Motivic Homotopy Theory 1

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Outline

- Review of presheaves and localizations on ∞-categories
- Construction of the unstable motivic homotopy category
- Construction of the stable motivic homotopy category

Presheaves on ∞-categories

$\mathcal{S}$ is the ∞-category of spaces.

Definition

Let $\mathcal{C}$ be an ∞-category. The ∞-category $\mathcal{PS}h(\mathcal{C})$ of presheaves on $\mathcal{C}$ is the $Fun(\mathcal{C}^{op}, \mathcal{S})$.

Facts :

- The Yoneda embedding

$$\begin{aligned} j : \mathcal{C} &\to \mathcal{PS}h(\mathcal{C}) \\ A &\to Map_{\mathcal{C}}(-, A) \end{aligned}$$

  is fully faithful.
- $\mathcal{PS}h(\mathcal{C})$ has all small limits and colimits.
- Problem : It has too many colimits.

Localizations of ∞-categories

Definition

- If $\mathcal{C}$ is a category and W is a nice collection of arrows, then we can construct $\mathcal{C}[W^{-1}]$, the localization of $\mathcal{C}$ with respect to W.
- If L is the localization functor, and $\mathcal{D}$ is any ∞-category, then $Fun(\mathcal{C}[W^{-1}], \mathcal{D}) \to Fun(\mathcal{C}, \mathcal{D})$ is a fully faithful functor.

If C is an object in $\mathcal{C}$, a sieve is a sub-presheaf of $j(C)$.
If $f : j(Y) \to j(Z)$ is a morphism and U is a sieve on Z, then f^*U is a sieve on Y.

Definition

A Grothendieck topology on $\mathcal{C}$ is a collection $J(Y)$ of sieves on each object Y, such that the following conditions are true :

(i) $j(Y) \in J(Y)$.

(ii) If S is a sieve on Y and $f : Y \to Z$ is a morphism, then $f^*(S) \in J(Z)$.

(iii) If $S \in J(Y)$ and T is any sieve on Z, and for any $U \in \mathcal{C}$, and $f : W \to Z \in S(U)$, we have $f^*(T)$ is a covering on W, then $T \in J(Z)$.

Localizing presheaves with respect to $W =$ the covering sieves gives us a category of sheaves, $\mathcal{S}h(\mathcal{C})$.

- A W-local object D is such that for every $f : X \to Y$ in W, $f^* : Hom(Y, D) \to Hom(X, D)$ is a bijection.
- It is equivalent to see sheaves as the full subcategory of $\mathcal{PS}h(\mathcal{C})$ consisting of W-local objects.
- This gives us an inclusion functor $\mathcal{S}h(\mathcal{C}) \to \mathcal{PS}h(\mathcal{C})$ which is a right adjoint to the sheafification functor $\mathcal{PS}h(\mathcal{C}) \to \mathcal{S}h(\mathcal{C})$.

Schemes

Definition

- An affine scheme is a locally ringed space that is isomorphic to the spectrum of a commutative ring.
- A scheme is a locally ringed space that can be covered by affine schemes.

- Here we consider the smooth schemes over a nice scheme S.
- For instance, the smooth schemes over $S = Spec\mathbb{C}$ are exactly the schemes whose $\mathbb{C}$-points form a manifold.

We take simplicial presheaves on Sm_S : $sPre(S) = Fun(S^{op}, Sp)$. To sheafify our presheaves, we need a good Grothendieck topology.

Definition

Let Sm_S be the category of smooth schemes over a scheme S.

- an étale cover $p : U \to X$ is Nisnevich if it is surjective on k-points for all fields k.
- the Nisnevich topology on X is the topology generated by the finite families of étale morphisms $\{p_i : U_i \to X\}_{i \in I}$ such that there is a finite sequence $\emptyset \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_1 = X$ of finitely presented closed subschemes of X such that

$$\bigcup_{i \in I} p_i^{-1}(Z_m \backslash Z_{m+1}) \longrightarrow Z_m \backslash Z_{m+1}$$

  admits a section for $0 \leq m \leq n-1$.

The Nisnevich topology is weaker than the étale topology, even when $S = Spec\overline{k}$.
For instance, the n-th power map ($|n| > 1$) $\mathbb{C}^* \to \mathbb{C}^*$ is an étale cover, but not a Nisnevich cover.
The map on the residue fields of the generic point is

$$\mathbb{C}(x) \to \mathbb{C}(x)$$
$$x \to x^n$$

and this is not an isomorphism.
The localization of $sPre(S)$ with respect to the Nisnevich site is a category of sheaves, $Sh_{Nis}(S)$.

Distinguished squares

A pullback diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

of S-schemes in Sm_S is an elementary distinguished square if i is an open Zariski immersion, p is étale, and $p^{-1}(X \backslash U) \to X \backslash U$ is an isomorphism. The Nisnevich topology is generated by these elementary distinguished squares.

A presheaf F on Sm_S is a Nisnevich-fibrant sheaf if and only if for any elementary distinguished square, the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(U \times_X V) \end{array}$$

is cartesian and $F(\emptyset)$ is a final object.
We will actually consider Spc_S, the category of fibrant Nisnevich sheaves. This is equivalent to localizing $sPre(S)$ with respect to the collection $\{p^{-1}U \cup_U V \to X, \emptyset_{initial} \to \emptyset_{scheme}\}$

Perfectly balanced, as all things should be

The Nisnevich topology combines a lot of the nice properties of both étale and Zariski.

- An étale property : Any smooth pair (X, Y) in the Nisnevich topology is locally equivalent to a pair $(\mathbb{A}^n, \mathbb{A}^m)$.
- An étale property : Nisnevich cohomology can be computed using Cech cochains.
- A Zariski property : It's generated by distinguished squares.
- A Zariski property : the Krull dimension and Nisnevich cohomological dimension of a scheme are the same.

Inverting $\mathbb{A}_1$

- In classical homotopy, the unit interval is contractible.
- We want to keep only the sheaves such that the projection $X \times \mathbb{A}_1 \to X$ induces an equivalence $F(X \times \mathbb{A}_1) \to F(X)$.
- We localize the Nisnevich sheaves by the collection of morphisms $\{\mathbb{A}_1 \times X \to X\}$.
  This gives us the **unstable motivic homotopy category** $\mathcal{H}(S)$.

Universal property

The functor $Fun(\mathcal{H}(S), \mathcal{D}) \to Fun(Sm/S, \mathcal{D})$ is fully faithful.
A map from $\mathcal{H}(S)$ to $\mathcal{D}$ is the same as a map from Sm/S to $\mathcal{D}$ that inverts $\mathbb{A}_1$ and has Nisnevich descent.
A map $S \to T$ of base schemes induces $f^* : Sm/T \to Sm/S$, and this gives us a functor $\mathcal{H}(T) \to \mathcal{H}(S)$.

$$\begin{array}{ccc} Sm/T & \longrightarrow & \mathcal{H}(T) \\ \downarrow{\scriptstyle f^*} & & \downarrow \\ Sm/S & \longrightarrow & \mathcal{H}(S) \end{array}$$

Spheres

Let $\mathbb{G}_m = \mathbb{A}_1 \backslash \{0\}$. We want $T = \mathbb{A}_1/\mathbb{G}_m$ to be a sphere.

$$T \simeq S^1 \wedge \mathbb{G}_m.$$

This gives us the idea for the following generalization. The spheres in $\mathbb{A}_1$-homotopy theory are :

$$S^{p+q,q} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q}.$$

The suspension functor is also bigraded : $\Sigma^{p+q,q} X = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q} \wedge X$.

The following square is a homotopy pushout :

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}_1 \\ \downarrow & & \downarrow \\ \mathbb{A}_1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

Since $\mathbb{A}_1$ is contractible, $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$.
$\mathbb{A}^n \backslash \{0\} \simeq S^{2n-1,n}$.

- In the ordinary world, stabilizing the ∞-category of spectra is essentially inverting the suspension functor Σ.
- In the motivic world, you have to invert the bigraded suspension functors.

Definition

The stable motivic homotopy category over S is the homotopy limit of the diagram :

$$\dots \xrightarrow{Hom(T,-)} H_*(S) \xrightarrow{Hom(T,-)} H_*(S) \xrightarrow{Hom(T,-)} H_*(S).$$