Topological Hochschild Homology Talk 4

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Outline

- Review of THH, cyclic bar construction, trace maps
- Cyclotomic spaces to cyclotomic spectra
- Computations: $\mathrm{TC}(\mathbb{F}_p)$, relationship to $\mathrm{THH}(\mathbb{F}_p)$

Overall goal

Compare Nikolaus-Scholze TC with Bökstedt-Hsiang-Madsen TC

Review

Define topological Hochschild homology as the geometric realization of the cyclic bar construction of a unital associative ring or ring spectrum A

$$\mathrm{THH}(A) := |B_{\mathrm{cyc}}(A)| =$$

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; A \otimes A \otimes A \circlearrowright C_3 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; A \otimes A \circlearrowright C_2 \;\substack{\longrightarrow\\\longrightarrow}\; A.$$

Review

We can consider ordinary rings as associative ring spectra. Given $R \in \mathrm{Alg}(\mathrm{Sp})$ be an associative ring spectrum, $\mathrm{THH}(R) := \mathrm{HH}(R/\mathrm{Sp})$. $\mathrm{Ass}_{act}^{\otimes}$ a symmetric monoidal category with objects finite sets.

- A morphism from S to T is given by a map $S \to T$ together with a linear ordering of the preimages for each $t \in T$.
- Composition is defined by composition of maps according to lexicographic ordering on preimages.
- Becomes symmetric monoidal under disjoint union.

We can then define $\mathrm{THH}(R)$ as the geometric realization of

$$\cdots \;\substack{\textstyle\rightarrow\\[-2pt]\textstyle\rightarrow\\[-2pt]\textstyle\rightarrow\\[-2pt]\textstyle\rightarrow}\; HR \otimes_{\mathbb{S}} HR \otimes_{\mathbb{S}} HR \;\substack{\textstyle\rightarrow\\[-2pt]\textstyle\rightarrow\\[-2pt]\textstyle\rightarrow}\; HR \otimes_{\mathbb{S}} HR \;\rightrightarrows\; HR$$

Proposition [NS17 IV.4.2] For any associative and unital ring A, the map

$$\pi_i \mathrm{THH}(HA) \to \pi_i H\mathrm{HH}(A) = H_i \mathrm{HH}(A)$$

is an isomorphism for $i \leq 2$

Proof.

In degrees ≤ 2 of $\mathrm{THH}(HA)$, the truncation only depends on

$$\tau_{\leq 0}(HA \otimes HA \otimes HA) \simeq \tau_{\leq 0} H\left(A \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} A \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} A\right),$$

$$\tau_{\leq 1}(HA \otimes HA) \simeq \tau_{\leq 1} H\left(A \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} A\right)$$

$$\tau_{\leq 2} HA \simeq \tau_{\leq 2} HA$$

.

□

Theorem. (Bökstedt). $\mathrm{THH}(\mathbb{F}_p) = \mathbb{F}_p[x]$.
Proof. Use Hopkins-Mahowald theorem about Thom spectra (*cf.* Weinan's talk).

Comparison to HH

Note that the canonical map

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x] \to \mathbb{F}_p\langle x\rangle = \mathrm{HH}_*(\mathbb{F}_p)$$

sends x to x and so it is zero in degrees $\geq 2p$. This is part of why THH is a 'better' invariant in characteristic p than HH relative to $\mathbb{Z}$.

THH as a cyclotomic spectrum

A cyclotomic spectrum is an S^1 -equivariant spectrum E with fixed points for all the finite cyclic-groups $C_p = Z/pZ \hookrightarrow S^1$ inside the circle group, and equipped with S^1 -equivariant identifications $E^{C_p} \overset{\approx}{\to} E$ of the C_p-fixed points with the full object.

Definition

A cyclotomic structure on a spectrum X is given by a $\mathbb{T}$-action together with a $\mathbb{T}$-equivariant map $\varphi_p : X \to X^{tC_p}$ for every prime p, the Frobenius.

- For every ring (or ring spectrum) R the spectrum $\mathrm{THH}(R)$ is a cyclotomic spectrum.
- Tate spectrum $X^{tG} := \mathrm{cofib}(X_{hG} \to X^{hG})$.

THH as a cyclotomic spectrum

Motivation. We have an abelian group A and want a diagonal map

$$A \to (A \otimes \ldots \otimes A)^{C_p}$$

$$a \mapsto \otimes \ldots \otimes a\,.$$

Problem: This assignment isn't additive.
Solution: Quotient by elements in the image of the norm map

$$(A \otimes \ldots \otimes A)^{C_p} / \text{ norms}$$

given for any C_p-module M by

$$M_{C_p} \to M^{C_p} \quad m \mapsto \sum_{g \in C_p} gm\,.$$

"Algebraic Tate cohomology" isomorphic to the group

$$\hat{H}^0\left(C_p, A \otimes \ldots \otimes A\right).$$

THH as a cyclotomic spectrum

Analogous statement in homotopy theory

Tate Diagonal

Theorem. [NS17] III.1.7 For every spectrum X there is a map $\Delta_p : X \to (X \otimes_{\mathbb{S}} \ldots \otimes_{\mathbb{S}} \mathrm{X})^{tC_p}$ with the following properties:

- The map is natural in X, that is it extends to a natural transformation between functors $\mathrm{Sp} \to \mathrm{Sp}$
- The transformation is lax symmetric monoidal.

- Δ_p is unique with respect to these properties.
- Δ_p exhibits for every bounded below spectrum X the spectrum $(X \otimes_{\mathbb{S}} \ldots \otimes_{\mathbb{S}} X)^{tC_p}$ as the p-completion of X.

Classical TC from THH

The Dennis trace $tr : K(A) \to \mathrm{THH}(A)$ lifts to homotopy spaces

$$\mathrm{tr}_{p^n} : K(A) \longrightarrow \mathrm{THH}(A)^{C_{p^n}}.$$

fixed under cyclic action for powers $n \geq 0$ of a prime p.

- Complicated construction: form TC as $\operatorname{holim}_n \mathrm{THH}(A)^{C_p n}$ of

$$\cdots \underset{R}{\overset{F}{\rightrightarrows}} \mathrm{THH}(A)^{C_{p^{n+1}}} \underset{R}{\overset{F}{\rightrightarrows}} \mathrm{THH}(A)^{C_{p^n}} \underset{R}{\overset{F}{\rightrightarrows}} \cdots$$

- Simplifies (results of) computations: cyclotomic trace

Theorem (Bökstedt-Hsiang-Madsen, 93)

The cyclotomic trace factors through the lifted Dennis trace

$$\mathrm{trc} : K(A) \longrightarrow \mathrm{TC}(A; p)$$

to yield topological cyclic homology.

Theorem

(McClure-Schwänzl-Vogt [MSV97]). *For an* $\mathbb{E}_\infty$*-ring spectrum* R *the map* $R \to \mathrm{THH}(R)$ *is initial among all maps from* R *to an* $\mathbb{E}_\infty$*-ring spectrum equipped with a* $\mathbb{T}$*-action (through* $\mathbb{E}_\infty$*-maps). That is the map* $R \to \mathrm{THH}(R)$ *induces an equivalence*

$$\mathrm{Map}^{\mathbb{T}}_{\mathbb{E}_\infty}(\mathrm{THH}(R), A) \to \mathrm{Map}_{\mathbb{E}_\infty}(R, A)$$

for each $\mathbb{E}_\infty$-algebra A with a $\mathbb{T}$-action.

Proof.

Use two key facts about $\mathbb{E}_\infty$-algebras

- (1) The coproduct in the category of $\mathbb{E}_\infty$-algebras is the tensor product
- (2) Filtered colimits of $\mathbb{E}_\infty$-algebras are formed on underlying spectra.

$$\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$$

$$X \simeq \varinjlim X_\bullet$$

□

The initial object is the colimit over S^1 of the constant diagram with value R in the category of $\mathbb{E}_\infty$-rings. Using the simplicial circle $\Delta^1/\partial\Delta^1$, with $n+1$-simplices in dimension n, we see that this colimit is equivalent to the colimit of the simplicial spectrum given in degree n by the $n+1$-fold coproduct of R with itself, which is $R^{\otimes n+1} = B_{\mathrm{cyc}}(R)$.

There is a unique $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-map φ_p as in the diagram

$$\begin{array}{ccc} R & \longrightarrow & \mathrm{THH}(R) \\ \Big\downarrow{\scriptstyle \Delta_p} & & \Big\downarrow{\scriptstyle \varphi_p} \\ (R \otimes \ldots \otimes R)^{tC_p} & \longrightarrow & \mathrm{THH}(R)^{tC_p} \end{array}$$

where the $\mathbb{T}$ -action on $\mathrm{THH}(R)^{tC_p}$ is given under the isomorphism $\mathbb{T} \cong \mathbb{T}/C_p$ by the residual action.

Proof.

Follows from the universal property of THH given in [MSV97] and the fact that $\mathrm{THH}(R)^{tC_p}$ admits a canonical $\mathbb{T}/C_p = \mathbb{T}$-action. □

Computing TC

Nikolaus-Scholze TC

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \left(X^{t\mathbb{T}}\right)^{\wedge}$$

The Nikolaus-Scholze construction of TC comes with

- a canonical map from the homotopy fixed points to the Tate construction $\mathrm{can} : X^{hG} \to X^{tG}$,
- and a nullhomotopy of the composite $X_{hG} \to X^{hG} \to X^{tG}$.

Computing TC

We have an LES

$$\cdots \longrightarrow \mathrm{TC}_i(\mathbb{F}_p) \longrightarrow \mathrm{TC}_i^-(\mathbb{F}_p) \xrightarrow{\mathrm{can}-\varphi} \mathrm{TP}_i(\mathbb{F}_p) \longrightarrow \cdots$$

We claim that the map $\mathrm{can} - \varphi$ is an isomorphism in all nonzero degrees. This follows from the fact that it is the difference of an isomorphism and a map which is divisible by p between two copies of $\mathbb{Z}_p$. In degree 0 it is the zero map. So $\mathrm{TC}_*(\mathbb{F}_p)$ is only nonzero in degrees 0 and -1 and in both cases given by $\mathbb{Z}_p$. Thus we obtain topological cyclic homology as

$$\mathrm{TC}_*(\mathbb{F}_p) = \mathbb{Z}_p[\epsilon]/\epsilon^2 \quad |\epsilon| = 1.$$