Simplicial Commutative Rings

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Outline

The following motivating example is given at the beginning of Lurie's thesis [Lur04].

Theorem (Bezout's Theorem for transversal intersection)

Let C and C' be projective curves in $\mathbb{CP}^2$ with degree n and m respectively. If C and C' meet transversely(denote $C \perp C'$), then $\#(C \cap C') = nm$.

- Conceptual Understanding:
  - Let $[C]$ and $[C']$ be the fundamental classes of C and C' respectively, then $[C] \cup [C'] = nm \in H^4(\mathbb{P}^2; \mathbb{Z}) \simeq \mathbb{Z}$.
  - $C \cap C'$ is of dimension 0, i.e. it constitutes of discrete points. In the transversal case, $[C \cap C'] = [C] \cup [C']$.
- Geometric Picture:

Theorem

(Bezout's Theorem, Ver. 2) Let C and C' be plane projective curves of degree n and m, respectively. Then we have

$$nm = \sum_{p \in C \cap C'} \dim_{\mathbb{C}} (\mathcal{O}_C \otimes_{\mathbb{P}^2} \mathcal{O}_{C'})_p \tag{1}$$

Question:

1. Can it be generalized to higher dimensions?
2. Do we still have $[C \cap C'] = [C] \cup [C']$?

Answer to Question 1:

Add higher *Tor* groups to generalize the multiplicity in the curve case. [Ser00]

Theorem (Serre's Intersection Formula)

Let C and C' be as above. Then we have

$$nm = \sum_{p \in C \cup C'} \left(\sum_{i=0}^{\infty} dim\ Tor_i^{\mathcal{O}_{\mathbb{P}^{a+b},p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p}) \right).$$

Answer to question 2:

- Let C be $x = 0$, C' be $x = y^2$, we have $[C] \cup [C'] = 2$ as usual but $[C \cap C'] = \cdots\cdots 1!$
- Fortunately, we can find a way to fix this:
  - Consider in a Derived setting;
  - Regard $Tor_i^{\mathcal{O}_{\mathbb{P}^{a+b},p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p})$ as homotopy groups of $\mathcal{O}_{C,p} \otimes^L_{\mathcal{O}_{\mathbb{P}^n,p}} \mathcal{O}_{C',p}$;
  - Regard $[C]$ as a "virtual fundamental class" instead of the classical fundamental class;
  - In the setting above, the cup product encodes the information of the total derived functor $\otimes^L$.
- To achieve this, we need a "generalized ring" structure.

There are several candidates

- $\mathcal{SCR}_{R/}$: (∞-)category of topological commutative rings endowed with and R-algebra structure;
    - objects can be modeled by simplicial commutative R-algebras
- $\mathcal{DGA}_R$: (∞-)category of commutative differential graded R-algebras;
- $\mathcal{EI}_{R/}$: (∞-)category of E_∞-R-algebras;

Which one to use?

- E_∞-spectra are useful in applying algebraic ideas to homotopy theory;
- Simplicial commutative rings bring homotopy theoretical ideas to algebra and are easier to compute;
- $\mathcal{DGA}$: Lurie said it's not conceptual in this setting. Quillen had used it for rational homotopy theory since it works well with $\mathbb{Q}$ coefficients. Toën and Vezzosi have also used this guy in their papers.

See [Lur04] Ch 2.6 for more detailed comparisons.

We begin following [Mat12]

Definition (Category SCR)

*A **simplicial commutative ring** $A_\bullet$ is a simplicial object in the category of commutative rings.*

- *$A_\bullet$ a simplicial set;*
- *each A_n have the structure of a commutative ring;*
- *simplicial operators are ring morphisms.*

*A **morphism** between simplicial rings is a morphism of simplicial sets $A \to B$ where $A_n \to B_n$ is a ring morphism for each n.*

Example

- *An ordinary ring R can be viewed as a constant simplicial ring $A_\bullet$: each A_n is R and each simplicial operators are the identity.*
- *Let $X_\bullet$ be a simplicial set. Define the simplicial commutative ring $R[X_\bullet]$ by letting $(R[X_\bullet])_n$ to be the polynomial algebra $R[\{X_n\}]$.*

Homotopy Groups of SCRs:

- Simplicial rings are simplicial groups and Kan complexes. Therefore Given $R_\bullet$ in SCR, we can talk about its homotopy groups w.r.t a basepoint $0 \in R_0$;
- The homotopy groups $\pi_* R_\bullet$ forms a graded commutative ring
- We have $\pi_0 R_\bullet \simeq R_0/(d_1 - d_0)R_1$ as a ring.
- We can regard higher homotopy groups as analogies of nilpotent elements in classical algebraic geometry. The map $R_\bullet \to \pi_0 R_\bullet$ for a simplical ring $R_\bullet$ is like the canonical map $R \to R_{red}$ for an ordinary ring R.

The category SCR is enriched in simplicial sets:

- Define $K_\bullet \otimes R_\bullet$ for $K \in sSet$ and $R_\bullet \in SCR$ by letting $(K \otimes X)_n = \bigotimes_{K_n} R_n$. From this we can have a simplicial structure on SCR.
- Given $X_\bullet$ and $Y_\bullet$ in SCR, we can define a simplicial set $\underline{hom}(X_\bullet, Y_\bullet) \in sSet$ by letting $\underline{hom}(X_\bullet, Y_\bullet)_n = hom_{SCR}(K_\bullet \otimes X_\bullet, Y_\bullet)$.

Therefore we have

Definition (Simplicial Homotopy)

*A **simplicial homotopy** between two morphisms*

$$f, g : X_\bullet \to Y_\bullet$$

is a morphism

$$\Delta^1 \otimes X_\bullet \to Y_\bullet$$

which restricts to f on $\Delta^{\{0\}} \otimes X_\bullet$ and to g on $\Delta^{\{1\}} \otimes X_\bullet$.

Theorem

There is a cofibrantly generated simplicial model structure on SCR with:

- *Fibrations* $Fib(SCR) = \{$*fibrations of the underlying simplicial sets*$\}$;
- *Weak equivalences* $W(SCR) = \{$*weak equivalences of the underlying simplicial sets*$\}$;
- *Cofibrations are determined by the above two.*

Moreover, we have

- *the free-forgetful adjunction is a Quillen adjunction between SSet and SCR.*
- *SCR is a monoidal model category w.r.t the tensor product. This further gives a "derived tensor product" in the homotopy category.*

We can do the similar things for the category $SCR_{R/}$ of all simplicial commutative R-algebras.

- Let $F\colon SCR \to \mathcal{A}$ be a left Quillen functor, where $\mathcal{A}$ is a model category. We can define a **derived functor** $\mathbb{L}F$ of functor F by letting $\mathbb{L}F(X_\bullet) = F(\tilde{X}_\bullet)$, where $\tilde{X}_\bullet$ is a cofibrant replacement of $X_\bullet$.
- Simiarly, we can define a **derived tensor product** on the homotopy category. Let $A_\bullet \to B_\bullet$ and $A_\bullet \to C_\bullet$ be morphisms in SCR, the derived tensor product $\otimes^{\mathbb{L}}$ is defined as

$$B_\bullet \otimes^{\mathbb{L}}_{A_\bullet} C_\bullet = \tilde{B}_\bullet \otimes_{A_\bullet} \tilde{C}_\bullet,$$

  where $A_\bullet \to \tilde{B}_\bullet$ and $A_\bullet \to \tilde{C}_\bullet$ are cofibrant replacements for $A_\bullet \to B_\bullet$ and $A_\bullet \to C_\bullet$.
- Let $S = \mathcal{O}_C \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{P}^{a+b}}} \mathcal{O}_{C'}$, we have

$$\pi_i S = Tor_i^{\mathcal{O}_{\mathbb{P}^{a+b}}}(\mathcal{O}_C, \mathcal{O}_{C'}).$$

Example

Let $X_\bullet$ be a simplicial set. Want to compute the homotopy groups of the free simplicial commutative ring $\mathbb{Z}[X_\bullet]$.

- *$\mathbb{Z}[X_\bullet]$ is the (dimensionalwise) symmetric algebra of the free simplicial abelian group $\mathbb{Z}X_\bullet$.*
- *We have a weak equivalence*

$$\mathbb{Z}X_\bullet \simeq \bigoplus K(H_n(X_\bullet), n)$$

  of simplicial groups, where K are the Eilenberg-MacLane spaces.
- *Assume that the symmetric algebra is independent of homotopy type, we have a weak equivalence*

$$\mathbb{Z}[X_\bullet] = Sym^\bullet \mathbb{Z}X_\bullet \simeq \bigotimes \mathbb{L}Sym^\bullet K(H_n(X_\bullet), n)$$

  in SCR.
- *Therefore we can compute the homotopy groups of $\mathbb{Z}[X_\bullet]$ from $\mathbb{L}Sym^k$ and the Künneth formula, although this is quite hard.*

Example

- *We can use the bar construction of SCR to compute simplicial resolutions.*
- *This kind of computation is useful for computing cotangent complexes, which are important in the deformation theory of schemes.*

Definition

A (affine) ***derived scheme*** *is a topological space X with a sheaf of topological rings $\mathcal{O}_X$ s.t $(X, \mathcal{O}_X) \cong (Spec\ A, \mathcal{O}_A)$ for a topological commutative ring A.*

- *Spec* A as a topological space is just *Spec* $\pi_0 A$ in the ordinary sense;
- in ordinary algebraic geometry we have $\mathcal{O}_{Spec\ A}(U_f) = A[f^{-1}]$;
- in derived algebraic geometry it is also this case, but the operation of adding the inverse of f to A should be taken in the world of topological commutative rings.

More about this story: [Lur04] [Lur05]

Thank you!

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