The Hopf condition for bilinear forms (Proseminar)

XiaoLin Danny Shi

May, 2020

Background

- The set of squares is closed under multiplication: $x^2y^2 = z^2$

Background

- The set of squares is closed under multiplication: $x^2 y^2 = z^2$
- Sum of two squares is also closed under multiplication:

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = z_1^2 + z_2^2$$

where $z_1 = x_1 y_1 - x_2 y_2, z_2 = x_2 y_1 + x_1 y_2$

Background

- The set of squares is closed under multiplication: $x^2 y^2 = z^2$
- Sum of two squares is also closed under multiplication:

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = z_1^2 + z_2^2$$

  where $z_1 = x_1 y_1 - x_2 y_2, z_2 = x_2 y_1 + x_1 y_2$
- "law of moduli" for complex numbers

Background

- The set of squares is closed under multiplication: $x^2 y^2 = z^2$
- Sum of two squares is also closed under multiplication:

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = z_1^2 + z_2^2$$

  where $z_1 = x_1 y_1 - x_2 y_2, z_2 = x_2 y_1 + x_1 y_2$
- "law of moduli" for complex numbers
- $|\alpha\beta| = |\alpha| \cdot |\beta|$, with $\alpha = x_1 + ix_2$, $\beta = y_1 + iy_2$

- Euler (1748): tried to prove every positive integer is a sum of four squares

- Euler (1748): tried to prove every positive integer is a sum of four squares
- $(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2$

$$\begin{aligned}
z_1 &= x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 \\
z_2 &= x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4 \\
z_3 &= x_3 y_1 + x_4 y_2 + x_1 y_3 - x_2 y_4 \\
z_4 &= x_4 y_1 - x_3 y_2 + x_2 y_3 + x_1 y_4
\end{aligned}$$

- Euler (1748): tried to prove every positive integer is a sum of four squares
- $(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2$

$$\begin{aligned} z_1 &= x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 \\ z_2 &= x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4 \\ z_3 &= x_3 y_1 + x_4 y_2 + x_1 y_3 - x_2 y_4 \\ z_4 &= x_4 y_1 - x_3 y_2 + x_2 y_3 + x_1 y_4 \end{aligned}$$

- After Hamilton's discovery of the quaternion algebra $\mathbb{H}$ in 1843, "law of moduli" for quaternions

- Euler (1748): tried to prove every positive integer is a sum of four squares
- $(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2$

$$
\begin{aligned}
z_1 &= x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 \\
z_2 &= x_2 y_1 + x_1 y_2 - x_4 y_3 + x_3 y_4 \\
z_3 &= x_3 y_1 + x_4 y_2 + x_1 y_3 - x_2 y_4 \\
z_4 &= x_4 y_1 - x_3 y_2 + x_2 y_3 + x_1 y_4
\end{aligned}
$$

- After Hamilton's discovery of the quaternion algebra $\mathbb{H}$ in 1843, "law of moduli" for quaternions
- 1848: Octonion algebra was discovered by Graves and Cayley, giving a 8-square identity

A 16-square identity?

- For the next 50 years, various mathematicians tried to construct a 16-square identity

A 16-square identity?

- For the next 50 years, various mathematicians tried to construct a 16-square identity
- All failed $\implies$ start to suspect that none can exist

A 16-square identity?

- For the next 50 years, various mathematicians tried to construct a 16-square identity
- All failed $\Longrightarrow$ start to suspect that none can exist
- 50 years later, in 1898, Hurwitz proved it by using ideas in linear algebra

Definition

A **composition formula** *of size* $[r, s, n]$ *is a formula of the type*

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 \cdots + z_n^2$$

where $X = (x_1, \ldots x_r)$ *and* $Y = (y_1, \ldots y_s)$ *are systems of indeterminates, and each* $z_k = z_k(X, Y)$ *is a bilinear form in* X *and* Y*, with coefficients in a field* F*.*

Definition

A **composition formula** *of size* $[r, s, n]$ *is a formula of the type*

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 \cdots + z_n^2$$

where $X = (x_1, \ldots x_r)$ *and* $Y = (y_1, \ldots y_s)$ *are systems of indeterminates, and each* $z_k = z_k(X, Y)$ *is a bilinear form in* X *and* Y*, with coefficients in a field* F*.*

- We can view X, Y, Z as column vectors

Definition

A **composition formula** *of size* $[r, s, n]$ *is a formula of the type*

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 \cdots + z_n^2$$

where $X = (x_1, \ldots x_r)$ *and* $Y = (y_1, \ldots y_s)$ *are systems of indeterminates, and each* $z_k = z_k(X, Y)$ *is a bilinear form in* X *and* Y*, with coefficients in a field* F*.*

- We can view X, Y, Z as column vectors
- Then $z_1^2 + z_2^2 + \cdots + z_n^2 = Z^T Z$

Hurwitz matrix equations

Theorem (Hurwitz 1898)

There exist $n \times s$ *matrices* $A_1, \ldots, A_r$ *over* F *satisfying*

$$\begin{aligned} A_i^T A_i &= I_s, \\ A_i^T A_j + A_j^T A_i &= 0 \text{ whenever } i \neq j \end{aligned}$$

if and only if there is a composition formula over F *of size* $[r, s, n]$*.*

Hurwitz matrix equations

Theorem (Hurwitz 1898)

There exist $n \times s$ matrices A_1, ..., A_r over F satisfying

$$\begin{aligned} A_i^T A_i &= I_s, \\ A_i^T A_j + A_j^T A_i &= 0 \text{ whenever } i \neq j \end{aligned}$$

if and only if there is a composition formula over F of size $[r, s, n]$.

- To make things simpler, make $s = n$ (work with square matrices)

Hurwitz matrix equations

Theorem (Hurwitz 1898)

There exist $n \times s$ matrices $A_1, \ldots, A_r$ over F satisfying

$$\begin{aligned} A_i^T A_i &= I_s, \\ A_i^T A_j + A_j^T A_i &= 0 \text{ whenever } i \neq j \end{aligned}$$

if and only if there is a composition formula over F of size $[r, s, n]$.

- To make things simpler, make $s = n$ (work with square matrices)
- The existence of these matrices put restrictions on r and n.

By doing some linear algebra, we get

Theorem (Hurwitz 1898)

If there exists a composition of size $[r, n, n]$*, then:*

1. *if* $r \geq 2$*, then* $2|n$*;*
2. *if* $r \geq 3$*, then* $4|n$*;*
3. $2^{r-2} \leq n^2$.

By doing some linear algebra, we get

Theorem (Hurwitz 1898)

If there exists a composition of size $[r, n, n]$, *then:*

1. *if* $r \geq 2$, *then* $2|n$;
2. *if* $r \geq 3$, *then* $4|n$;
3. $2^{r-2} \leq n^2$.

Corollary (Hurwitz 1898)

If a composition formula of size $[n, n, n]$ *exists, then* $n = 1, 2, 4, 8$.

- These ideas were pushed further by Hurwitz (1923) (over $\mathbb{C}$) and Radon (1922) (over $\mathbb{R}$)

- These ideas were pushed further by Hurwitz (1923) (over $\mathbb{C}$) and Radon (1922) (over $\mathbb{R}$)
- They determined all possible sizes for compositions of type $[r, n, n]$

- These ideas were pushed further by Hurwitz (1923) (over $\mathbb{C}$) and Radon (1922) (over $\mathbb{R}$)
- They determined all possible sizes for compositions of type $[r, n, n]$

Theorem (Hurwitz–Radon)

There exists a composition of size $[r, n, n]$ if and only if $r \leq \rho(n)$, where $\rho(n)$ is determined by the following rules:

- *If $n = 1$, 2, 4, or 8, then $\rho(n) = n$;*
- *If k is odd, then $\rho(2^m k) = \rho(2^m)$;*
- $\rho(16n) = 8 + \rho(n)$.

- These ideas were pushed further by Hurwitz (1923) (over $\mathbb{C}$) and Radon (1922) (over $\mathbb{R}$)
- They determined all possible sizes for compositions of type $[r, n, n]$

Theorem (Hurwitz–Radon)

There exists a composition of size $[r, n, n]$ if and only if $r \leq \rho(n)$, where $\rho(n)$ is determined by the following rules:

- *If $n = 1, 2, 4,$ or 8, then $\rho(n) = n$;*
- *If k is odd, then $\rho(2^m k) = \rho(2^m)$;*
- $\rho(16n) = 8 + \rho(n)$.

Hurwitz only worked with $\mathbb{C}$, but his ideas work over any field F where $2 \neq 0$

- After Hurwitz and Radon, several mathematicians found new proofs of their theorem

- After Hurwitz and Radon, several mathematicians found new proofs of their theorem
- Eckmann (1943): representation theory of certain finite groups

- After Hurwitz and Radon, several mathematicians found new proofs of their theorem
- Eckmann (1943): representation theory of certain finite groups
- Dubisch (1946) and Lee (1948): representation theory of Clifford algebras

- After Hurwitz and Radon, several mathematicians found new proofs of their theorem
- Eckmann (1943): representation theory of certain finite groups
- Dubisch (1946) and Lee (1948): representation theory of Clifford algebras
- The first results about compositions of size $[r, s, n]$ when $r, s < n$ were obtained by topologists

Connection with topology

- An n-dimensional algebra for which the norm is multiplicative ($|xy| = |x| \cdot |y|$) provides a composition formula of size $[n, n, n]$

Connection with topology

- An n-dimensional algebra for which the norm is multiplicative ($|xy| = |x| \cdot |y|$) provides a composition formula of size $[n, n, n]$
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$: real division algebras of dimensions 1, 2, 4, 8

Connection with topology

- An n-dimensional algebra for which the norm is multiplicative ($|xy| = |x| \cdot |y|$) provides a composition formula of size $[n, n, n]$
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$: real division algebras of dimensions 1, 2, 4, 8
- Question: can there be any other finite dimensional division algebras over $\mathbb{R}$?

Connection with topology

- An n-dimensional algebra for which the norm is multiplicative ($|xy| = |x| \cdot |y|$) provides a composition formula of size $[n, n, n]$
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{Q}$: real division algebras of dimensions 1, 2, 4, 8
- Question: can there be any other finite dimensional division algebras over $\mathbb{R}$?

Definition

Let $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a real bilinear map of size $[r, s, n]$.

- *f is **normed** if $|f(x, y)| = |x| \cdot |y|$ for every $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$;*
- *f is **nonsingular** if $f(x, y) = 0$ implies that either $x = 0$ or $y = 0$*

Connection with topology

- An n-dimensional algebra for which the norm is multiplicative ($|xy| = |x| \cdot |y|$) provides a composition formula of size $[n, n, n]$
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{Q}$: real division algebras of dimensions 1, 2, 4, 8
- Question: can there be any other finite dimensional division algebras over $\mathbb{R}$?

Definition

Let $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a real bilinear map of size $[r, s, n]$.

- *f is* **normed** *if $|f(x, y)| = |x| \cdot |y|$ for every $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$;*
- *f is* **nonsingular** *if $f(x, y) = 0$ implies that either $x = 0$ or $y = 0$*

- A composition formula is exactly a normed bilinear map of that size

Connection with topology

- An n-dimensional algebra for which the norm is multiplicative ($|xy| = |x| \cdot |y|$) provides a composition formula of size $[n, n, n]$
- $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{Q}$: real division algebras of dimensions 1, 2, 4, 8
- Question: can there be any other finite dimensional division algebras over $\mathbb{R}$?

Definition

Let $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a real bilinear map of size $[r, s, n]$.

- *f is* **normed** *if $|f(x, y)| = |x| \cdot |y|$ for every $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$;*
- *f is* **nonsingular** *if $f(x, y) = 0$ implies that either $x = 0$ or $y = 0$*

- A composition formula is exactly a normed bilinear map of that size
- A nonsingular bilinear map of size $[n, n, n]$ is exactly an n-dimensional real division algebra

Hopf–Stiefel Theorem

- 1930s: Stiefel and Hopf attacked the problem using topology

Hopf–Stiefel Theorem

- 1930s: Stiefel and Hopf attacked the problem using topology
- A nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ induces a map of real projective spaces $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$

Hopf–Stiefel Theorem

- 1930s: Stiefel and Hopf attacked the problem using topology
- A nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ induces a map of real projective spaces $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$
- Hopf applied ring structure in cohomology

Hopf–Stiefel Theorem

- 1930s: Stiefel and Hopf attacked the problem using topology
- A nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ induces a map of real projective spaces $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$
- Hopf applied ring structure in cohomology
- Stiefel applied characteristic classes of vector bundles

Hopf–Stiefel Theorem

- 1930s: Stiefel and Hopf attacked the problem using topology
- A nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ induces a map of real projective spaces $\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$
- Hopf applied ring structure in cohomology
- Stiefel applied characteristic classes of vector bundles
- Both approach gave the same result

Hopf–Stiefel Theorem

Theorem (Hopf–Stiefel 1940)

If there exists a nonsingular bilinear map of size $[r, s, n]$ *over* $\mathbb{R}$*, then*

$$(x + y)^n = 0$$

in the ring $\mathbb{F}_2[x, y]/(x^r, y^s)$*. In other words,* $\binom{n}{k}$ *is even whenever* $n - s < k < r$*.*

Hopf–Stiefel Theorem

Theorem (Hopf–Stiefel 1940)

If there exists a nonsingular bilinear map of size $[r, s, n]$ over $\mathbb{R}$, then

$$(x + y)^n = 0$$

in the ring $\mathbb{F}_2[x, y]/(x^r, y^s)$. In other words, $\binom{n}{k}$ is even whenever $n - s < k < r$.

- Example: a nonsingular bilinear $[3, 5, 6]$ cannot exist over $\mathbb{R}$ because $\binom{6}{3}$ is odd

Hopf–Stiefel Theorem

Theorem (Hopf–Stiefel 1940)

If there exists a nonsingular bilinear map of size $[r, s, n]$ over $\mathbb{R}$, then

$$(x + y)^n = 0$$

in the ring $\mathbb{F}_2[x, y]/(x^r, y^s)$. In other words, $\binom{n}{k}$ is even whenever $n - s < k < r$.

- Example: a nonsingular bilinear $[3, 5, 6]$ cannot exist over $\mathbb{R}$ because $\binom{6}{3}$ is odd

Corollary

If there is an n-dimensional real division algebra, then n is a power of 2

- The final answer was proved in 1958 as a consequence of Bott Periodicity Theorem (topological K-theory)

- The final answer was proved in 1958 as a consequence of Bott Periodicity Theorem (topological K-theory)
- Topologists studied nonsingular bilinear maps for other reasons as well

- The final answer was proved in 1958 as a consequence of Bott Periodicity Theorem (topological K-theory)
- Topologists studied nonsingular bilinear maps for other reasons as well
- A map of size $[r, n, n]$ provides a system of $r-1$ linearly independent vector fields on S^{n-1}

- The final answer was proved in 1958 as a consequence of Bott Periodicity Theorem (topological K-theory)
- Topologists studied nonsingular bilinear maps for other reasons as well
- A map of size $[r, n, n]$ provides a system of $r - 1$ linearly independent vector fields on S^{n-1}
- A nonsingular $[r, r, n]$ provides an immersion of $\mathbb{RP}^{r-1}$ into $\mathbb{R}^{n-1}$

- The final answer was proved in 1958 as a consequence of Bott Periodicity Theorem (topological K-theory)
- Topologists studied nonsingular bilinear maps for other reasons as well
- A map of size $[r, n, n]$ provides a system of $r - 1$ linearly independent vector fields on S^{n-1}
- A nonsingular $[r, r, n]$ provides an immersion of $\mathbb{RP}^{r-1}$ into $\mathbb{R}^{n-1}$
- Study of homotopy groups of spheres

The Hopf Condition

Theorem (Hopf–Stiefel 1940)

If a sums-of-squares formula of type $[r, s, n]$ *exists over* $\mathbb{R}$*, then* $\binom{n}{i}$ *must be even for* $n - r < i < s$*.*

The Hopf Condition

Theorem (Hopf–Stiefel 1940)

If a sums-of-squares formula of type $[r, s, n]$ exists over $\mathbb{R}$, then $\binom{n}{i}$ must be even for $n - r < i < s$.

- The condition on the binomial coefficients is commonly called the "Hopf condition"

The Hopf Condition

Theorem (Hopf–Stiefel 1940)

If a sums-of-squares formula of type $[r, s, n]$ *exists over* $\mathbb{R}$*, then* $\binom{n}{i}$ *must be even for* $n - r < i < s$.

- The condition on the binomial coefficients is commonly called the "Hopf condition"
- Let F be a field, and let $r *_F s$ denote the smallest n for which a sum-of-squares formula of type $[r, s, n]$ exists

The Hopf Condition

Theorem (Hopf–Stiefel 1940)

If a sums-of-squares formula of type $[r, s, n]$ exists over $\mathbb{R}$, then $\binom{n}{i}$ must be even for $n - r < i < s$.

- The condition on the binomial coefficients is commonly called the "Hopf condition"
- Let F be a field, and let $r *_F s$ denote the smallest n for which a sum-of-squares formula of type $[r, s, n]$ exists
- The Hopf condition gives a lower bound for $r *_F s$ when $F = \mathbb{R}$

The Hopf Condition

Theorem (Hopf–Stiefel 1940)

If a sums-of-squares formula of type $[r, s, n]$ exists over $\mathbb{R}$, then $\binom{n}{i}$ must be even for $n - r < i < s$.

- The condition on the binomial coefficients is commonly called the "Hopf condition"
- Let F be a field, and let $r *_F s$ denote the smallest n for which a sum-of-squares formula of type $[r, s, n]$ exists
- The Hopf condition gives a lower bound for $r *_F s$ when $F = \mathbb{R}$
- What about for other fields?

The Hopf Condition

Theorem (Hopf–Stiefel 1940)

If a sums-of-squares formula of type $[r, s, n]$ *exists over* $\mathbb{R}$*, then* $\binom{n}{i}$ *must be even for* $n - r < i < s$*.*

- The condition on the binomial coefficients is commonly called the "Hopf condition"
- Let F be a field, and let $r *_F s$ denote the smallest n for which a sum-of-squares formula of type $[r, s, n]$ exists
- The Hopf condition gives a lower bound for $r *_F s$ when $F = \mathbb{R}$
- What about for other fields?
- K. Y. Lam and T. Y. Lam proved that the Hopf condition holds for any field F of characteristic 0

- What about fields of other characteristic?

- What about fields of other characteristic?
- Does the Hopf condition holds for fields of other characteristic?

- What about fields of other characteristic?
- Does the Hopf condition holds for fields of other characteristic?
- Work has been done by Adems, Yuzinsky for special values of r, s, and n

- What about fields of other characteristic?
- Does the Hopf condition holds for fields of other characteristic?
- Work has been done by Adems, Yuzinsky for special values of r, s, and n
- Shapiro and Szyjewski proved a slightly weaker condition for arbitrary fields of char > 2 and arbitrary values of r, s, and n (computing the Chow groups of "deleted quadrics")

Hopf condition over arbitrary fields

Theorem (Dugger–Isaksen 2007)

If F is a field of characteristic not equal to 2, and a sum-of-squares formula of type $[r, s, n]$ exists over F, then $\binom{n}{i}$ must be even for $n - r < i < s$.

Hopf condition over arbitrary fields

Theorem (Dugger–Isaksen 2007)

If F is a field of characteristic not equal to 2, and a sum-of-squares formula of type $[r, s, n]$ exists over F, then $\binom{n}{i}$ must be even for $n - r < i < s$.

- Their proof generalizes ideas from Hopf's original proof and uses motivic cohomology

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$
- Gives a bilinear map

$$\phi : F^r \times F^s \longrightarrow F^n$$

by sending $(x_1, \ldots x_r; y_1, \ldots, y_s) \mapsto (z_1, \ldots, z_n)$

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$
- Gives a bilinear map

$$\phi : F^r \times F^s \longrightarrow F^n$$

by sending $(x_1, \ldots x_r; y_1, \ldots, y_s) \mapsto (z_1, \ldots, z_n)$

- Let q be the quadratic form on F^k given by $q(w_1, \ldots, w_k) = w_1^2 + \cdots + w_k^2$

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$
- Gives a bilinear map

$$\phi : F^r \times F^s \longrightarrow F^n$$

  by sending $(x_1, \dots x_r; y_1, \dots, y_s) \mapsto (z_1, \dots, z_n)$
- Let q be the quadratic form on F^k given by $q(w_1, \dots, w_k) = w_1^2 + \cdots + w_k^2$
- $q(\phi(x, y)) = q(x)q(y)$ (normed)

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$
- Gives a bilinear map

$$\phi : F^r \times F^s \longrightarrow F^n$$

  by sending $(x_1, \dots x_r; y_1, \dots, y_s) \mapsto (z_1, \dots, z_n)$
- Let q be the quadratic form on F^k given by $q(w_1, \dots, w_k) = w_1^2 + \cdots + w_k^2$
- $q(\phi(x, y)) = q(x)q(y)$ (normed)
- When $F = \mathbb{R}$, $q(w) = 0 \iff w = 0$ (non-singular)

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$
- Gives a bilinear map

$$\phi : F^r \times F^s \longrightarrow F^n$$

  by sending $(x_1, \dots x_r; y_1, \dots, y_s) \mapsto (z_1, \dots, z_n)$
- Let q be the quadratic form on F^k given by $q(w_1, \dots, w_k) = w_1^2 + \cdots + w_k^2$
- $q(\phi(x, y)) = q(x)q(y)$ (normed)
- When $F = \mathbb{R}$, $q(w) = 0 \iff w = 0$ (non-singular)
- ϕ restricts to a map $(\mathbb{R}^r - 0) \times (\mathbb{R}^s - 0) \longrightarrow (\mathbb{R}^n - 0)$

Hopf's original proof

- Suppose we have a sum-of-squares formula of type $[r, s, n]$
- Gives a bilinear map

$$\phi : F^r \times F^s \longrightarrow F^n$$

  by sending $(x_1, \dots x_r; y_1, \dots, y_s) \mapsto (z_1, \dots, z_n)$
- Let q be the quadratic form on F^k given by $q(w_1, \dots, w_k) = w_1^2 + \cdots + w_k^2$
- $q(\phi(x, y)) = q(x)q(y)$ (normed)
- When $F = \mathbb{R}$, $q(w) = 0 \iff w = 0$ (non-singular)
- ϕ restricts to a map $(\mathbb{R}^r - 0) \times (\mathbb{R}^s - 0) \longrightarrow (\mathbb{R}^n - 0)$
- Since ϕ is bilinear, we can quotient by scalar multiplication:

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$$

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$$

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$$

- Apply mod 2 cohomology

$$\mathbb{F}_2[x]/x^n \longrightarrow \mathbb{F}_2[a]/a^r \otimes \mathbb{F}_2[b]/b^s$$

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$$

- Apply mod 2 cohomology

$$\mathbb{F}_2[x]/x^n \longrightarrow \mathbb{F}_2[a]/a^r \otimes \mathbb{F}_2[b]/b^s$$

- $x \mapsto a + b$ (ϕ is bilinear)

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$$

- Apply mod 2 cohomology

$$\mathbb{F}_2[x]/x^n \longrightarrow \mathbb{F}_2[a]/a^r \otimes \mathbb{F}_2[b]/b^s$$

- $x \mapsto a + b$ (ϕ is bilinear)
- Since $x^n = 0$, $(a + b)^n = 0$

$$\mathbb{R}P^{r-1} \times \mathbb{R}P^{s-1} \longrightarrow \mathbb{R}P^{n-1}$$

- Apply mod 2 cohomology

$$\mathbb{F}_2[x]/x^n \longrightarrow \mathbb{F}_2[a]/a^r \otimes \mathbb{F}_2[b]/b^s$$

- $x \mapsto a + b$ (ϕ is bilinear)
- Since $x^n = 0$, $(a + b)^n = 0$
- We get the Hopf condition

A crucial property of $\mathbb{R}$

- Note that the proof uses in a crucial way that a sum of squares in $\mathbb{R}$ is 0 if and only if when all the numbers are 0

A crucial property of $\mathbb{R}$

- Note that the proof uses in a crucial way that a sum of squares in $\mathbb{R}$ is 0 if and only if when all the numbers are 0
- This does not work for arbitrary fields (fields of characteristic p, or over $\mathbb{C}$)

A crucial property of $\mathbb{R}$

- Note that the proof uses in a crucial way that a sum of squares in $\mathbb{R}$ is 0 if and only if when all the numbers are 0
- This does not work for arbitrary fields (fields of characteristic p, or over $\mathbb{C}$)
- Our bilinear form gives us a map of schemes $\phi : \mathbb{A}^r \times \mathbb{A}^s \longrightarrow \mathbb{A}^n$

A crucial property of $\mathbb{R}$

- Note that the proof uses in a crucial way that a sum of squares in $\mathbb{R}$ is 0 if and only if when all the numbers are 0
- This does not work for arbitrary fields (fields of characteristic p, or over $\mathbb{C}$)
- Our bilinear form gives us a map of schemes $\phi : \mathbb{A}^r \times \mathbb{A}^s \longrightarrow \mathbb{A}^n$
- But it does not necessarily restrict to $(\mathbb{A}^r - 0) \times (\mathbb{A}^s - 0) \longrightarrow (\mathbb{A}^n - 0)$

A crucial property of $\mathbb{R}$

- Note that the proof uses in a crucial way that a sum of squares in $\mathbb{R}$ is 0 if and only if when all the numbers are 0
- This does not work for arbitrary fields (fields of characteristic p, or over $\mathbb{C}$)
- Our bilinear form gives us a map of schemes $\phi : \mathbb{A}^r \times \mathbb{A}^s \longrightarrow \mathbb{A}^n$
- But it does not necessarily restrict to $(\mathbb{A}^r - 0) \times (\mathbb{A}^s - 0) \longrightarrow (\mathbb{A}^n - 0)$
- So how to remedy this situation?

- Remove all the numbers whose sum of squares would give us 0

- Remove all the numbers whose sum of squares would give us 0
- Let Q_k be the projective quadric in $\mathbb{P}^{k+1}$, defined by the equation

$$w_1^2 + \cdots + w_{k+2}^2 = 0$$

- Remove all the numbers whose sum of squares would give us 0
- Let Q_k be the projective quadric in $\mathbb{P}^{k+1}$, defined by the equation

$$w_1^2 + \cdots + w_{k+2}^2 = 0$$

- The bilinear map ϕ induces

$$(\mathbb{P}^{r-1} - Q_{r-2}) \times (\mathbb{P}^{s-1} - Q_{s-2}) \longrightarrow (\mathbb{P}^{n-1} - Q_{n-2})$$

- Remove all the numbers whose sum of squares would give us 0
- Let Q_k be the projective quadric in $\mathbb{P}^{k+1}$, defined by the equation

$$w_1^2 + \cdots + w_{k+2}^2 = 0$$

- The bilinear map ϕ induces

$$(\mathbb{P}^{r-1} - Q_{r-2}) \times (\mathbb{P}^{s-1} - Q_{s-2}) \longrightarrow (\mathbb{P}^{n-1} - Q_{n-2})$$

- $DQ_k := \mathbb{P}^k - Q_{k-1}$ (the deleted quadric)

- Remove all the numbers whose sum of squares would give us 0
- Let Q_k be the projective quadric in $\mathbb{P}^{k+1}$, defined by the equation

$$w_1^2 + \cdots + w_{k+2}^2 = 0$$

- The bilinear map ϕ induces

$$(\mathbb{P}^{r-1} - Q_{r-2}) \times (\mathbb{P}^{s-1} - Q_{s-2}) \longrightarrow (\mathbb{P}^{n-1} - Q_{n-2})$$

- $DQ_k := \mathbb{P}^k - Q_{k-1}$ (the deleted quadric)
- Apply mod 2 motivic cohomology to the map

$$DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Remove all the numbers whose sum of squares would give us 0
- Let Q_k be the projective quadric in $\mathbb{P}^{k+1}$, defined by the equation

$$w_1^2 + \cdots + w_{k+2}^2 = 0$$

- The bilinear map ϕ induces

$$(\mathbb{P}^{r-1} - Q_{r-2}) \times (\mathbb{P}^{s-1} - Q_{s-2}) \longrightarrow (\mathbb{P}^{n-1} - Q_{n-2})$$

- $DQ_k := \mathbb{P}^k - Q_{k-1}$ (the deleted quadric)
- Apply mod 2 motivic cohomology to the map

$$DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- $H^{*,*}(DQ_n; \mathbb{Z}/2)$ is very close to being a truncated polynomial algebra

- Remove all the numbers whose sum of squares would give us 0
- Let Q_k be the projective quadric in $\mathbb{P}^{k+1}$, defined by the equation

$$w_1^2 + \cdots + w_{k+2}^2 = 0$$

- The bilinear map ϕ induces

$$(\mathbb{P}^{r-1} - Q_{r-2}) \times (\mathbb{P}^{s-1} - Q_{s-2}) \longrightarrow (\mathbb{P}^{n-1} - Q_{n-2})$$

- $DQ_k := \mathbb{P}^k - Q_{k-1}$ (the deleted quadric)
- Apply mod 2 motivic cohomology to the map

$$DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- $H^{*,*}(DQ_n; \mathbb{Z}/2)$ is very close to being a truncated polynomial algebra
- Repeat Hopf's argument

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes
- Takes values in bi-graded commutative rings of characteristic 2

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes
- Takes values in bi-graded commutative rings of characteristic 2
- Set $\mathbb{M} = H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2)$

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes
- Takes values in bi-graded commutative rings of characteristic 2
- Set $\mathbb{M} = H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2)$
- The map induced by $X \to \operatorname{Spec} F$ makes $H^{*,*}(X;\mathbb{Z}/2)$ into a $\mathbb{M}$-algebra

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes
- Takes values in bi-graded commutative rings of characteristic 2
- Set $\mathbb{M} = H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2)$
- The map induced by $X \to \operatorname{Spec} F$ makes $H^{*,*}(X;\mathbb{Z}/2)$ into a $\mathbb{M}$-algebra
- $\mathbb{M}$ can be very complicated (and is in general unknown)

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes
- Takes values in bi-graded commutative rings of characteristic 2
- Set $\mathbb{M} = H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2)$
- The map induced by $X \to \operatorname{Spec} F$ makes $H^{*,*}(X;\mathbb{Z}/2)$ into a $\mathbb{M}$-algebra
- $\mathbb{M}$ can be very complicated (and is in general unknown)
- Is it known that $\mathbb{M}^{0,0} \cong \mathbb{Z}/2$, $\mathbb{M}^{0,1} \cong \mathbb{Z}/2$, and the generator $\tau \in \mathbb{M}^{0,1}$ is not nilpotent

Mod 2 Motivic cohomology

- $H^{*,*}(-;\mathbb{Z}/2)$ is a contravariant functor defined on smooth F-schemes
- Takes values in bi-graded commutative rings of characteristic 2
- Set $\mathbb{M} = H^{*,*}(\operatorname{Spec} F;\mathbb{Z}/2)$
- The map induced by $X \to \operatorname{Spec} F$ makes $H^{*,*}(X;\mathbb{Z}/2)$ into a $\mathbb{M}$-algebra
- $\mathbb{M}$ can be very complicated (and is in general unknown)
- Is it known that $\mathbb{M}^{0,0} \cong \mathbb{Z}/2$, $\mathbb{M}^{0,1} \cong \mathbb{Z}/2$, and the generator $\tau \in \mathbb{M}^{0,1}$ is not nilpotent
- If every element of F is a square, it follows form the proof of the Milnor conjecture that $\mathbb{M} \cong \mathbb{Z}/2[\tau]$

Motivic cohomology of deleted quadrics

- There are inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$

Motivic cohomology of deleted quadrics

- There are inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$
- Under these inclusions, $Q_{n-2} \hookrightarrow Q_{n-1}$

Motivic cohomology of deleted quadrics

- There are inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$
- Under these inclusions, $Q_{n-2} \hookrightarrow Q_{n-1}$
- Therefore we get inclusions $DQ_{n-1} \hookrightarrow DQ_n$

Motivic cohomology of deleted quadrics

- There are inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$
- Under these inclusions, $Q_{n-2} \hookrightarrow Q_{n-1}$
- Therefore we get inclusions $DQ_{n-1} \hookrightarrow DQ_n$
- From now on, we will assume every element of F is a square

Motivic cohomology of deleted quadrics

- There are inclusions $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$
- Under these inclusions, $Q_{n-2} \hookrightarrow Q_{n-1}$
- Therefore we get inclusions $DQ_{n-1} \hookrightarrow DQ_n$
- From now on, we will assume every element of F is a square
- This is harmless, because if we have a sum of squares formula of type $[r, s, n]$ over F, that will remain true of we extend F

Theorem (Dugger–Isaksen)

Assume every element of F is a square and char$(F) \neq 2$.

1. *If $n = 2k+1$, then*

$$H^{*,*}(DQ_n; \mathbb{Z}/2) \cong \mathbb{M}[a, b]/(a^2 = \tau b, b^{k+1}),$$

   where a has degree $(1,1)$ and b has degree $(2,1)$

Theorem (Dugger–Isaksen)

Assume every element of F is a square and char$(F) \neq 2$.

1. *If $n = 2k+1$, then*

$$H^{*,*}(DQ_n;\mathbb{Z}/2) \cong \mathbb{M}[a,b]/(a^2 = \tau b, b^{k+1}),$$

   where a has degree $(1,1)$ and b has degree $(2,1)$

2. *If $n = 2k$, then*

$$H^{*,*}(DQ_n;\mathbb{Z}/2) \cong \mathbb{M}[a,b]/(a^2 = \tau b, b^{k+1}, ab^k).$$

Theorem (Dugger–Isaksen)

Assume every element of F is a square and $\mathrm{char}(F) \neq 2$.

1. *If $n = 2k+1$, then*

$$H^{*,*}(DQ_n; \mathbb{Z}/2) \cong \mathbb{M}[a,b]/(a^2 = \tau b, b^{k+1}),$$

   where a has degree $(1,1)$ and b has degree $(2,1)$

2. *If $n = 2k$, then*

$$H^{*,*}(DQ_n; \mathbb{Z}/2) \cong \mathbb{M}[a,b]/(a^2 = \tau b, b^{k+1}, ab^k).$$

3. *The map $H^{*,*}(DQ_{n+1}; \mathbb{Z}/2) \to H^{*,*}(DQ_n; \mathbb{Z}/2)$ sends $a \mapsto a$ and $b \mapsto b$*

Theorem (Dugger–Isaksen)

Assume every element of F is a square and $\mathrm{char}(F) \neq 2$.

1. *If* $n = 2k+1$, *then*

$$H^{*,*}(DQ_n;\mathbb{Z}/2) \cong \mathbb{M}[a,b]/(a^2 = \tau b, b^{k+1}),$$

   where a has degree $(1,1)$ *and b has degree* $(2,1)$

2. *If* $n = 2k$, *then*

$$H^{*,*}(DQ_n;\mathbb{Z}/2) \cong \mathbb{M}[a,b]/(a^2 = \tau b, b^{k+1}, ab^k).$$

3. *The map* $H^{*,*}(DQ_{n+1};\mathbb{Z}/2) \to H^{*,*}(DQ_n;\mathbb{Z}/2)$ *sends* $a \mapsto a$ *and* $b \mapsto b$

Corollary

In $H^{*,*}(DQ_n;\mathbb{Z}/2)$, *we have* $a^{n+1} = 0$ *and* $a^i \neq 0$ *for* $i \leq n$

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Apply $H^{*,*}(-;\mathbb{Z})$ to get

$$p^* : H^{*,*}(DQ_{n-1};\mathbb{Z}/2) \longrightarrow H^{*,*}(DQ_{r-1};\mathbb{Z}/2) \otimes_{\mathbb{M}} H^{*,*}(DQ_{s-1};\mathbb{Z}/2)$$

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Apply $H^{*,*}(-; \mathbb{Z})$ to get

$$p^* : H^{*,*}(DQ_{n-1}; \mathbb{Z}/2) \longrightarrow H^{*,*}(DQ_{r-1}; \mathbb{Z}/2) \otimes_{\mathbb{M}} H^{*,*}(DQ_{s-1}; \mathbb{Z}/2)$$

- $p^*(a) = a_1 + a_2$

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Apply $H^{*,*}(-;\mathbb{Z})$ to get

$$p^* : H^{*,*}(DQ_{n-1};\mathbb{Z}/2) \longrightarrow H^{*,*}(DQ_{r-1};\mathbb{Z}/2) \otimes_{\mathbb{M}} H^{*,*}(DQ_{s-1};\mathbb{Z}/2)$$

- $p^*(a) = a_1 + a_2$
- Since $a^n = 0$, $(a_1 + a_2)^n = 0$

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Apply $H^{*,*}(-;\mathbb{Z})$ to get

$$p^* : H^{*,*}(DQ_{n-1};\mathbb{Z}/2) \longrightarrow H^{*,*}(DQ_{r-1};\mathbb{Z}/2) \otimes_{\mathbb{M}} H^{*,*}(DQ_{s-1};\mathbb{Z}/2)$$

- $p^*(a) = a_1 + a_2$
- Since $a^n = 0$, $(a_1 + a_2)^n = 0$
- $\binom{n}{i}$ is even for $n - r < i < s$

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Apply $H^{*,*}(-; \mathbb{Z})$ to get

$$p^* : H^{*,*}(DQ_{n-1}; \mathbb{Z}/2) \longrightarrow H^{*,*}(DQ_{r-1}; \mathbb{Z}/2) \otimes_{\mathbb{M}} H^{*,*}(DQ_{s-1}; \mathbb{Z}/2)$$

- $p^*(a) = a_1 + a_2$
- Since $a^n = 0$, $(a_1 + a_2)^n = 0$
- $\binom{n}{i}$ is even for $n - r < i < s$
- DONE!

Proof of the Hopf condition

- Assume we have a sums of squares formula of type $[r, s, n]$ over F
- Get a map

$$p : DQ_{r-1} \times DQ_{s-1} \longrightarrow DQ_{n-1}$$

- Apply $H^{*,*}(-;\mathbb{Z})$ to get

$$p^* : H^{*,*}(DQ_{n-1};\mathbb{Z}/2) \longrightarrow H^{*,*}(DQ_{r-1};\mathbb{Z}/2) \otimes_{\mathbb{M}} H^{*,*}(DQ_{s-1};\mathbb{Z}/2)$$

- $p^*(a) = a_1 + a_2$
- Since $a^n = 0$, $(a_1 + a_2)^n = 0$
- $\binom{n}{i}$ is even for $n - r < i < s$
- DONE!
- Remark: if we compute the Chow ring $CH^*(DQ_n)$ (which is what Shapiro–Szyjewki did), it corresponds to the subring of $H^{*,*}(DQ_n;\mathbb{Z}/2)$ generated by b. Get a weaker Hopf condition.

Thank you!