Topological Hochschild Homology, Talk 2

Computations of $\mathrm{HH}(\mathbb{F}_p)$ and $\mathrm{THH}(\mathbb{F}_p)$

Weinan Lin

The University of Chicago

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Review on $\mathrm{HH}(R)$

Let R be an associative and unital ring.

Definition

If R is flat, the *Hochschild homology groups* $\mathrm{HH}(R)$ of are defined as the homology groups of the chain complex

$$\mathrm{HH}(R) = (\cdots \to R \otimes R \otimes R \to R \otimes R \to R)$$

whose differential is given by

$$d(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i \cdots \otimes x_i x_{i+1} \otimes \cdots + (-1)^n x_n x_0 \otimes \cdots \otimes x_{n-1}$$

Review on $\mathrm{HH}(R)$

If R is not flat, we can replace R with a flat resolution $R_\bullet$ which is a DGA and define $\mathrm{HH}(R)$ as $\mathrm{HH}(R_\bullet)$. Equivalently, we replace tensor products $\otimes$ in $\mathrm{HH}(R)$ with derived tensor products $\otimes^L$.

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In previous talk, we have obtained the following equivalence using the resolution of R as an R-R bimodule

$$\mathrm{HH}(R) \simeq R \otimes^L_{R \otimes^L R^{op}} R.$$

We will use this to compute the Hochschild homology $\mathrm{HH}_*(\mathbb{F}_p)$.

Computation of $\mathrm{HH}_*(\mathbb{F}_p)$

Proposition

We get that

$$\mathrm{HH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[x, x^2/2!, x^3/3!, \ldots] = \mathbb{F}_p\langle x \rangle$$

is the free divided power algebra on a single generator x in degree 2.

In order to obtain $\mathrm{HH}_*(\mathbb{F}_p)$ we first compute the chain complex

$$\mathrm{HH}(\mathbb{F}_p) \simeq \mathbb{F}_p \otimes^L_{\mathbb{F}_p \otimes^L \mathbb{F}_p^{op}} \mathbb{F}_p.$$

Computation of $\mathrm{HH}_*(\mathbb{F}_p)$

The finite field $\mathbb{F}_p$ is not flat over $\mathbb{Z}$. However, we can find the flat resolution of $\mathbb{F}_p$

$$\mathbb{Z}[\varepsilon]/\varepsilon^2, \quad d\varepsilon = p,\ |\varepsilon| = 1.$$

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To compute $\mathbb{F}_p \otimes_A^L \mathbb{F}_p$ we resolve $\mathbb{F}_p$ as an A-algebra by

$$A\langle x\rangle = \frac{A[x_1, x_2, \dots]}{x_i x_j = \binom{i+j}{i} x_{i+j}}, \quad dx_i = \epsilon x_{i-1},\ |x_i| = 2i.$$

Tensoring this over A with $\mathbb{F}_p$ we get the desired result.

Review on $\mathrm{THH}(R)$

Let R be an associative ring spectrum. We define $\mathrm{THH}(R)$ as the geometric realization of a simplicial spectrum:

$$\mathrm{THH}(R) = \left| \cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} R \wedge R \wedge R \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} R \wedge R \substack{\longrightarrow\\\longrightarrow} R \right|,$$

and $\mathrm{THH}_*(R) = \pi_*(\mathrm{THH}(R))$.

If R is algebraic, $\mathrm{THH}(R) := \mathrm{THH}(HR)$.

THH of Thom Spectra

Theorem (Mahowald)

The Thom spectrum of the unique non-trivial double loop map

$$\Omega^2 S^3 \to BO \tag{1}$$

is equivalent to $H\mathbb{F}_2$.

The map (1) may be obtained by thrice looping the composite

$$\mathbb{H}P^\infty \simeq BSp(1) \to BSp \simeq B^5 O \xrightarrow{\eta} B^4 O$$

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Theorem (Hopkins)

For odd prime p, $H\mathbb{F}_p$ is equivalent to the Thom spectrum of certain 2-fold loop map

$$\Omega^2(S^3) \to BF_{(p)}.$$

THH of Thom Spectra

Theorem 1 (Blumberg-Cohen-Schlichkrull)

If the spherical fibration $f : X \to BF$ is a 2-fold loop map, then there is a stable equivalence

$$\mathrm{THH}(T(f)) \simeq T(f) \wedge T(\eta \circ Bf),$$

where $T(\eta \circ Bf)$ denotes the Thom spectrum of $BX \xrightarrow{Bf} B^2F \xrightarrow{\eta} BF$ and

$$\eta : B^2F \simeq \mathrm{Map}_*(S^2, B^4F) \xrightarrow{\eta^*} \mathrm{Map}_*(S^3, B^4F) \simeq BF.$$

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Theorem 2 (Blumberg-Cohen-Schlichkrull)

There is a stable equivalence

$$\mathrm{THH}(H\mathbb{F}_p) \simeq H\mathbb{F}_p \wedge \Omega(S^3)_+$$

for each prime p.

THH of Thom Spectra

Proof of Theorem 2 from Theorem 1

Consider the 2-fold loop map $f : \Omega^2(S^3) \to BF_{(p)}$, we get that

$$\mathrm{THH}(H\mathbb{F}_p) \simeq \mathrm{THH}(T(f)) \simeq H\mathbb{F}_p \wedge T(\eta \circ Bf).$$

Note that $T(\eta \circ Bf)$ is $H\mathbb{F}_p$-orientable since

$$\Omega S^3 \to BF_{(p)} \to B\mathbb{Z}_{(p)}^\times \to B\mathbb{F}_p^\times$$

is null homotopic because ΩS^3 is simply connected. Therefore we have the Thom isomorphism

$$H\mathbb{F}_p \wedge T(\eta \circ Bf) \simeq H\mathbb{F}_p \wedge \Omega(S^3)_+.$$

Corollary

Taking the homotopy groups of the equivalence in Theorem 2 we get

$$\pi_* \mathrm{THH}(\mathbb{F}_p) \cong H_*(\Omega S^3, \mathbb{F}_p) \cong \mathbb{F}_p[x]$$

where x has degree 2.