \Sigma_+^\infty BU[\beta^{-1}]$$
$$KU = \Sigma_+^\infty \mathbb{C}P^\infty[\beta^{-1}]$$

Theorem (Gepner-Snaith)

The same theorems hold motivically! In particular:

$$PMGL = \Sigma_+^\infty BGL[\beta^{-1}]$$
$$KGL = \Sigma_+^\infty \mathbb{P}^\infty[\beta^{-1}]$$

This can be used to give quick proofs of Conner-Floyd and check that $PMGL$ and KGL are E_∞ rings.

The Nilpotence Theorem

Theorem (Devinatz-Hopkins-Smith, conjectured by Ravenel)

MU_* detects nilpotence: that is, for any ring spectrum, an element in $\pi_*(R)$ is nilpotent if and only if its image in $MU_*(R)$ is zero.

Corollary (Nishida)

(Almost) every element in the homotopy groups of spheres is nilpotent.

This is a VERY important theorem with numerous consequences. So we'd like a similar theorem in the motivic world.

Counterexample (Morel)

The motivic Hopf map $\eta \in \pi_{1,1}(S)$ is not nilpotent but does map to 0 in $MGL_{1,1}(S)$. So MGL does not detect nilpotence.

"Naive" Motivic Cobordism

We have the spectrum MGL, but we'd like a geometric interpretation of the groups $MGL^{2n,n}(X)$.

Definition (Naive)

Instead of "manifolds over" a smooth variety X, we have projective morphisms $Y \to X$ with

1. Y irreducible and smooth (over the field k)
2. $\dim Y - \dim X = n$.

We say Y is cobordant to Z if there is a projective morphism $W \to X \times \mathbb{P}^1$ whose fibers over $X \times 0$ and $X \times 1$ are Y and Z.

Problem: this doesn't look anything like topological cobordism!

Example (Levine-Morel)

In the above construction with $k = \mathbb{C}$, a curve of genus g is not cobordant to $(1 - g)\mathbb{P}^1$.

Motivic Cobordism

Definition (Levine-Pandharipande)

A double point cobordism is a projective morphism $f : W \to X \times \mathbb{P}^1$ of smooth schemes such that:

1. $Y := f^{-1}(X \times 1)$ is smooth.
2. $Z := f^{-1}(X \times 0)$ is the union of two smooth, transversely-intersecting subschemes A and B.

In this case, we denote by $\mathbb{P}(f)$ the projectivisation of the normal bundle of $A \cap B$ inside either A or B.

Definition (Levine-Pandharipande, simplifying Levine-Morel)

The algebraic cobordism ring $\Omega^*(X)$ is generated as before, with relations:

- $[Y \cup_X Z] = [Y] + [Z]$
- For each double point cobordism, $[Y] = [A] + [B] - [\mathbb{P}(f)]$.

Motivic Cobordism

Theorem (Levine, using unpublished work of Hopkins-Morel)

For smooth, quasi-projective varieties X, there is a natural isomorphism

$$\Omega^*(X) = MGL^{2*,*}(X)$$

Sketch of proof for fields k embedding into $\mathbb{C}$:

1. Ω^* comes with a universal property giving us natural maps $\Omega^*(X) \to MGL^{2*,*}(X) \to MU^{2*}(X(\mathbb{C}))$ for free.
2. The composition is an isomorphism by previous work of Morel-Levine, so we only need to check that $\Omega^*(X) \to MGL^{2*,*}(X)$ is a surjection.
3. When X is Spec of a field, use a spectral sequence of Hopkins-Morel to check directly.
4. Induct on the dimension of X, using localization sequences to glue together information from lower-dimensional subvarieties.