MOTIVIC HOMOTOPY THEORY: CONJECTURES-COMPUTATIONS-COMPARISONS

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Some Useful References.

- Dugger, Notes on the Milnor conjectures.
- Dugger, Navigating the motivic world.
- Dugger and Isaksen, The motivic Adams spectral sequence
- Frankland and Spitzweck, Towards the dual motivic Steenrod algebra in positive characteristic.
- Heller and Ormsby, Galois equivariance and stable motivic homotopy theory.
- Levine, A comparison of motivic and classical stable homotopy theories.
- Levine, Algebraic cycle complexes
- Powell, Steenrod operations in motivic cohomology.

Quadratic Forms. Let k have characteristic $\neq 2$. Then quadratic forms on k correspond to symmetric bilinear forms which themselves correspond to symmetric matrices via $q(x) = x^T Ax$. Equivalence is defined by $A \sim P^T BP$ for invertible P. Non degeneracy corresponds to invertibility of A. A can be diagonalized so

$$q(x) = a_1 x_1^2 + \cdots + a_n x_n^2 = \langle a_1, ..., a_n \rangle$$

- The Grothendieck-Witt ring $GW(k)$ is defined by group completing the monoid of nondegenerate quadratic forms mod equivalence. By Witt cancellation, this monoid injects in $GW(k)$.
- The Witt ring $W(k)$ is defined as $GW(k)$ mod the ideal spanned by the hyperbolic plane $\langle 1, -1 \rangle$. In $W(k)$, $\langle -1 \rangle = -\langle 1 \rangle$ and $W(k)$ is 2-torsion iff $\sqrt{-1} \in k$. Note that said ideal is spanned by $\langle a, -a \rangle$ and $\langle a, -a \rangle \simeq \langle 1, -1 \rangle$ (exercise) hence it's the additive subgroup generated by $\langle 1, -1 \rangle$ so we have

$$0 \to \mathbb{Z}\langle 1, -1 \rangle \to GW(k) \to W(k) \to 0$$

- We have a rank map $GW(k) \to \mathbb{Z}$ that descends to $W(k) \to \mathbb{Z}/2$ with kernel I.
- Let $Gr_I(k) = \oplus_{n \geq 0} I^n / I^{n+1}$, $I^0 = W(k)$.

  We have a graded ring map

$$K_*^M(k)/2 \to Gr_I(k)$$

determined at $* = 1$ as $(a) \mapsto \langle a, -1 \rangle$ (the LHS is defined in a later section). It's a conjecture of Milnor that this is an isomorphism (proven by looking at the Motivic Adams SS).

The Dress map. Fix a Galois extension E/k and $G = Gal(E/k)$. Then

$$G/H \mapsto GW(E^H)$$

is a Tambara functor with

- $Res^H_K : GW(E^H) \to GW(E^K)$ given by extension of scalars $E^H \subseteq E^K$
- $\mathrm{Tr}^H_K : GW(E^K) \to GW(E^H)$ given by composing with trace $Tr_{E^K/E^H} : E^K \to E^H$.
- $N^H_K : GW(E^K) \to GW(E^H)$ determined by $N^H_K(\langle a \rangle) = \langle Nm^H_K(a) \rangle$.

The Dress map is the initial map from the Burnside Tambara functor

$$A(-) \to GW(E^-)$$

is determined by the transfer of 1, so it must be

$$[H/K] \mapsto \mathrm{Tr}^H_K(\langle 1 \rangle)$$

This is the so called trace form given by

$$x \mapsto Tr_{E^K/E^H}(x^2)$$

We will realize it topologically later.

The stable motivic category. The stable motivic category $SH(k)$:

- Symmetric monoidal stable presentable ∞-cat.
- Symmetric monoidal exact functor $c : Sp \to SH(k)$ given by the constant sheaf.
- Symmetric monoidal left exact functor $\Sigma^\infty_+ : Sm/k \to SH(k)$. Also based version $\Sigma^\infty : (Sm/k)_* \to SH(k)$.
- Two types of spheres: $S^{1,0} = c(S^1) = \Sigma 1$ and $S^{1,1} = \Sigma^\infty G_m$. In general

$$S^{p,q} = (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge q}$$

All these spheres are inverted ($p, q \in \mathbb{Z}$). Moreover, $\Sigma^\infty A^1_k \simeq 0$.

$$H(k) = L_{\mathcal{F}\times A^1_k \to \mathcal{F}} sSh(Sm/k)$$

$$SH(k) = \lim(\cdots \xrightarrow{\Omega^{2,1}} H_*(k) \xrightarrow{\Omega^{2,1}} H_*(k))$$

$H_*(k)$ uses pointed sheaves i.e. having a chosen sheaf map $\mathrm{Spec}(k) \to \mathcal{F}$.

0.1. **Realization.** Let's talk about c. It induces the rank map

$$\mathbb{Z} = End_{Sp}(S) = \pi_0(S) \to \pi_{0,0}(S) = End_{SH(k)}(S) = GW(k)$$

Mark Levine proves that if k is algebraically closed of characteristic 0, then c is fully faithful.

If k embeds in $\mathbb{C}$, there is an exact symmetric monoidal Betti Realization functor $Re : SH(k) \to Sp$ defined by

$$Re(\Sigma^\infty X) = \Sigma^\infty X(\mathbb{C})$$

We have that $Re \circ c = 1$.

There is an exact symmetric monoidal functor $c_{L/k} : Sp^G \to SH(k)$ where L/k is a finite extension with Galois group G. Indeed, a G-orbit G/H corresponds to the finite étale $\mathrm{Spec}(L^H)$ and we define $c_{L/k}$ from orbits to smooth schemes over k through this Galois correspondence and then stabilize. On $\pi_{0,0}$ it induces the Dress map

$$A(G) \to GW(k)$$

which is the unique map of Tambara functors. If the extension is $\mathbb{C}$ / $\mathbb{R}$ then,

$$A(C_2) = GW(\mathbb{R}) = \mathbb{Z}[C_2]$$

Heller and Ormsby show that if k is real closed then $c_{k[i]/k} : Sp^{C_2} \to SH(k)$ is fully faithful after completing at (p, η) where p is any prime and η is the motivic Hopf element induced by $A^2 \setminus 0 \to P^1$.

If k embeds in $\mathbb{R}$, there is an exact symmetric monoidal Betti Realization functor $Re^{C_2} : SH(k) \to Sp^{C_2}$ such that

$$Re^{C_2}(\Sigma^\infty X) = \Sigma^\infty X(\mathbb{C})$$

where the C_2 action is conjugation. We have that $Re^{C_2} \circ c_{k[i]/k} = 1$.

We have the compatibility between change of group and change of ring functors:

$$\begin{array}{ccc} SH(\mathbb{R}) & \xrightarrow{Re^{C_2}} & Sp^{C_2} \\ \downarrow{\scriptstyle \mathbb{C}\otimes_R -} & & \downarrow \\ SH(\mathbb{C}) & \xrightarrow{Re} & Sp \end{array}$$

Motivic cohomology. The motivic co/homology groups of X in Sm/k:

$$H^{p,q}(X;\mathbb{Z}) = H^p(X;\mathbb{Z}(q)) = \mathrm{Hom}_{DM_k}(M(X), \mathbb{Z}(q)[p]) = CH^q(X, 2q-p)$$

For $p = q = n \geq 0$ and $X = k$ we get Milnor K-theory

$$H^{n,n}(k;\mathbb{Z}) = K_n^M(k)$$

Note: $H^{p,q}(X;\mathbb{Z}) = 0$ if $q < 0$ or $p > 2q$ or $p - q > \dim(X)$

- Hyperhomology definition: There are complexes $\mathbb{Z}(q)$ conjectured by Beilinson–Lichtenbaum in the derived category of abelian Zariski (or Nishnevich) sheaves on X and

$$H^p(X;\mathbb{Z}(q)) = \mathbb{H}^p_{\Gamma_{Zar}}(X_{Zar}, \mathbb{Z}(q))$$

- Higher Chow groups: Let $\Delta^n = k[t_0, ..., t_n]/(\sum_i t_i = 1)$ a cosimplicial k-scheme. A q-algebraic cycle of X is a formal integral combination of q-dimensional irreducible subvarieties of X. Let $z^q(X \times \Delta_n)$ be the subgroup of q-cycles in $X \times \Delta_n$ spanned by varieties W with $codim_{X\times F}(W \cap X \times F) \geq n$ for every face F. Then $z^q(X \times \Delta_*)$ is a simplicial abelian group and

$$CH^q(X, n) = H_n(z^q(X \times \Delta_*))$$

For $n = 0$ we get the Chow groups $CH^q(X)$.

- Milnor K-theory is the graded ring

$$K_*^M(k) = \frac{T_{\mathbb{Z}}(k^\times)}{a \otimes (1-a)}$$

where we quotient by the two-sided ideal generated by $a \otimes (1-a)$, $a \neq 0, 1$ (Steinberg relation).

The Galois symbol. Let us describe a map

$$K_n^M(k)/\ell \to H_{et}^n(k, \mu_\ell^{\otimes n})$$

For $n = 1$ the Kummer sequence

$$1 \to \mu_\ell \to G_m \to G_m \to 1$$

and Hilbert's Theorem 90 give

$$H_{et}^1(k, \mu_\ell) = k^\times/(k^\times)^\ell$$

An a on the right corresponds to a Steinberg symbol (a) on the left, satisfying the Steinberg relation: If $a = b^\ell$ then $(a) = 0$ by the iso above. Otherwise write $a = b^\ell$ over an extension L/k; then $Nm_{L/k}(1-b) = \prod(1 - b\zeta^i) = 1 - a$ so

$$(a)(1-a) = (a)Nm_{L/k}(1-b) = Nm_{L/k}((a)(1-b)) = Nm_{L/k}((b^\ell(1-b)) = 0$$

So tensoring up the map and using the Steinberg relation for cup products in $H_{et}^1(k, \mu_n) \otimes H_{et}^1(k, \mu_n) \to H_{et}^2(k, \mu_n \otimes \mu_n)$ defines the norm residue map.

The Hilbert symbol. If k contains a primitive ℓ-th root of unity, $\mu_\ell = \mathbb{Z}/\ell$ and $\mu_\ell^{\otimes n} = \mathbb{Z}/\ell$. In particular we get

$$K_2^M(k)/\ell \to H_{et}^2(k; \mathbb{Z}/\ell)$$

Now assume k is also a local field (finite extension of $\mathbb{R}, \mathbb{Q}_p, \mathbb{F}_p((x))$); class field theory says that

$$H^2(k, \mu_\ell) = \mu_\ell(k)$$

the group of ℓ-th roots of unity of k. The map

$$K_2^M(k)/\ell \to \mu_\ell(k)$$

is called the Hilbert symbol.

We have that $(x, y) = 1 \iff y$ is in the image of $Nm_{k[\sqrt[\ell]{a}]/k}$.

In the case of $\ell = 2$, we get

$$K_2^M(k)/2 \to \{\pm 1\}$$

and $(a, b) = 1 \iff z^2 = ax^2 + by^2$ has a nonzero solution for $(x, y, z) \in k^3$.

The Hilbert symbol can also be defined through the local Artin symbol which is the isomorphism

$$\psi : \frac{k^\times}{(k^\times)^\ell} \to Gal(L/k)$$

where L is obtained from k by adjoining all ℓ-th roots. Then,

$$\psi(y)(\sqrt[\ell]{x}) = (x, y)\sqrt[\ell]{x}$$

If k is a number field, we get a Hilbert symbol $(a,b)_v$ for each of the completions of k at a valuation v. The Artin Reciprocity Theorem for the global Artin symbol implies that

$$\prod_v (a,b)_v = 1$$

$\mathbb{Q}$ has completions the local fields $\mathbb{Q}_p$ and $\mathbb{R}$ so we get $(a,b)_p$ and $(a,b)_\infty$. For $\ell = 2$:

- $(a,b)_\infty = 1 \iff a$ or b are positive.
- If $a = 2^k a'$ and $b = 2^l b'$ for odd a', b' then

$$(a,b)_2 = \left(\frac{-1}{a'}\right)\left(\frac{2}{a'}\right)^l \left(\frac{-1}{b'}\right)\left(\frac{2}{b'}\right)^k$$

  using the Jacobi symbol.
- If p is odd and $a = p^k a', b = p^l b'$ for a', b' coprime to p then

$$(a,b)_p = \left(\frac{-1}{p}\right)^{kl}\left(\frac{a'}{p}\right)^l \left(\frac{b'}{p}\right)^k$$

Then

$$(a,b)_\infty \prod_p (a,b)_p = 1$$

generalizes Quadratic Reciprocity (take a, b to be distinct primes).

0.2. **Conjectures.** Conjectures (proven by Voevodsky):

- Milnor conjecture:

$$K_n^M(k)/2 = H_{et}^n(k, \mathbb{Z}/2)$$

  for k of characteristic not 2.
- Bloch-Kato conjecture:

$$K_n^M(k)/\ell = H_{et}^n(k, \mu_\ell^{\otimes n})$$

  for ℓ invertible in k.
- Beilinson–Lichtenbaum conjecture:

$$H^{p,q}(k; \mathbb{Z}/\ell) = H_{et}^p(k; \mu_\ell^{\otimes q})$$

  if $0 \le p \le q$ and ℓ invertible in k.

The isomorphism in all cases is the norm-residue map.

Beilinson–Soulé vanishing conjecture (open!): If $p < 0$, $H^{p,q}(X; \mathbb{Z}) = 0$.

In $\mathbb{Z}/2$ coefficients,

$$H_{et}^p(k; \mathbb{Z}/2) = H_c^*(Gal(k^s/k); \mathbb{Z}/2)$$

where k^s is the separable closure of k. We note that $H^{p,q}(\mathrm{Spec}(k), \mathbb{Z}/2) = 0$ if $p > q$ or $q < 0$. Therefore,

$$H^{*,*}(\mathrm{Spec}(k), \mathbb{Z}/2) \hookrightarrow H_c^*(Gal(k^s/k); \mathbb{Z}/2) \otimes \mathbb{Z}/2[\tau]\ ,\ |\tau| = (0,1)$$

as the elements where $p \le q$.

Eg

$$H^{p,q}(\mathrm{Spec}(\mathbb{C}); \mathbb{Z}/2) = \mathbb{Z}/2[\tau]$$
$$H^{p,q}(\mathrm{Spec}(\mathbb{R}), \mathbb{Z}/2) = \mathbb{Z}/2[\rho, \tau]\ ,\ |\rho| = (1,1)$$

Application: AHSS:

$$E_2^{p,q} = H^{p-q,-q}(X;\mathbb{Z}/2) \implies K_{-p-q}(X)$$

Motivic Steenrod. Voevodsky defines bistable power operations $P^i : H^{*,*}(-;\mathbb{Z}/\ell) \to H^{*,*}(-\mathbb{Z}/\ell)$ of bidegree $(2i(\ell-1), i(\ell-1))$ satisfying analogues of instability, identity, the Cartan and Adem relations. Together with the Bocksteins, admissible monomials of them generate the Steenrod algebra $H\mathbb{Z}_\ell^{*,*}H\mathbb{Z}_\ell$ as a module over $H\mathbb{Z}_\ell^{*,*}$ as long as ℓ is invertible in k.

The dual Steenrod $H\mathbb{Z}_{\ell *,*}H\mathbb{Z}_\ell$ has classes τ_i, ξ_i of bidegrees $(2\ell^i - 1, \ell - i)$ and $(2\ell^i - 2, \ell - i)$. Their products, where each τ_i has power ≤ 1, span $H\mathbb{Z}_{\ell *,*}H\mathbb{Z}_\ell$ over $H\mathbb{Z}_{\ell *,*}$ (if ℓ is invertible in k) and define the Milnor basis in the Steenrod algebra.

If ℓ is the characteristic of k, these are only subalgebras and actually retracts of the actual answers.

$\mathbb{C}$ motivic.

- $H_{*,*}(\mathrm{Spec}(\mathbb{C}); \mathbb{F}_2) = \mathbb{F}_2[\tau]$ for $|\tau| = (0, -1)$.
- Realization preserves Eilenberg MacLane spectra, hence induces $H^{p,q}(X) \to H^p(X(\mathbb{C}))$. It sends $\tau \mapsto 1$:
$$H^{0,1}(\mathrm{Spec}(\mathbb{C})) = H^0_{et}(\mathrm{Spec}(\mathbb{C})) = H^0_{sing}(\mathbb{C}) = \mathbb{F}_2$$
- This gives a map
$$H^{*,*}(X)_\tau \to H^*(X(\mathbb{C}))[\tau^\pm]$$
that is an isomorphism if $X = S, H\mathbb{F}_2$.
- $H_{*,*}H = H_{*,*}[\tau_i, \xi_i]/(\tau_i^2 = \tau\xi_{i+1})$. Killing τ gives the odd prime Steenrod. Realization is
$$\tau_i \mapsto \xi_{i+1}$$
$$\xi_i \mapsto \xi_i^2$$
(up to conjugates). In fact,
$$(H_{*,*}H)_\tau = H_*H \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau^\pm]$$

$\mathbb{R}$ motivic.

- $H_{*,*}(\mathrm{Spec}(\mathbb{R}); \mathbb{F}_2) = \mathbb{F}_2[\tau, \rho]$ for $|\tau| = (0, -1)$ and $|\rho| = (1, 1)$.
- Realization preserves Eilenberg MacLane spectra (using the constant Mackey functor) hence induces $H^{p,q}(X) \to H^p_{C_2}(X(\mathbb{C}))$. In particular, $\tau \mapsto u_\sigma, \rho \mapsto a_\sigma$.
- $H_{*,*}H = H_{*,*}[\xi_i, \tau_i]/(\tau_i^2 = (\tau + \rho\tau_0)\xi_{i+1} + \rho\tau_{i+1})$. Realization sends these to the Hu-Kriz generators and
$$H_{*,*}H \otimes_{H_{*,*}} H_\bigstar^{C_2} = H_\bigstar H^{C_2}$$
Killing ρ is basechange to $\mathbb{C}$.