Representing Algebraic K-Theory

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References

- "A primer for unstable motivic homotopy theory" - Antieau and Elmanto
- "Higher algebraic K-theory of schemes and of derived categories" - Thomason and Trobaugh

Vector Bundles in Algebraic Topology

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Example

Suppose X is connected with universal cover $\tilde{X}$. Then $\tilde{X} \to X$ is a principal $\pi_1(X)$-bundle.

Theorem

If X is paracompact and Hausdorff, then there is a bijection $[X, BG] \cong \mathrm{Tors}_G(X)$.

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Therefore,

$$\mathrm{Vect}_n^{\mathbb{R}}(X) \cong [X, BGL_n(\mathbb{R})].$$

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is a monoid under direct sum. Let $KO^0(X)$ be the group completion of $\mathrm{Vect}^{\mathbb{R}}(X)$.

There is an isomorphism

$$KO^0(X) \cong [X, \mathbb{Z} \times BGL(\mathbb{R})]$$

where

$$BGL(\mathbb{R}) := \varinjlim_n BGL_n(\mathbb{R}).$$

$KO^0(X)$ can be extended to cohomology theories

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These cohomology theories are useful in topology.
Is there an analogous functor in algebraic geometry?

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Here, B denotes the realization of the nerve of the category.
Higher algebraic K-theory is defined to be $K_n(R) := \pi_n(K(R))$.

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A *vector bundle* on X is a locally free $\mathcal{O}_X$-module whose rank at each point is finite. Let $\mathbf{VB}(X)$ denote the category of vector bundles on X.

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Example

When $X = \operatorname{Spec} R$, then $\mathbf{VB}(X)$ is equivalent to the category of finitely generated projective R-modules.

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In general, we need a slightly different definition given by Thomason-Trobaugh, which I won't discuss here.

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- Sm_S will denote the category of finitely presented smooth schemes over S.
- $L_{Nis}sPre(Sm_S)$ is the category of simplicial presheaves with the Nisnevich-local model structure.
- $Spc_S^{\mathbb{A}^1} := L_{\mathbb{A}^1}L_{Nis}sPre(Sm_S)$ is the unstable $\mathbb{A}^1$-homotopy category.

Properties of Algebraic K-Theory

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$$K(X) \xrightarrow{\simeq} \varprojlim \left(\prod K(U_i) \rightrightarrows \prod K(U_i \times_X U_j) \cdots \right)$$

Therefore, $K(-)$ is a fibrant object in $L_{Nis}sPre(Sm_S)$.

More Properties of Algebraic K-Theory

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Notation

We will now assume that S is regular, noetherian.

Representability

Yoneda

Suppose $F : \mathcal{C}^{op} \to Kan$. Then F is the composition

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$\mathbb{A}^1$-homotopy theory allows us to represent K-theory but this isn't a satisfying answer.

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Remark

The set of rank n vector bundles is not an $\mathbb{A}^1$-invariant so we won't be able to classify them by $\mathbb{A}^1$-homotopy classes of maps.

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A map $BGL(R) \to H$ where H is an H-space factors through $BGL(R) \to BGL(R)^+$.

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Definition

Let $X \in sPre(Sm_S)$. Define $\mathrm{Sing}^{\mathbb{A}^1} X(R)$ to be the realization of the simplicial space

$$X(R[\Delta^0]) \leftarrow X(R[\Delta^1]) \leftleftarrows \cdots$$

where $R[\Delta^n] := R[t_0, \cdots, t_n] / \left(\sum_{i=0}^n t_i = 1\right)$.

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- $\mathrm{Sing}^{\mathbb{A}^1} X$ is $\mathbb{A}^1$-invariant: it sends $U \to U \times_S \mathbb{A}^1$ to weak equivalences.
- $\mathrm{Sing}^{\mathbb{A}^1} BGL(R)$ is an H-space.
- The map $\mathrm{Sing}^{\mathbb{A}^1} BGL(R) \to \mathrm{Sing}^{\mathbb{A}^1} BGL(R)^+$ is a weak equivalence.