Motives, p-adic Hodge theory, and topological Hochschild homology

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Mathematics in isolation

Jean Leray (1906-1998):

1. Research before WWII: PDEs with some applications from topology
2. 1940-1945: Prisoner of war camp, in Austria.
3. Does research in algebraic topology from scratch, because he didn't want to help German authorities and their war-related research.
4. Comes up with sheaves and spectral sequences (Leray sp. sec.)
5. Modern methods in algebraic topology are born!

Mathematics in isolation

André Weil (1906-1998):

1. Research in algebraic geometry and number theory.
2. 1939-1940: Gets sentenced to jail by the French authorities, because he didn't want to fight in the war.
3. Comes up with his well known conjectures.
4. Introduces the Jacobian variety and solves his conjectures for the case of curves.
5. The journey of modern algebraic geometry begins!

Weil's program

- Hasse-Weil zeta function of a smooth, projective variety X over $\mathbb{F}_p$:

$$\zeta_X(t) := \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}$$

- In order to solve the Riemann hypothesis for algebraic varieties over finite fields, Weil envisioned a program.
- The crucial part of this is the existence of a Lefschetz-type fixed point formula with respect to a conjectural cohomology theory for X.
- Grothendieck: That cohomology theory is étale cohomology, with ℓ-adic coefficients!

Étale cohomology

- A k-algebra homomorphism

$$B \to A = B[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

  is étale, if the associated Jacobian determinant $\{\partial f_i/\partial x_j\}$ is invertible, i.e. étale maps can be thought as algebraic maps that are "locally invertible" or satisfy an algebraic form of the "inverse function theorem".
- Étale maps over a scheme X form a site ("generalization of topology").
- Étale cohomology:= the right derived functor of the global sections functor $\Gamma(\cdot, \mathcal{O}_{X,\text{ét}})$.
- ℓ-adic cohomology:= take etale cohomology of the sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$, take the inverse limit over n, and invert ℓ, to get something over $\mathbb{Q}_\ell$
- For X over $\mathbb{F}_p$, this only works for $\ell \neq p$!!

Weil cohomology theories

Fix a smooth, proper scheme X.

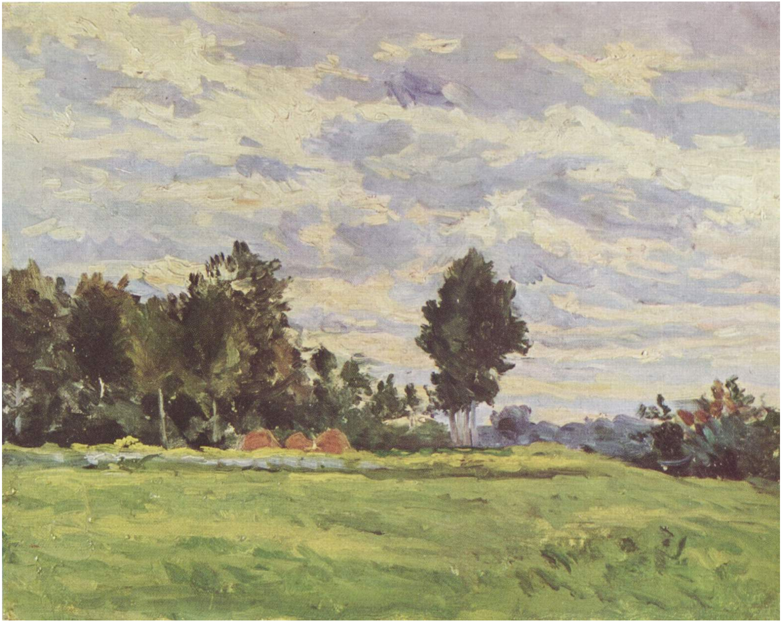
- Betti cohomology: $H_B^*(X) = H_{sing}^*(X(\mathbb{C}), \mathbb{C})$
- de Rham cohomology: $H_{\mathrm{dR}}^*(X)$
- ℓ-adic cohomology: $H^*(X, \overline{\mathbb{Q}}_\ell)$, lives over $\mathbb{Q}_\ell$.
- crystalline cohomology: $H_{\mathrm{cris}}^*(X)$

Grothendieck: There should be a common source of all these!

The motivic dream - Grothendieck

- Name comes from Paul Cézanne's impressionistic method of painting.

The motivic dream - Grothendieck



The motivic dream - Grothendieck

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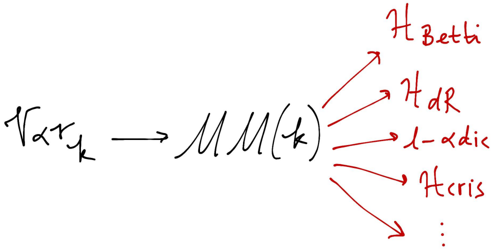
- "There exists a semisimple abelian category of pure (mixed) motives, which gives rise to the realizations of these cohomology theories.



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- Grothendieck: States his "standard conjectures" on algebraic cycles and motives.
- "There exists a semisimple abelian category of pure (mixed) motives, which gives rise to the realizations of these cohomology theories.
- A lot of the conectures about motives remain unproven - algebraic cycles are very difficult to manipulate!

The motivic dream - Beilinson, Deligne, and others...

- There should be a category of mixed motives (motivic sheaves) $\mathcal{MM}(k)$ on the category of algebraic varieties Var_k.
- Voevodsky: Partially realized this dream!
- Constructed a triangulated category $D\mathcal{M}(k)$, called "the derived category of mixed motives".
- Beilinson-Soulé vanishing conjecture:
  "There exists a conjectural t-structure, whose heart is the abelian category of mixed motives."

Classical Hodge theory

Fix X a smooth and projective complex variety.

Theorem (Period isomorphism)

$$H^*_{Betti}(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^*_{dR}(X)$$

Theorem (Hodge decomposition)

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X), \quad \overline{H^{p,q}} = H^{q,p}$$

p-adic Hodge theory I

- The main goal of the subject is to understand whether similar results hold for algebrogeometric objects of mixed-characteristic nature.
- Fix a smooth projective variety X over p-adic field K (finite extension of $\mathbb{Q}_p$).
- Fontaine (main developer of p-adic Hodge theory) conjectured similar comparison isomorhisms to the ones in the previous slide. These were proved by a variety of different methods, by: Faltings, Tsuji, Niziol, Scholze, Beilinson-Bhatt.

p-adic Hodge theory II

Theorem (Comparison isomorphisms)

$$H^*_{dR}(X/K) \otimes B_? \cong H^*_{ét}(X \times \overline{K}, \mathbb{Q}_p) \otimes B_?$$

- For $\mathbb{C}_K = \widehat{\overline{K}}$, we have Fontaine's period rings be:
  1. $B_{HT} := \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_K(i) \cong \mathbb{C}_K[t, t^{-1}]$, $\mathbb{C}_K$-algebra
  2. B_{dR} := filtered version of B_{HT}, with $gr^* B_{\mathrm{dR}} = B_{HT}$, complete field
  3. B_{cris} := Frobenius stable version of the above etc...
- The isomorphism respects the Galois group action and extra algebraic structure (graded, filtered, Frobenius, ...)

p-adic Hodge theory III

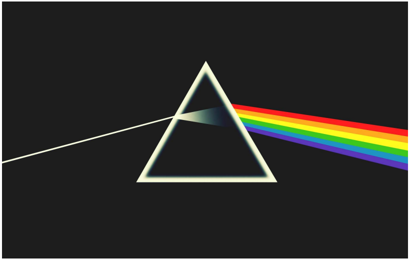
Here are some approaches:

1. Niziol: Using the theory of étale K-theory (developed by Thomason and others).
2. Scholze: Reworked older theories by Faltings and Fontaine, using his perfectoid rings approach:
   Perfectoid rings are a generalization of perfect rings into mixed characteristic, e.g. $\mathbb{C}_p$ or $\mathbb{Q}_p(p^{1/p^\infty})\hat{}$
3. Beilinson-Bhatt: First use of derived algebraic geometry tools in arithmetic geometry. Period rings and a "p-adic Poincaré lemma" via derived de Rham cohomology:

$$B_{\mathrm{dR}}^{+} \cong \widehat{\mathrm{dR}}_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} \widehat{\otimes} \mathbb{Q}_p$$

p-adic Hodge theory IV

- The last piece of the story is the new theory of prismatic cohomology by Bhatt-Scholze, which realizes a part of the motivic dream of Grothendieck (this is also with integral, not rational coefficients as in Fontaine's classical approach).



p-adic Hodge theory IV

- The last piece of the story is the new theory of prismatic cohomology by Bhatt-Scholze, which realizes a part of the motivic dream of Grothendieck (this is also with integral, not rational coefficients as in Fontaine's classical approach).
- But, how does topological Hochschild homology relate to all these?
- In fact, the first "attack" for introducing prismatic cohomology was done via the use of THH. Why?

Prismatic cohomology via *THH*, I

- Reason: Our little jewel, Böckstedt's computation.
  It sees integral coefficients as opposed to classical *HH*!!

$$THH(\mathbb{F}_p) \cong \mathbb{F}_p[u]$$

- Main computation in the paper of Bhatt-Morrow-Scholze - they lift this to the case of a perfectoid ring R:

$$THH(R; \mathbb{Z}_p) \cong R[u]$$

Prismatic cohomology via *THH*, II

How is this done?

1. Using the HKR filtration, prove that $\pi_i HH(u; \mathbb{Z}_p) \cong R$ for even $i \geq 0$ and 0 elsewhere.
2. Using the lemma relating *THH* and *HH*

$$THH(A) \otimes_{THH(\mathbb{Z})} \mathbb{Z} \cong HH(A)$$

3. Build up, using the above and base change arguments for *HH* and *THH*, from the case of $\mathbb{F}_p$ to perfect rings, and finally to the case of perfectoid rings.

Prismatic cohomology via *THH*, III

- If $A \to B$ a faithfully flat map of S-algebras, then wedge powers of the cotangent complex satisfy faithfully flat descent:

$$\wedge^i L_{A/S} \cong \lim \wedge^i L_{B \otimes_A \ldots \otimes_A B/S}$$

- Using the strategies of the previous slide (base change arguments and HKR) one can see that all variants of *HH* and *THH* satisfy faithfully flat descent.
- These ideas allow to expand "calculations" to a larger class of rings (quasiregular-semiperfectoid or quasisyntomic rings).

Prismatic cohomology via THH, IV

- For $S \in QRSPerfd_R$, one can *define* the Nygaard completed prismatic cohomology as:

$$\widehat{\Delta}_S := \pi_0 TC^-(S; \mathbb{Z}_p) \cong \pi_0 TP(S; \mathbb{Z}_p)$$

- Then extend to the case of a quasi-syntomic ring A, by:

$$\widehat{\Delta}_A := R\Gamma_{syn}(A, \pi_0 TC^-(\cdot; \mathbb{Z}_p))$$

Prismatic cohomology via THH, V

- For X^{top} space, we have the Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = H^{p-q}(X^{top}, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X^{top})$$

- For X smooth variety, we have a *motivic* Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

involving motivic cohomology and algebraic K-theory.

Prismatic cohomology via THH, VI

Theorem (BMS motivic filtrations)

For A quasisyntomic ring, we have filtrations giving rise to spectral sequences:

$$E_2^{p,q} = H^{p-q}(\mathcal{N}^{-q}\widehat{\Delta}_A) \Rightarrow \pi_{-p-q}THH(A;\mathbb{Z}_p)$$
$$E_2^{p,q} = H^{p-q}(\mathcal{N}^{\geq -q}\widehat{\Delta}_A\{-q\}) \Rightarrow \pi_{-p-q}TC^-(A;\mathbb{Z}_p)$$
$$E_2^{p,q} = H^{p-q}(\widehat{\Delta}_A\{-q\}) \Rightarrow \pi_{-p-q}TP(A;\mathbb{Z}_p)$$

These should be "*motivic*", ie independent of the p-completion.

The end

Thank you!