Topological Hochschild Homology Talk 3

Tate Construction

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Structure of the talk

- Tate construction
- Tate cohomology and spectral sequences
- Relations with cyclotomic spectra and THH

Tate construction

- G is a finite group
  GSp: Category of genuine G-spectra
  Sp: Category of nonequivariant spectra
- $(-)^{tG} : GSp \to Sp$
- For any $X \in GSp$, we have a norm map $N : X_{hG} \to X^{hG}$.
  Homotopy orbit: $X_{hG} := EG_+ \wedge_G X$
  Homotopy fixed point: $X^{hG} := F(EG_+, X)^G$
  $X^{tG} := Cofib(N)$

Tate diagram

- Cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$.
  For any $X \in GSp$, $\epsilon : X \cong F(S^0, X) \to F(EG_+, X)$.
- Tate diagram:

$$\begin{array}{ccccc} X \wedge EG_+ & \longrightarrow & X & \longrightarrow & X \wedge \widetilde{EG} \\ \downarrow & & \downarrow & & \downarrow \\ F(EG_+, X) \wedge EG_+ & \longrightarrow & F(EG_+, X) & \longrightarrow & F(EG_+, X) \wedge \widetilde{EG} \end{array}$$

- Fact: $(-) \wedge EG_+$, $F(EG_+, -)$ only depends on the nonequivariant homotopy type.
  $\Rightarrow X \wedge EG_+ \cong F(EG_+, X) \wedge EG_+$

Tate diagram

Take the G-fixed point of

$$\begin{array}{ccccc} X \wedge EG_+ & \longrightarrow & X & \longrightarrow & X \wedge \widetilde{EG} \\ \downarrow & & \downarrow & & \downarrow \\ F(EG_+, X) \wedge EG_+ & \longrightarrow & F(EG_+, X) & \longrightarrow & F(EG_+, X) \wedge \widetilde{EG} \end{array}$$

Adams Isomorphism: $E \in GSp$, $Y \in Sp \subset Sp^G$. Then there is a natrual correspondence $[i^* Y, E]^G \cong [Y, E/G]$.

$$\begin{array}{ccccc} X_{hG} & \longrightarrow & X^G & \longrightarrow & (X \wedge \widetilde{EG})^G \\ \downarrow \cong & & \downarrow & & \downarrow \\ X_{hG} & \longrightarrow & X^{hG} & \longrightarrow & X^{tG} \end{array}$$

If $G = C_p$ with prime p, the upper right corner will be the geometric fixed point $\Phi^G(X)$.

Tate Construction

- $X_{hG} \to X^{hG} \to X^{tG}$ only depends on the non-equivariant homotopy type of X.
- $\Rightarrow$ Let $\iota : \mathbb{R}^\infty \to$ complete G-universe. Then the bottom row of the Tate diagrams of $X \in GSp$ and $\iota_* \iota^* X$ are equivalent.
- Sp^G: "Category of naive G-spectra" or "Category of spectra with G-actions"
  $Sp^G \cong Fun(\mathcal{B}G, Sp)$.
- $(-)_{hG}$ and $(-)^{hG}$ become $hocolim_{\mathcal{B}G}$ and $holim_{\mathcal{B}G}$.
  Left and right adjoint to $Sp \to Sp^G$, or equivalently,
  $F(*, Sp) \to Fun(\mathcal{B}G, Sp)$ induced by the unique functor $\mathcal{B}G \to *$.
- Norm map $X_{hG} \to X^{hG}$ can be constructed categorically.

Categorical Tate construction

- Functor $F : A \to B$, inducing $F^* : Fun(B, C) \to Fun(A, C)$, with special additional conditions.
  There exist left and right adjoints $F_!, F_*$ of F^*. And we can define norm map $F_! \to F_*$.
- Example 1: $C = Ab, A = \mathcal{B}G, B = *$. $F^* : Ab \to G - mod$. $F_!, F_*$ becomes the usual orbit and fixed points functors.
  Norm map $M_G \to M^G$ sending each $[m]$ to $\sum_{g \in G} gm$.
- Example 2: H is normal subgroup of G. Let $C = Sp, A = \mathcal{B}G, B = \mathcal{B}(G/H)$. $F_!, F_*$ agree with $(-)_{hH}, (-)^{hH}$.

$$\begin{array}{ccccc} X \wedge EG_+ & \longrightarrow & X & \longrightarrow & X \wedge \widetilde{EG} \\ \downarrow & & \downarrow & & \downarrow \\ F(EG_+, X) \wedge EG_+ & \longrightarrow & F(EG_+, X) & \longrightarrow & F(EG_+, X) \wedge \widetilde{EG} \end{array}$$

Get relative Tate spectrum functor $(-)^{tH} : Sp^G \to Sp^{G/H}$.

Spectral sequences

- $... \to X^{n-1} \to X^n \to X^{n+1} \to ....$
  $X^{-\infty}, X^{\infty}$ as holim and hocolim.
- Homotopy SS: $E^2_{p,q} = \pi_{p+q}(Cofib(X^q \to X^{q+1}))$ converging conditionally to $\pi_{p+q}(Cofib(X^{-\infty} \to X^{\infty}))$.
- $X \in Sp^G$. Let $X^q = \tau_{\geq q} X$ such that $\pi_i(X^q) = \pi_i(X)$ if $i \geq q$, otherwise it is zero.
  (In the case that $X \in GSp$, you want the same condition on $\underline{\pi}_i(X^q)$.)
  Get sequence $... \to X^q \to X^{q+1} \to ...$ with $X = X^{-\infty}$ and $* = X^{\infty}$.
- $Cofib(X^q \to X^{q+1}) = \Sigma^q H\pi_q(X)$.
  $E^2_{p,q} = \pi_{p+q}(\Sigma^q H\pi_q(X)) \Rightarrow \pi_{p+q}(X)$.
- Apply $(-)_{hG} \to (-)^{hG} \to (-)^{tG}$. Cofiber sequences are preserved!

Spectral sequences

- Homotopy orbit SS:

$$E^2_{p,q} = \pi_{p+q}((\Sigma^q H\pi_q(X))_{hG}) \cong H_p(G;\pi_q(X)) \Rightarrow \pi_{p+q}(X_{hG})$$

- Homotopy fixed point SS:

$$E^2_{p,q} = \pi_{p+q}((\Sigma^q H\pi_q(X))^{hG}) \cong H^{-p}(G;\pi_q(X)) \Rightarrow \pi_{p+q}(X^{hG})$$

- Tate SS:

$$E^2_{p,q} = \pi_{p+q}((\Sigma^q H\pi_q(X))^{tG}) \cong \hat{H}^{-p}(G;\pi_q(X)) \Rightarrow \pi_{p+q}(X^{tG})$$

- Tate cohomology: $\hat{H}^n(G;M)$ is defined as $H^n(G;M)$ if $n \geq 1$; $H_{-n+1}(G;M)$ if $n \leq -2$, with $\hat{H}^0(G;M) = coker(N)$ and $\hat{H}^{-1}(G;M) = ker(N)$.
Here $N : H_0(G;M) \to H^0(G;M)$ sending each a to $\sum_{g\in G} ga$.

Tate cohomology

- $H_*(G;M) := Tor_*^{\mathbb{Z}[G]}(\mathbb{Z}, M)$, $H^*(G;M) := Ext^*_{\mathbb{Z}[G]}(\mathbb{Z}, M)$.
- Projective $\mathbb{Z}[G]$ resolution:

$$... \to R_1 \to R_0 \to \mathbb{Z} \to 0$$

- Assume this resoluton is free. Take the "dual":

$$0 \to \mathbb{Z} \to Hom_{\mathbb{Z}}(R_0, \mathbb{Z}) \to Hom_{\mathbb{Z}}(R_1, \mathbb{Z}) \to ...$$

- $Hom_{\mathbb{Z}}(R_i, \mathbb{Z})$ has a natural free $\mathbb{Z}[G]$-module structure.
  The "dual" sequence is still an exact sequence of $\mathbb{Z}[G]$-modules.
- Glue up: Take $R_{-i} := Hom_{\mathbb{Z}}(R_{i-1}, \mathbb{Z})$ for $i > 0$, we have a **complete resolution**

$$... \to R_1 \to R_0 (\to \mathbb{Z}) \to R_{-1} \to R_{-2} \to ...$$

- $\hat{H}^n(G;M) = H^n(Hom_{\mathbb{Z}[G]}(R_*, M))$.

Multiplicative structure

- "Cup product": $\hat{H}^p(G; M_1) \otimes \hat{H}^q(G; M_2) \to \hat{H}^{p+q}(G; M_1 \otimes M_2)$.
- Sketch construction: Consider a free complete resolution

$$... \to R_1 \to R_0(\to \mathbb{Z}) \to R_{-1} \to R_{-2} \to ...$$

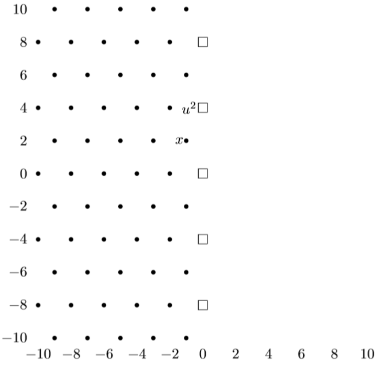
Let $d : R_i \to R_{i-1}$, $\epsilon : R_0 \to \mathbb{Z}$. We want to find maps $F_{p,q} : R_{p+q} \to R_p \otimes R_q$ such that $F_{p,q}d = (d \otimes id)F_{p+1,q} + (id \otimes d)F_{p,q+1}$, $((d \otimes id)F_{p,-p} + (id \otimes d)F_{p-1,1-p})d = 0$, and $(\epsilon \otimes \epsilon)F_{0,0} = \epsilon$. Get $F_{0,0}$ first. Induct on $|p|$ to get all $F_{p,-p}$. Then induct on $|p+q|$ to get general $F_{p,q}$.

- Back to the HFPSS and TSS:

$$HFPSS : E^2_{p,q} = H^{-p}(G; \pi_q(X)) \Rightarrow \pi_{p+q}(X^{hG})$$

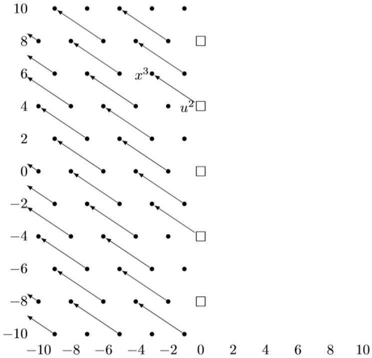
$$TSS : E^2_{p,q} = \hat{H}^{-p}(G; \pi_q(X)) \Rightarrow \pi_{p+q}(X^{tG})$$

Example of compuation

- Example: Computing KU^{tC_2}.
- $\pi_*(KU) = \mathbb{Z}[u^\pm]$, $|u| = 2$.
  $gu = -u$, where g is the generator of C_2.
- HFPSS: $E^2_{*,*} = \mathbb{Z}[x, u^{\pm 2}]/(2x)$, $|u^2| = (0, 4), |x| = (-1, 2)$.
  TSS: $E^2_{*,*} = (\mathbb{Z}/2)[x^\pm, u^{\pm 2}]$.
- 

Example of computation

HFPSS: $E^2_{*,*} = \mathbb{Z}[x, u^{\pm 2}]/(2x)$, $|u^2| = (0, 4), |x| = (-1, 2)$.
TSS: $E^2_{*,*} = (\mathbb{Z}/2)[x^{\pm}, u^{\pm 2}]$.

- $d_2 = 0$ for both. d_3 is computable for HFPSS (by the map from ANSS for S^0).
  $d_3(x) = 0$, $d_3(u^2) = x^3$.
- 

- $KU^{tG} = *$.

Relation with cyclotomic and THH

- Definition: $X \in Comm(\mathbb{T}Sp)$ is cyclotomic if $X \cong \Phi^{C_n} X$ for any n + compatibility conditions.
  $X \in Comm(C_{p^\infty}Sp)$ is p-cyclotomic if $X \cong \Phi^{C_p} X$.
  Use $CycSp$, $CycSp_p$ to denote the two categories.
- Consider the category $\mathcal{C}_p$ of $X \in Sp^{C_{p^\infty}}$ with map $X \to X^{tC_p}$.
- Natural functor $CycSp_p \to \mathcal{C}_p$ from $X \cong \Phi^{C_p} X \to X^{tC_p}$. It has a right adjoint and becomes a Quillen equivalence.
  Similarly for $CycSp$ and $\mathcal{C}$ of $X \in Sp^{\mathbb{T}}$ with a map for each p.
- $THH(A)$ is the underlying naive spectrum of some cyclotomic spectrum $\iff$ There exists $THH(A) \to THH(A)^{tC_p}$ for each p.

Tate diagonal

- p is a prime. $T_p : Sp \to Sp$, sending X to $(X \wedge X \wedge \ldots \wedge X)^{tC_p}$. The Tate diagonal Δ_p is a special natural transformation from *id* to T_p.
- In order to find $THH(A) \to THH(A)^{tC_p}$, consider