∞-Categories Are Not Scary

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April 21, 2020

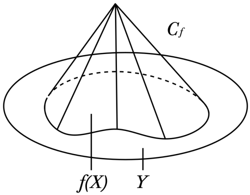
"Perhaps the purpose of categorical algebra is to show that which is trivial is trivially trivial." – Peter Freyd

Some Disclaimers

- I denote the smash product of spectra by $\otimes$ and the 1-object category with a group G as morphisms by BG. Sometimes it's important to know if a collection of things forms a set – I'm choosing not to mention these "size issues."
- I believe that anybody who cares about categories should care about ∞-categories for exactly the same reasons, and this talk reflects that belief. There are topologists I deeply respect who do not share this belief, at least one of whom is here.
- By far the most common models for ∞-categories in the literature are the quasicategories of Boardman-Vogt, expanded upon by Joyal and Lurie. In this talk, we'll use a handwavy idea in place of a serious model.
- This presentation is long and we might not get through the whole thing. The notes will be online and I'm always happy to talk more!

How do you work "up to homotopy"?

- Given a map $f : X \to Y$ of spaces, we as topologists like to construct the *mapping cone*, or *cofiber*, of f:



- This has a nice universal property! A map $Cf \to Z$ is the same as a map $Y \to Z$ and a chosen nullhomotopy of the composite $X \to Y \to Z$.
- This is inherently 2-categorical: we made use not only of the morphisms, but the homotopies between them!

How do you work "up to homotopy"?

- Given a space X and a group G acting on X, we can construct the homotopy fixed points

$$X^{hG} = \text{Maps}(EG, X).$$

- These also have a universal property: a map $Y \to X^{hG}$ is the same as a map $f : Y \to X$, and a choice of homotopy $H_g : f \to g \cdot f$ for each $g \in G$, and a choice of homotopy of homotopies $h \cdot H_g \circ H_h \to H_{gh}$ for each $g, h \in G$, and a choice of homotopy of homotopies of homotopies...
- The language of ∞-categories lets us organize all this information: we simply say X^{hG} is the (homotopy) limit of the functor $BG \to \text{Spaces}$ exhibiting X as a G-space.
  Exercise: what is the 1-categorical limit of this functor?

Groupoids

Recall that a homotopy *n*-type is a space X such that every map $S^m \to X$ is nullhomotopic for $m > n$.

- A homotopy 0-type is a disjoint union of contractible spaces, thus equivalent to a set (aka a 0-groupoid.)
- A homotopy 1-type is a disjoint union of $K(\pi, 1)$s.
- Similarly, every groupoid is equivalent to a disjoint union of BGs.
- In other words, the information needed to specify a groupoid or a homotopy 1-type is the same: a collection (possibly with repeats) of groups!
- Harder: the information needed to specify a 2-groupoid (that is, a bicategory with every morphism and 2-morphism invertible) is the same as specifying a homotopy 2-type.

∞-Groupoids

- An ∞-category is meant to be like a category, but with (invertible) homotopies between morphisms, and homotopies between those homotopies, and so on.
- If we ask that every morphism also be invertible, we get what's called a ∞-groupoid.
- Given a topological space X, we get an ∞-groupoid for free: the objects are points in X, the morphisms are paths in X, the homotopies are homotopies of paths, and so on.
- Conversely, given an ∞-groupoid, we can build a CW-complex: 0-cells are objects, 1-cells are morphisms, 2-cells come from homotopies, and so on.
- Grothendieck's Homotopy Hypothesis asserts that ∞-groupoids and spaces are the same thing!

What is an ∞-category?

- Ordinary categories are enriched in sets.
- Given objects X, Y, we have a set $\mathrm{Hom}(X, Y)$
- The composition operation induces an associative map of sets

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

What is an ∞-category?

- Topological categories are enriched in topological spaces.
- Given objects X, Y, we have a topological space $\mathrm{Hom}(X, Y)$
- The composition operation induces a continuous, associative map of spaces

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

What is an ∞-category?

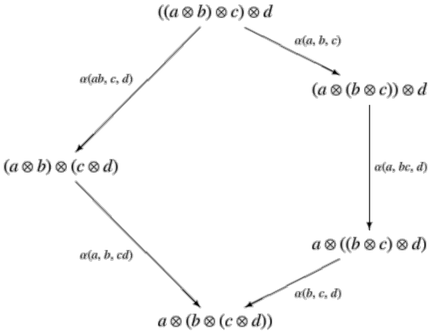
- ∞-categories are enriched in homotopy types, which are like spaces or simplicial sets "up to homotopy".
- Given objects X, Y, we have a homotopy type $\mathrm{Hom}(X, Y)$
- The composition operation induces an associative up to coherent homotopy map of homotopy types

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

(Note that "actually" associative is meaningless since we're dealing with homotopy types, so "up to coherent homotopy" is as good as we can do.)

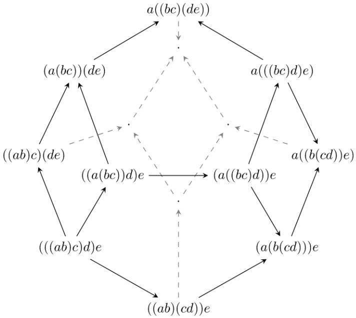
Coherent Homotopy?

- For operations on sets, associativity is a property: $(ab)c = a(bc)$.
- In categories, associativity is extra data $\alpha(a, b, c) : (ab)c \to a(bc)$ satisfying the properties:



Coherent Homotopy?

- In bicategories, the pentagon equation now becomes the additional data of a 2-cell, and there are further properties:



- In tricategories...

Coherent Homotopy?

- The main technical difficulty of ∞-categories comes from having to keep track of all the "extra" homotopies.
- Unsurprisingly, the tools that let us solve this problem for ∞-categories are related to similar tools in (e.g.) loop space theory:

Examples

- *An ∞-category with a unique (up to homotopy) object is the same as an A_∞ space.*
- *More generally: an ∞ category with a unique (up to homotopy) $k-$cell for $1 \leq k < n$ is the same as a (grouplike if $n > 1$) E_n space!*

Don't be evil

- In ordinary category theory, an evil definition is one that doesn't respect equivalence of categories.
- It's (usually) not okay to ask if two objects are equal – only if they're isomorphic (and really we ask *for a specified isomorphism* between the two.)
- It is okay to ask if two morphisms are equal, since Hom-sets are preserved by equivalences of categories.

Example (Evil definition of the empty set)

The empty set is the unique set ϕ such that if $X \simeq \phi$, then $X = \phi$.

Example (Non-evil definition of the empty set)

The empty set is the "unique" set ϕ such that, if $f, g : \phi \to X$ are morphisms, then $f = g$.

Don't be evil

- In ∞-category theory, we don't ask if two objects are equal – only if they're isomorphic.
- We don't ask if two morphisms are equal – only if they're homotopic. (Really, for a specified homotopy between them.)
- We don't ask if two homotopies are equal – only if they're (higher) homotopic.
- Whereas universal properties in categories specify objects "up to unique isomorphism", in ∞-categories objects will be specified "up to a contractible space of choices."
- So any two choices of the object are isomorphic, witnessed by an isomorphism that's unique up to homotopy, witnessed by a homotopy that's unique up to...

Category theory "up to homotopy"

All your favorite category ideas generalize!

- A *functor* between ∞-categories is a map on objects and Hom-spaces preserving identity and composition up to coherent homotopy.
- A *natural transformation* $F \to G$ is a collection of morphisms $F(c) \to G(c)$ for each c in the domain commuting with all the morphisms in the domain (up to coherent homotopy).
- Adjoint functors are pairs $F : C \to D$ and $G : D \to C$ with a natural equivalence $\mathrm{Hom}(Fx, y) \to \mathrm{Hom}(x, Gy)$.
- An *equivalence* $F : C \to D$ is a functor with a friend $G : D \to C$, so that FG and GF are both naturally isomorphic to the identity. F is an equivalence if and only if it is essentially surjective and fully faithful.
- Yoneda's lemma still holds! (With the caveat that representable functors now land in Spaces, not Set.)

The homotopy category

- Given a set, we get a homotopy type by treating it as a discrete space!
- This gives us a (fully faithful) inclusion of categories into ∞-categories, called (for quasicategories) the *nerve*.
- The nerve functor has a left adjoint $\tau_{\leq 1}$, usually called the "homotopy category" functor.
- If we think of morphisms $X \to Y$ as points in a space representing $\mathrm{Hom}(X, Y)$, we can think of homotopies as paths in this space.
- The homotopy category of an ∞-category is thus built by replacing each space $\mathrm{Hom}(X, Y)$ with its set of path components and keeping the same composition.
- Lots of things are defined on the homotopy category level, like isomorphisms! But other things are lost, like cofibers :(

Examples of ∞-categories

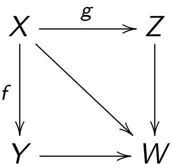
- Most ∞-categories “in nature” come from simplicial or topological categories, or ∞-categorical constructions applied to them.
- (You can think of the passage from a topological category to an ∞-category as a sort of cofibrant replacement, if you like that sort of thing.)
- The ∞-category of Spaces has CW-complexes as objects. The homotopy type of $\mathrm{Hom}(X, Y)$ comes from the mapping space $\mathrm{Maps}(X, Y)$.
- Fix a ring R. The derived ∞-category of R has (fibrant) chain complexes of projective modules as objects. The homotopy type of $\mathrm{Hom}(X, Y)$ comes from the chain complex $\underline{\mathrm{Hom}}(X, Y)$

Limits/Colimits

- One of the most important constructions in category theory is that of limits and colimits, and this remains true in the ∞-case.
- Most of the usual properties remain true: limits commute with limits, right adjoints preserve limits, you can write the limit of a category-shaped diagram as an equalizer of products, etc.
- You can find the general definition of homotopy limits/colimits in any notes on quasicategories, but instead I'd like to do a single example in detail.

Pushouts

Fix maps $f : X \to Y$ and $g : X \to Z$ in an ordinary category. Recall that a *pushout* of f and g is an initial object among commuting squares:



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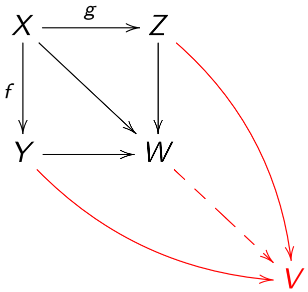
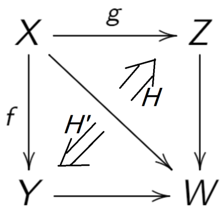


In words, a map out of W is the same as compatible maps out of X, Y, and Z.

Homotopy Pushouts

Fix maps $f : X \to Y$ and $g : X \to Z$ in an ∞-category. A (homotopy) pushout of f and g is an initial object among squares:



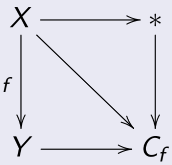
(Notice that we are *choosing* homotopies H and H'.)

In words, a map out of the pushout is the same as maps out of X, Y, and Z, *and choices of homotopies making those maps compatible.*

Homotopy Pushouts

Example

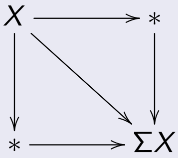
Let $f : X \to Y$ be a morphism in a pointed ∞-category (e.g. pointed spaces). The cofiber C_f of f is the pushout:



A map $C_f \to Z$ is the same as a map $Y \to Z$ with a specified nullhomotopy of the composite $X \to Y \to Z$.

Example

Let X be any object of a pointed ∞-category. The suspension $\Sigma X$of X is the pushout



A map $\Sigma X \to Y$ is the same as a map $X \to Y$ with two specified nullhomotopies.

Exercise: show that the suspensions in the ∞-category of spaces and the derived ∞-category are what you'd expect.

Universal Properties of ∞-categories

Theorem (Lurie, but maybe earlier?)

The ∞-category of spaces is formed by freely adjoining colimits to the contractible ∞-category $*$.

Theorem (Lurie)

The (connective) derived ∞-category of a ring R is formed by freely adjoining colimits of simplicial diagrams to the (ordinary) category of projective R-modules

Theorem (Elmendorf)

The ∞-category of G-spaces is formed by freely adjoining colimits to the (ordinary) category of G-orbits.

Localizations of ∞-Categories

- Given a ring R and a nice subset $S \subseteq R$, we can construct the localization $S^{-1}R$. A map out of $S^{-1}R$ is the same as a map out of R taking element of S to units.

Examples

- $(\mathbb{Z} - 0)^{-1}\mathbb{Z} = \mathbb{Q}$.
- $\{0\}^{-1}\mathbb{Z} = 0$.
- $\{2\}^{-1}\mathbb{Z} = \{\frac{a}{b} \mid b = 2^k\}$

- Given a category C and a nice collection W of arrows of C, we can (modulo set theory) construct the localization $C[W^{-1}]$. A functor out of $C[W^{-1}]$ is the same as a functor out of C taking arrows in of W to isomorphisms.

Localizations of ∞-Categories

Examples

- *Localizing a monoid (viewed as a one-object category) at all of its arrows gives its group completion.*
- *Localizing the category of spaces at the collection of weak equivalences gives the usual homotopy category.*
- *Localizing the category of chain complexes at the collection of quasi-isomorphisms gives the usual derived category.*
- *Localizing a presheaf category at the collection of covering sieves in some Grothendieck topology J gives the category of J-sheaves.*

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Localizations of ∞-Categories

- Given an ∞-category C and a nice collection W of arrows of C, we can (modulo set theory) construct the localization $C[W^{-1}]$. A functor out of $C[W^{-1}]$ is the same as a functor out of C taking arrows in of W to isomorphisms.

Examples

- *Localizing an A_∞ space (viewed as a one-object ∞-category) at all of its points gives its group completion.*
- *Localizing the (ordinary) category at the collection of weak equivalences gives the ∞-category of spaces.*
- *Localizing the (ordinary) category of chain complexes at the collection of quasi-isomorphisms gives the derived ∞-category.*
- *Localizing the ∞-category of spectra at the E_*-equivalences gives the ∞-category of E-local spectra.*

The ∞-category of Spectra

Denote the ∞-category of spaces by Spaces. To reach the stable world, we'd like to invert the suspension functor Σ, or equivalently its right adjoint Ω.

Definition (Lurie)

The ∞-category of spectra is the (homotopy) limit of the diagram

$$\cdots \xrightarrow{\Omega} \text{Spaces}_* \xrightarrow{\Omega} \text{Spaces}_* \xrightarrow{\Omega} \text{Spaces}_*$$

Exercise: What would we get if we replaced Ω with Σ, or the limit with a colimit?

A convenient category of spectra?

Theorem (Lewis 1991)

There is no category S such that:

1. *S is symmetric monoidal under a smash product $\otimes$.*
2. *There is an adjoint pair* $(\Sigma^\infty : \text{Spaces} \to S, \Omega^\infty : S \to \text{Spaces})$.
3. *$\Sigma^\infty S^0$ is a unit for $\otimes$.*
4. *Σ^∞ is colax monoidal.*
5. *There is a natural weak equivalence* $\Omega^\infty \Sigma^\infty X \to \operatorname{colim} \Omega^n \Sigma^n X$.

This is unfortunate because it implies that there is no category of spectra with all the properties we might expect. But maybe up to coherent homotopy...

A convenient category of spectra?

Theorem (Lurie, others)

The ∞-category of spectra...

- *...is symmetric monoidal under a smash product $\otimes$.*
- *...comes with a functor Ω^∞ to* Spaces, *with a left adjoint Σ^∞.*
- *... has $\Sigma^\infty S^0$ as a unit for $\otimes$.*

Furthermore,

- *Σ^∞ is symmetric monoidal.*
- *There is a natural weak equivalence* $\Omega^\infty \Sigma^\infty X \to \operatorname{colim} \Omega^n \Sigma^n X$.

The difficulties that arise from Lewis' theorem can't even be phrased in the language of ∞-categories.

Stable ∞-Categories

Definition

An ∞-category C is stable if:

1. C has a zero object (that is, an object which is both initial and terminal.)
2. Every morphism in C has a cofiber and a fiber. (A fiber of f is a pullback of f along the unique map from zero.)
3. A sequence $X \to Y \to Z$ in C is a cofiber sequence if and only if it is a fiber sequence.

The prototypical example of a stable ∞-category is the ∞-category of spectra! Other examples include collections of module spectra or derived ∞-categories.

Stable ∞-Categories

A stable ∞-category C come with a variety of nice properties, including:

- Finite coproducts in C coincide with finite products
- The homotopy category of C is uniquely enriched in abelian groups.
- (And C itself is naturally enriched in spectra!)
- The suspension functor $\Sigma : C \to C$ is an equivalence, with inverse Ω (the fiber of the unique morphism $0 \to X$.)
- The homotopy category of C is naturally a triangulated category.

The smash product of spectra

For a commutative ring R, the category of R-modules comes with a natural symmetric monoidal structure given by the tensor product.

Definition (Traditional)

$M \otimes N$ is the unique $R-$module such that $\mathrm{Hom}_R(M \otimes N, T)$ is naturally isomorphic to the set of bilinear maps $M \times N \to T$.

Definition (Computational)

$\otimes$ is the unique functor $\mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Mod}_R$ such that:

1. $R \otimes M \simeq M \otimes R \simeq M$ for all R-modules M.
2. $\otimes$ preserves colimits in each variable

The smash product of spectra

We can make the same definition, but for spectra!

Theorem (Lurie, Others??)

$\otimes$ is the unique symmetric monoidal ∞-functor $Sp \times Sp \to Sp$ such that:

1. $S \otimes M \simeq M \otimes S \simeq M$ *for all* M.
2. $\otimes$ *preserves homotopy colimits in each variable*

This looks very abstract, but an argument I learned from Dylan Wilson shows that this recovers our favorite "explicit" definitions. We'll sketch this for prespectra.

The smash product of spectra

1. Check that the functor $\Sigma^\infty : \mathrm{Spaces}_* \to \mathrm{Sp}$ is symmetric monoidal. (Using "Spaces is freely generated by colimits.")
2. Write a prespectrum $X = \{\Sigma X_i \to X_{i+1}\}$ as

$$X = \mathrm{colim}\, \Sigma^{-k}\Sigma^\infty X_k.$$

3. So

$$\begin{aligned} X \otimes Y &= (\mathrm{colim}_k\, \Sigma^{-k}\Sigma^\infty X_k) \otimes (\mathrm{colim}_\ell\, \Sigma^{-\ell}\Sigma^\infty Y_\ell.) \\ &= \mathrm{colim}_{k,\ell}\, \Sigma^{-k-\ell}\Sigma^\infty X_k \wedge Y_\ell \\ &= \mathrm{colim}_k\, \Sigma^{-2k}\Sigma^\infty X_k \wedge Y_k \end{aligned}$$

So we can identify the smash product of prespectra X and Y with W, where $W_{2k} = X_k \wedge Y_k$, and $W_{2k+1} = \Sigma W_{2k}$

Modules

- A(n A_∞) *ring spectrum* R is a monoid object in the symmetric monoidal ∞-category of spectra.
- A *module* over a ring spectrum R is a spectrum M and maps $R \otimes M \to M$ coherent with the multiplication on R.
- The ∞-category Mod_R of R-modules is naturally symmetric monoidal, where we define $\otimes_R$ to have unit R and preserve homotopy colimits in each variable.

Theorem (EKMM, Schwede-Shipley, others?)

Let R be an ordinary ring. The symmetric monoidal ∞-category of HR-modules is equivalent to the (unbounded) derived ∞-category of R.

Algebraic K-theory of (Connective) Rings.

Fix an ordinary commutative ring R.

- A (finitely-generated) *projective* R-module is a direct summand of R^n for some n.
- We obtain a set X_R from the category $\mathrm{Mod}_R^{\mathrm{proj}}$ by passing to isomorphism classes.
- X_R gets a commutative monoid structure by taking direct sums of modules. Group-completing X_R with respect to this structure gives us a group $K_0(R)$.
- How to define higher $K_i(R)$?

Algebraic K-theory of (Connective) Rings.

Fix a connective, commutative ring spectrum R.

- A (finitely-generated) *projective* R-module is a direct summand of R^n for some n.
- We obtain an ∞-groupoid X_R (hence a homotopy type) from the ∞-category $\mathrm{Mod}_R^{\mathrm{proj}}$ by throwing out all non-invertible morphisms.
- X_R gets a "commutative monoid" structure by taking direct sums of modules. Group-completing X_R with respect to this structure gives us a connective spectrum $K^{\mathrm{add}}(R)$.

Theorem (Lurie)

The natural map $K^{add}(R) \to K(R)$ is a homotopy equivalence.