THH and Witt vectors (THH talk 5)

Foling Zou

University of Chicago

April 30, 2020

Previously

- For a ring R or E_1-ring spectrum R, one (Bökstedt–Hsiang–Madsen) defines an S^1-spectrum $\mathrm{THH}(R)$.
- $\mathrm{THH}(R)$ has a cyclotomic structure, which allows one to define $\mathrm{TC}(R)$.
- $$\begin{array}{ccc} & \nearrow & \mathrm{TC}(R;p) \\ & & \downarrow \\ \mathrm{K}(R;p) & \longrightarrow & \mathrm{THH}(R) \end{array}$$

TC in a nutshell

- (Blumberg–Mandell) $\mathrm{TC}(X;p) = \mathrm{hofib}(F - \mathrm{id} : \mathrm{TR}(X;p) \to \mathrm{TR}(X;p))$, where

$$\begin{array}{ll} \mathrm{TR}(R;p) & = \lim(\cdots \xrightarrow{R} (\mathrm{THH}(R))^{C_{p^{n+1}}} \xrightarrow{R} (\mathrm{THH}(R))^{C_{p^n}} \xrightarrow{R} \cdots) \\ \downarrow F & \\ \mathrm{TR}(R;p) & = \lim(\cdots \xrightarrow{R} (\mathrm{THH}(R))^{C_{p^n}} \xrightarrow{R} (\mathrm{THH}(R))^{C_{p^{n-1}}} \xrightarrow{R} \cdots) \end{array}$$

- (Nikolaus–Sholze)

$$\mathrm{TC}(R;p) = \mathrm{hoeq}\left(can, \phi_p : \begin{array}{ccc} \mathrm{TC}^-(R) & \longrightarrow & \mathrm{TP}(R)^\wedge \\ \| & & \| \\ (\mathrm{THH}(R))^{\mathrm{h}S^1} & & (\mathrm{THH}(R))^{\mathrm{t}S^1} \end{array} \right)$$

p-cyclotomic spectra

Definition (sometimes by S^1, we mean $\mathscr{F}_{p^\infty} = \{C_p, \cdots, C_{p^n}, \cdots\}$)

- In the Blumberg–Mandell (classical) sense:
  - X: S^1-spectrum;
  - $+$ an S^1-map $X \to \Phi^{C_p}(X)$ which is an equivalence on fixed points for the subgroups $C_{p^n} \subset S^1$.
- In the Nikolaus–Sholze (∞-category) sense:
  - X: spectrum with S^1-action;
  - $+$ an S^1-map $\varphi : X \to X^{tC_p}$.

Motivation

Let $\mathscr{L}X = \mathrm{Map}(S^1, X)$ for a space X. Then $\mathscr{L}X$ has an S^1-action and $(\mathscr{L}X)^{C_p} \cong \mathscr{L}X$. In fact, $\Sigma^\infty_{S^1}(\mathscr{L}X)_+$ is a cyclotomic spectra.

Example

$\mathrm{THH}(R)$ is a cyclotomic spectrum.

Tate square

For a G-spectrum X, there is a comparison of fiber sequences:

$$\begin{array}{ccccc} X_{\mathrm{h}G} & \longrightarrow & X^G & \longrightarrow & (\widetilde{EG} \wedge X)^G \\ \| & & \downarrow & & \downarrow \\ X_{\mathrm{h}G} & \xrightarrow{N} & X^{\mathrm{h}G} & \longrightarrow & X^{\mathrm{t}G} \end{array}$$

Taking $G = C_{p^n}$, we get a pullback diagram:

$$\begin{array}{ccc} X^{C_{p^n}} & \longrightarrow & (\Phi^{C_p}(X))^{C_{p^{n-1}}} \\ \downarrow & & \downarrow \\ X^{\mathrm{h}C_{p^n}} & \longrightarrow & X^{\mathrm{t}C_{p^n}} \end{array}$$

Definition

- In the Blumberg–Mandell (classical) sense:
  - X: S^1-spectrum;
  - $+$ an S^1-map $X \to \Phi^{C_p}(X)$ which is an equivalence on fixed points for the subgroups $C_{p^n} \subset S^1$.
- In the Nikolaus–Sholze (∞-category) sense:
  - X: spectrum with S^1-action;
  - $+$ an S^1-map $\varphi : X \to X^{\mathrm{t}C_p}$.

$$\begin{array}{ccc} X^{C_{p^n}} & \longrightarrow & (\Phi^{C_p}(X))^{C_{p^{n-1}}} \\ \downarrow & & \downarrow \\ X^{\mathrm{h}C_{p^n}} & \longrightarrow & X^{\mathrm{t}C_{p^n}} \end{array} \qquad \begin{array}{ccc} X^{C_{p^n}} & \longrightarrow & X^{C_{p^{n-1}}} \\ \downarrow & & \downarrow \\ X^{\mathrm{h}C_{p^n}} & \longrightarrow & (X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-1}}} \end{array}$$

Tate orbit lemma: $X^{tC_{p^n}} \simeq (X^{tC_p})^{hC_{p^{n-1}}}$ if X is bounded below.

Idea to comparing the two definitions:

- If we have a BM X, we can define φ to be $X \to \Phi^{C_p} X \to X^{\mathrm{t}C_p}$.
- If we have a NS X, we can define $X^{C_{p^n}}$ inductively as a pullback and build an S^1-spectrum from these fixed points. (Ref: Krause–Nikolaus, Prop 9.2)

Remark: "Tate stairs" (all squares are pullbacks)

$$\begin{array}{ccc} X^{C_{p^n}} & \longrightarrow & X^{C_{p^{n-1}}} \\ \downarrow & & \downarrow \\ X^{\mathrm{h}C_{p^n}} & \xrightarrow{\text{can}} & (X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-1}}} \end{array}$$

Remark: “Tate stairs” (all squares are pullbacks)

$$
\begin{array}{ccccc}
X^{C_{p^n}} & \longrightarrow & X^{C_{p^{n-1}}} & \longrightarrow & X^{C_{p^{n-2}}} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & X^{\mathrm{h}C_{p^{n-1}}} & \xrightarrow{\text{can}} & (X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-2}}} \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_p} & & \\
X^{\mathrm{h}C_{p^n}} & \xrightarrow{\text{can}} & (X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-1}}} & &
\end{array}
$$

Remark: “Tate stairs” (all squares are pullbacks)

$$
\begin{array}{ccccccc}
X^{C_{p^n}} & \xrightarrow{R} & X^{C_{p^{n-1}}} & \longrightarrow & X^{C_{p^{n-2}}} & \longrightarrow & \dots X \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_p} \\
 & & & & X^{\mathrm{h}C_{p^{n-2}}} & \xrightarrow{\text{can}} & \dots X^{\mathrm{t}C_p} \\
 & & & & \Big\downarrow{\scriptstyle \varphi_p} & & \\
 & & X^{\mathrm{h}C_{p^{n-1}}} & \xrightarrow{\text{can}} & (X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-2}}} & & \\
 & & \Big\downarrow{\scriptstyle \varphi_p} & & & & \\
X^{\mathrm{h}C_{p^n}} & \xrightarrow{\text{can}} & (X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-1}}} & & & &
\end{array}
$$

Definition

$$X^{C_{p^n}} \simeq X^{\mathrm{h}C_{p^n}} \times_{(X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-1}}}} X^{\mathrm{h}C_{p^{n-1}}} \times_{(X^{\mathrm{t}C_p})^{\mathrm{h}C_{p^{n-2}}}} \times \cdots \times_{X^{\mathrm{t}C_p}} X$$

Theorem [Hesselholt–Madsen]

For a connective commutative ring spectrum R, there are isomorphisms of rings

$$\pi_0(\mathrm{THH}(R))^{C_{p^n}} \cong W_{<p^n>}(\pi_0 R).$$

Moreover, this isomorphisms are compatible with F, R, V (to be defined).

Remark: If R is an E_1-ring spectrum, we still have isomorphisms of abelian groups.

Corollary

For any (associative) ring A,

$$\mathrm{TC}_{-1}(A; p) \cong W(A)_F.$$

(Fact: TC is (-2)-connected.)

Compare with

$$\mathrm{TC}_*(\mathbb{F}_p) \cong \mathbb{Z}_p[\epsilon]/\epsilon^2, \ |\epsilon| = -1.$$

Summary

Goal

$$\pi_0(\mathrm{THH}(R))^{C_{p^n}} \cong W_{<p^n>}(\pi_0 R).$$

- Analogy

Rings	Ring Spectra
$R_0 = \pi_0(R)$	R
$W_{\langle 1\rangle}(R_0)$	$\mathrm{THH}(R)$
$W_{\langle p^n\rangle}(R_0)$	$\mathrm{THH}(R)^{C_{p^n}}$

- Remark: $W_{\langle 1\rangle}(R_0) \cong \mathrm{HH}_0(R_0)$.
- Goal one: define a "Witt vector structure" (R, F, V, w, τ) on RHS.
- Goal two: compare RHS to LHS.

Witt vector: Big, S-truncated, p-typical

- Invented for: studying cyclic extension of fields in number theory.
- Appears in: p-adic Hodge theory; chromatic homotopy theory.
- Input and output: (W is right adjoint to the forgetful functor.)

$$
\begin{aligned}
W: \quad & \mathrm{CommRing} \quad \to \quad \Lambda{-}\mathrm{Ring} \quad (\to \quad \mathrm{CommRing}); \\
W_{\langle p^\infty \rangle}: \quad & \mathrm{CommRing} \quad \to \quad \delta{-}\mathrm{Ring} \quad (\to \quad \mathrm{CommRing});
\end{aligned}
$$

- W commutes with split coeqlizers. So it suffices to construct them on free rings.
- Big: coordinates indexed by $\mathbb{N}$ (NOT including 0).
- S-truncated: coordinates indexed only by $S \subset \mathbb{N}$.
- p-typical: $S = \langle p^\infty \rangle = \{1, p, p^2, \cdots\}$.
- n-truncated p-typical: $S = \langle p^n \rangle = \{1, p, \cdots, p^n\}$.

Coordinates: Witt, ghost, generating functions

p-typical Witt vectors

$$\prod_{k=0}^{n} R = W_{\langle p^n \rangle}(R) \xrightarrow{w} \prod_{k=0}^{n} R.$$

$$\begin{aligned} w_0 &= x_0; \\ w_1 &= x_0^p + p x_1; \\ w_2 &= x_0^{p^2} + p x_1^p + p^2 x_2; \cdots \end{aligned}$$

- - The LHS gives the Witt coordinates $(x_0, x_1, \cdots)$. It is NOT a ring map.
  - The RHS gives the ghost coordinates $(w_0, w_1, \cdots)$. It is a ring map.
  - If R is torsion free, w is injective.
  - Image of w can be identified by Dwork's lemma.
- Remark (Krause–Nikolaus): The formulas can be recovered by $-\mathrm{dlog}$ if we identify

$$(x_0, x_1, \cdots) \leftrightarrow \prod_{k=0}^{\infty}(1 - x_k t^{p^k}); \quad (w_0, w_1, \cdots) \leftrightarrow \sum_{k=0}^{\infty} w_k t^{p^k - 1}.$$

Structures on p-typical Witt vectors

Reduction: $R(w_0, w_1, \cdots, w_n) = (w_0, w_1, \cdots, w_{n-1})$

$R(x_0, x_1, \cdots, x_n) = (x_0, x_1, \cdots, x_{n-1})$

Frobenius: $F(w_0, w_1, \cdots, w_n) = (w_1, w_2, \cdots, w_n)$

Verschiebung: $V(w_0, w_1, \cdots, w_n) = (0, pw_0, pw_1, \cdots, pw_n)$

$V(x_0, x_1, \cdots, x_n) = (0, x_0, x_1, \cdots, x_n)$

Teichmüler: $\tau : R \to W_{\langle p^\infty \rangle}(R), \quad \tau(r) = (r, r^p, r^{p^2}, \cdots)$

$\tau_{p^n} : R \to W_{\langle p^n \rangle}(R), \quad \tau(r) = (r, r^p, r^{p^2}, \cdots, r^{p^n})$

- (Ref:9.8) $RF = FR$, $RV = VR$, $FV = p$ $(FV = \sum_{\sigma \in C_p} \sigma)$.
- (Ref:9.11) $R\tau_{p^n} = \tau_{p^{n-1}}$, $F\tau_{p^n} = \tau_{p^{n-1}} \circ (-)^p$.
- (Ref:B5) $(x_0, x_1, \cdots) = \sum_{k=0}^{\infty} V^k \tau(x_k)$.

Summary

Goal

$$\pi_0(\mathrm{THH}(R))^{C_{p^n}} \cong W_{<p^n>}(\pi_0 R).$$

- Analogy

Rings	Ring Spectra
$R_0 = \pi_0(R)$	R
$W_{\langle 1\rangle}(R_0)$	$\mathrm{THH}(R)$
$W_{\langle p^n\rangle}(R_0)$	$\mathrm{THH}(R)^{C_{p^n}}$

- Remark: $W_{\langle 1\rangle}(R_0) \cong \mathrm{HH}_0(R_0)$.
- Goal one: define a "Witt vector structure" (R, F, V, w, τ) on RHS.
- Goal two: compare RHS to LHS.

First goal

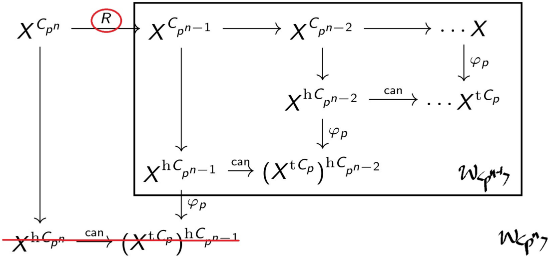
- Want to define
  - $R : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n-1}}}$;
  - $F : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n-1}}}$;
  - $V : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n+1}}}$;
  - $\tau : R \to \mathrm{THH}(R)^{C_{p^n}}$; ⋆
  - Ghost coordinate $w : \mathrm{THH}(R)^{C_{p^n}} \to \prod_{k=0}^{n} \mathrm{THH}(R)$;
  - Witt coordinates $\prod_{k=0}^{n} R \to \mathrm{THH}(R)^{C_{p^n}}$. ⋆
- Warning on names:

Witt	Equivariant homotopy	In brief
R	from Frobenius $\varphi_p : X \to X^{tC_p}$	the upper leg in "Tate"
F	restriction $X^{C_p} \to X$	⋆
V	transfer $X \to X^{C_p}$	⋆
$(\ FV = p\ )$	$(\ FV = \sum_{\sigma \in C_p} \sigma\ )$	

Note: ⋆=only on the 0-space. ⋆ = definition.

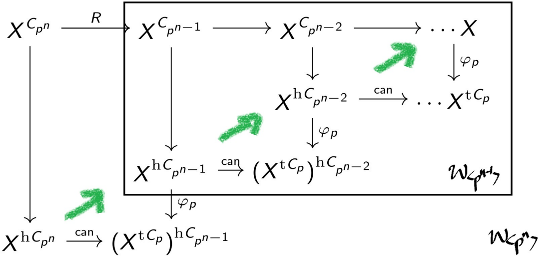
Reduction

Reduction is the upper leg in the Tate square.



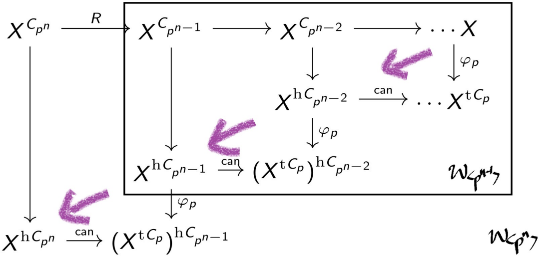
In the picture (pullback), just forget the last line. (This is analogous to the algebra case.)

Frobenius



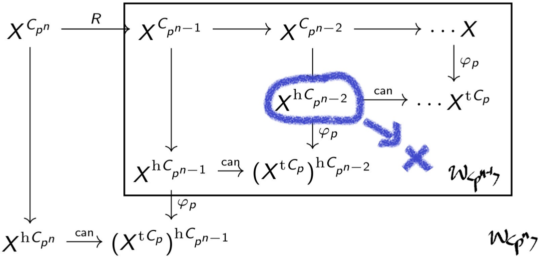
Remark: The arrows are induced by $X^{hC_p} \to X$, inclusion of fixed points.

Verschiebung



Remark: The arrows are induced by $X \to X_{\mathrm{h}C_p} \xrightarrow{N} X^{\mathrm{h}C_p}$, the transfer map.
V is well defined because $X \to X^{\mathrm{h}C_p} \xrightarrow{can} X^{\mathrm{t}C_p}$ is 0.

Ghost $w^{(n)} : X^{C_{p^n}} \to \prod_{k=0}^{n} X$

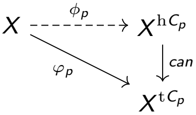


Definition

$$w_k^{(n)} = F^k R^{n-k}.$$

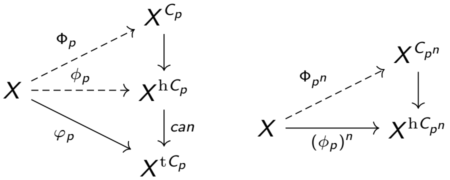
Teichmüler: ideally, $R \to \mathrm{THH}(R) \xrightarrow{?} \mathrm{THH}(R)^{C_{p^n}}$.

- Frobenius lift ϕ_p for a p-cyclotomic spectra X.



Teichmüler: ideally, $R \to \mathrm{THH}(R) \xrightarrow{?} \mathrm{THH}(R)^{C_{p^n}}$.

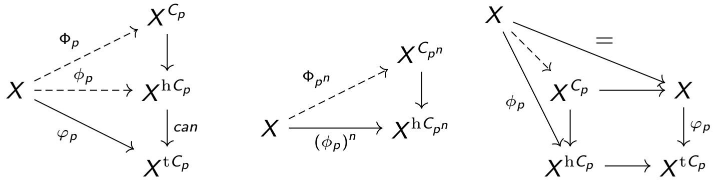
- Frobenius lift ϕ_p for a p-cyclotomic spectra X.
- Good case: If a cyclotomic spectra X admits a Frobenius lift ϕ_p,



Then there are canonical lifts Φ_{p^n} as displayed (Ref:9.9).

Teichmüler: ideally, $R \to \mathrm{THH}(R) \overset{?}{\to} \mathrm{THH}(R)^{C_{p^n}}$.

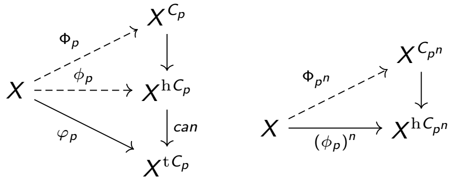
- Frobenius lift ϕ_p for a p-cyclotomic spectra X.
- Good case: If a cyclotomic spectra X admits a Frobenius lift ϕ_p,



Then there are canonical lifts Φ_{p^n} as displayed (Ref:9.9).
Proof: Tate diagram + induction.

Teichmüler: ideally, $R \to \mathrm{THH}(R) \overset{?}{\to} \mathrm{THH}(R)^{C_{p^n}}$.

- Frobenius lift ϕ_p for a p-cyclotomic spectra X.
- Good case: If a cyclotomic spectra X admits a Frobenius lift ϕ_p,



  Then there are canonical lifts Φ_{p^n} as displayed (Ref:9.9).

- $X = \mathrm{THH}(R)$ may not admit Frobenius lift;
  But $X = \mathrm{THH}(\mathbb{S}[G])$ always do. In fact, $\mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma^\infty_+ \mathscr{L}BG$ and φ_p is given by

$$\mathscr{L}BG \overset{``(-)^p"}{\to} (\mathscr{L}BG)^{\mathrm{h}C_p} \to (\mathscr{L}BG)^{\mathrm{t}C_p}.$$

- We have $\Phi_{p^n} : \mathrm{THH}(\mathbb{S}[\Omega^\infty R]) \to (\mathrm{THH}(\mathbb{S}[\Omega^\infty R]))^{C_{p^n}}$.

Teichmüler, cont.

- Adjunction

$$\mathbb{S} = \Sigma^{\infty}_{+} : E_1(\mathrm{Space}) \leftrightarrow E_1(\mathrm{Sp}) : \Omega^{\infty}$$

  - Ideally,

$$R \to \mathrm{THH}(R) \xrightarrow{?} \mathrm{THH}(R)^{C_{p^n}}$$

  - In fact,

$$\mathbb{S}[\Omega^{\infty} R] \to \mathrm{THH}(\mathbb{S}[\Omega^{\infty} R]) \xrightarrow{\Phi_{p^n}} (\mathrm{THH}(\mathbb{S}[\Omega^{\infty} R]))^{C_{p^n}}$$

Teichmüler, cont.

- Adjunction
$$\mathbb{S} = \Sigma^\infty_+ : E_1(\mathrm{Space}) \leftrightarrow E_1(\mathrm{Sp}) : \Omega^\infty$$
    - Ideally,
    $$R \to \mathrm{THH}(R) \xrightarrow{?} \mathrm{THH}(R)^{C_{p^n}}$$
    - In fact,
    $$\mathbb{S}[\Omega^\infty R] \to \mathrm{THH}(\mathbb{S}[\Omega^\infty R]) \xrightarrow{\Phi_{p^n}} (\mathrm{THH}(\mathbb{S}[\Omega^\infty R]))^{C_{p^n}} \xrightarrow{\epsilon} \mathrm{THH}(R)^{C_{p^n}}$$
    - Counit, E_1-map $\epsilon : \mathbb{S}[\Omega^\infty R] \to R$
- Adjoint to get
$$\tau_{p^n} : \Omega^\infty R \to \Omega^\infty(\mathrm{THH}(R)^{C_{p^n}})$$
- Remark: τ_1 is the expected map $(R \to \mathrm{THH}(R))$.
  Proof: play with the triangle identity of the adjunction.

Witt coordinate

Definition

$$I^{(n)} : \prod_{k=0}^{n} \Omega^\infty R \to \Omega^\infty(\mathrm{THH}(R)^{C_{p^n}});$$

$$(\alpha_k) \mapsto \sum_{k=0}^{n} V_{p^k} \tau_{p^{n-k}}(\alpha_k).$$

- Here, $\sum$ is addition on the 0-space of a spectrum.
- $I^{(n)}$ is only defined on the 0-space because τ is.
- $I^{(0)} = \tau_1$ is the canonical map $R \to \mathrm{THH}(R)$.

$$I^{(n)}: \prod_{k=0}^{n} \Omega^\infty R \to \Omega^\infty(\mathrm{THH}(R)^{C_{p^n}}).$$

Comparision and new goal

	Witt coordinate		abstract		ghost coordinate
topology	$\pi_0(\prod_{k=0}^n R)$	$\xrightarrow{\pi_0 I^{(n)}}$	$\pi_0(\mathrm{THH}(R)^{C_{p^n}})$	$\xrightarrow{\pi_0 w^{(n)}}$	$\pi_0(\prod_{k=0}^n \mathrm{THH}(R))$
	$\Vert$				$\Vert$
algebra	$\prod_{k=0}^n \pi_0 R$	$\xrightarrow{q}$	$W_{\langle p^n\rangle}(\pi_0 R)$	$\xrightarrow{w_{\langle p^n\rangle}}$	$\prod_{k=0}^n \pi_0 R$

- Want to show: the middle is an isomorphism.
- Step one: Reduce the general case to the torsion free case.
- If $\pi_0(R)$ is torsion free, we know q is surjective and w is injective in the algebra line. Step two: show the same thing for the topology line assuming torsion free. (Then the middle terms are isomorphic.)

Lemma (Ref:9.12)

The following sequence is exact:

$$\pi_0(\mathrm{THH}(R)) \xrightarrow{V^{n+1}} \pi_0(\mathrm{THH}(R)^{C_{p^{n+1}}}) \xrightarrow{R} \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \longrightarrow 0$$

Moreover, it is left exact if $\pi_0 R$ is p-torsion free.

- Why do we want this? For induction.
- How to see whether it should be R or F? In algebra, $W_{\langle 1\rangle} \xrightarrow{V^{n+1}} W_{\langle p^{n+1}\rangle} \xrightarrow{R} W_{\langle p^n\rangle} \to 0.$
  Using ghost coordinates,

$$\begin{aligned} V_{p^{n+1}}(r) &= (0, \cdots, 0, p^{n+1} r); \\ R_p(r_0, \cdots, r_{n+1}) &= (r_0, \cdots, r_{n-1}); \\ F_p(r_0, \cdots, r_{n+1}) &= (r_1, \cdots, r_n). \end{aligned}$$

Lemma (Ref:9.12)

The following sequence is exact:

$$\pi_0(\mathrm{THH}(R)) \xrightarrow{V^{n+1}} \pi_0(\mathrm{THH}(R)^{C_{p^{n+1}}}) \xrightarrow{R} \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \longrightarrow 0$$

Moreover, it is left exact if $\pi_0 R$ is p-torsion free.

- In topology, the Tate diagram gives fiber sequence of R:

$$\begin{array}{ccccc} \mathrm{THH}(R)_{\mathrm{h}C_{p^{n+1}}} & \longrightarrow & \mathrm{THH}(R)^{C_{p^{n+1}}} & \xrightarrow{R} & \mathrm{THH}(R)^{C_{p^n}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{THH}(R)_{\mathrm{h}C_{p^{n+1}}} & \xrightarrow{N} & \mathrm{THH}(R)^{\mathrm{h}C_{p^{n+1}}} & \longrightarrow & \mathrm{THH}(R)^{\mathrm{t}C_{p^{n+1}}} \end{array}$$

And $\pi_0\mathrm{THH}(R) \to \pi_0(\mathrm{THH}(R)_{\mathrm{h}C_{p^{n+1}}}) \to \pi_0(\mathrm{THH}(R)^{C_{p^{n+1}}})$ is the transfer map, so it is related to V.

$$I^{(n)} : \prod_{k=0}^{n} \Omega^\infty R \to \Omega^\infty(\mathrm{THH}(R)^{C_{p^n}}).$$

Comparision and new goal

	Witt coordinate		abstract		ghost coordinate
topology	$\pi_0(\prod_{k=0}^n R)$	$\xrightarrow{\pi_0 I^{(n)}}$	$\pi_0(\mathrm{THH}(R)^{C_{p^n}})$	$\xrightarrow{\pi_0 w^{(n)}}$	$\pi_0(\prod_{k=0}^n \mathrm{THH}(R))$
	$\Vert$				$\Vert$
algebra	$\prod_{k=0}^n \pi_0 R$	$\xrightarrow{q}$	$W_{\langle p^n \rangle}(\pi_0 R)$	$\xrightarrow{w_{\langle p^n \rangle}}$	$\prod_{k=0}^n \pi_0 R$

- Want to show: the middle is an isomorphism.
- Step one: Reduce the general case to the torsion free case.
- If $\pi_0(R)$ is torsion free, we know q is surjective and w is injective in the algebra line. Step two: show the same thing for the topology line assuming torsion free. (Then the middle terms are isomorphic.)

Step two: Show $\pi_0 I^{(n)}$ is surjective and $\pi_0 w^{(n)}$ is injective in the torsion free case.

Witt	$\prod_{k=0}^{n} \pi_0 R$
	$\downarrow \pi_0 I^{(n)}$
abstract	$\pi_0(\mathrm{THH}(R)^{C_{p^n}})$
	$\downarrow \pi_0 w^{(n)}$
ghost	$\prod_{k=0}^{n}(\pi_0 \mathrm{THH}(R))$
	topology

Step two: Show $\pi_0 I^{(n)}$ is surjective and $\pi_0 w^{(n)}$ is injective in the torsion free case.

$$\begin{array}{c} \pi_0(R) \\ \downarrow I^{(0)}=\tau_1 \\ \pi_0(\mathrm{THH}(R)) \\ \downarrow w^{(0)}=\mathrm{id} \\ \pi_0(\mathrm{THH}(R)) \end{array} \qquad\qquad \begin{array}{c} \prod_{k=0}^n \pi_0 R \\ \downarrow \pi_0 I^{(n)} \\ \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \\ \downarrow \pi_0 w^{(n)} \\ \prod_{k=0}^n (\pi_0 \mathrm{THH}(R)) \end{array}$$

base case

Step two: Show $\pi_0 I^{(n)}$ is surjective and $\pi_0 w^{(n)}$ is injective in the torsion free case.

$$
\begin{array}{lccccc}
\text{Witt} & \pi_0 R & \xrightarrow{\ ?\ } & \prod_{k=0}^{n+1} \pi_0 R & \xrightarrow{\ ?\ } & \prod_{k=0}^{n} \pi_0 R \\
 & \text{canonical}\downarrow & & \downarrow \pi_0 I^{(n+1)} & & \downarrow \pi_0 I^{(n)} \\
\text{abstract} & \pi_0(\mathrm{THH}(R)) & \xrightarrow{\ V\ } & \pi_0(\mathrm{THH}(R)^{C_{p^{n+1}}}) & \xrightarrow{\ R\ } & \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \\
 & ??\downarrow & & \downarrow \pi_0 w^{(n+1)} & & \downarrow \pi_0 w^{(n)} \\
\text{ghost} & \pi_0(\mathrm{THH}(R)) & \longrightarrow & \prod_{k=0}^{n+1}(\pi_0 \mathrm{THH}(R)) & \to & \prod_{k=0}^{n}(\pi_0 \mathrm{THH}(R)) \\
 & & & \text{inductive case} & & \text{inductive hypothesis}
\end{array}
$$

- The canonical map $R \to \mathrm{THH}(R)$ is also just $\tau_1 = I^{(0)}$;
- ? is i_{n+1} and $p_{0,\cdots,n}$ because by definition $V_{p^{n+1}} \tau_1 = I^{(n+1)} \circ i_{n+1}$.
- ?? is p^{n+1} because $w_{n+1} V_{p^{n+1}} = F_{p^{n+1}} V_{p^{n+1}} = p^{n+1}$.

Step two: Show $\pi_0 I^{(n)}$ is surjective and $\pi_0 w^{(n)}$ is injective in the torsion free case.

$$
\begin{array}{llcccc}
\text{Witt} & \pi_0 R & \xrightarrow{i_{n+1}} & \prod_{k=0}^{n+1} \pi_0 R & \xrightarrow{p_{0,\cdots,n}} & \prod_{k=0}^{n} \pi_0 R \\
 & \text{canonical}\downarrow & & \downarrow \pi_0 I^{(n+1)} & & \downarrow \pi_0 I^{(n)} \\
\text{abstract} & \pi_0(\mathrm{THH}(R)) & \xrightarrow{V} & \pi_0(\mathrm{THH}(R)^{C_{p^{n+1}}}) & \xrightarrow{R} & \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \\
 & p^{n+1}\downarrow & & \downarrow \pi_0 w^{(n+1)} & & \downarrow \pi_0 w^{(n)} \\
\text{ghost} & \pi_0(\mathrm{THH}(R)) & \longrightarrow & \prod_{k=0}^{n+1}(\pi_0 \mathrm{THH}(R)) & \to & \prod_{k=0}^{n}(\pi_0 \mathrm{THH}(R)) \\
 & & & \text{inductive case} & & \text{inductive hypothesis}
\end{array}
$$

- All rows are exact (The second row is the Lemma).
- By the snake lemma, p^{n+1} and $w^{(n)}$ being injective implies $w^{(n+1)}$ being injective.
- Similarly for $I^{(n+1)}$, expect that a prior $\pi_0 I$ may not be a group homomorphism. They are group homomorphism because wI is and w is injective. □

Summary

Goal

$$\pi_0(\mathrm{THH}(R))^{C_{p^n}} \cong W_{<p^n>}(\pi_0 R).$$

- Analogy

Rings	Ring Spectra
$R_0 = \pi_0(R)$	R
$W_{\langle 1 \rangle}(R_0)$	$\mathrm{THH}(R)$
$W_{\langle p^n \rangle}(R_0)$	$\mathrm{THH}(R)^{C_{p^n}}$

- Remark: $W_{\langle 1 \rangle}(R_0) \cong \mathrm{HH}_0(R_0)$.
- Goal one: define a "Witt vector structure" (R, F, V, w, τ) on RHS.
- Goal two: compare RHS to LHS.

First goal

- Want to define
  - $R : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n-1}}}$;
  - $F : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n-1}}}$;
  - $V : \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n+1}}}$;
  - $\tau : R \to \mathrm{THH}(R)^{C_{p^n}}$; $\star$
  - Ghost coordinate $w : \mathrm{THH}(R)^{C_{p^n}} \to \prod_{k=0}^{n} \mathrm{THH}(R)$;
  - Witt coordinates $\prod_{k=0}^{n} R \to \mathrm{THH}(R)^{C_{p^n}}$.$\star$
- Warning on names:

Witt	Equivariant homotopy	In brief
R	from Frobenius $\varphi_p : X \to X^{tC_p}$	the upper leg in "Tate"
F	restriction $X^{C_p} \to X$	$\star$
V	transfer $X \to X^{C_p}$	$\star$
$(\ FV = p\ )$	$(\ FV = \sum_{\sigma \in C_p} \sigma\ )$	

Note: $\star$=only on the 0-space. $\star$ = definition.

$$I^{(n)}: \prod_{k=0}^{n} \Omega^\infty R \to \Omega^\infty(\mathrm{THH}(R)^{C_{p^n}}).$$

Comparision and new goal

	Witt coordinate		abstract		ghost coordinate
topology	$\pi_0(\prod_{k=0}^n R)$	$\xrightarrow{\pi_0 I^{(n)}}$	$\pi_0(\mathrm{THH}(R)^{C_{p^n}})$	$\xrightarrow{\pi_0 w^{(n)}}$	$\pi_0(\prod_{k=0}^n \mathrm{THH}(R))$
	$\Vert$				$\Vert$
algebra	$\prod_{k=0}^n \pi_0 R$	$\xrightarrow{q}$	$W_{\langle p^n \rangle}(\pi_0 R)$	$\xrightarrow{w_{\langle p^n \rangle}}$	$\prod_{k=0}^n \pi_0 R$

- Want to show: the middle is an isomorphism.
- Step one: Reduce the general case to the torsion free case.
- If $\pi_0(R)$ is torsion free, we know q is surjective and w is injective in the algebra line. Step two: show the same thing for the topology line assuming torsion free. (Then the middle terms are isomorphic.)

The end

Thank you!