Topological Hochschild Homology, Talk 1

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Organization

- Hochschild homology: definitions, Hochschild-Kostant-Rosenberg theorem, Connes operator
- $\mathrm{HC}, \mathrm{HC}^{-}, \mathrm{HP}$: Cyclic, negative cyclic, and periodic homology
- A general construction, $\mathbb{T}$-action, Topological Hochschild homology

Background

Hochschild homology and computations of algebraic K-theory:

- (Goodwillie, 86) For a nilpotent extension of rings $f : B \to A$, we have a homotopy cartesian square:

$$\begin{array}{ccc} K(B) \otimes \mathbb{Q} & \longrightarrow & \mathrm{HC}(B|\mathbb{Q}) \\ \downarrow & & \downarrow \\ K(A) \otimes \mathbb{Q} & \longrightarrow & \mathrm{HC}(A|\mathbb{Q}) \end{array}$$

For p-adic version, need topological cyclic homology TC.

Background

Topological Hochschild homology:

- the topological version of Hochschild homology.
- (Bökstedt-Hsiang-Madsen, 93) Trace map: $K(R) \to \mathrm{TC}(R)$.
- (McCarthy, 97) For a nilpotent extension of rings $f : B \to A$, we have a homotopy cartesian square after completion at any prime p:

$$\begin{array}{ccc} K(B) & \longrightarrow & \mathrm{TC}(B) \\ \downarrow & & \downarrow \\ K(A) & \longrightarrow & \mathrm{TC}(A) \end{array}$$

- Integral p-adic Hodge theory, Noncommutative algebraic geometry and nc-Hodge theory...

Hochschild homology

- R: an associative and unital ring, flat over $\mathbb{Z}$
- Hochschild complex $\mathrm{HH}(R)$:

$$\cdots \to R \otimes R \otimes R \to R \otimes R \to R$$

  with differentials:

$$d(x_0 \otimes \cdots \otimes x_n) = \Sigma_0^{n-1} (-1)^i \cdots \otimes x_i x_{i+1} \otimes \cdots + (-1)^n x_n x_0 \otimes \cdots \otimes x_{n-1}$$

- Hochschild homology $\mathrm{HH}_*(R)$: homology of Hochschild complex

Examples

Let R be $\mathbb{Q}$:

- d: alternate between 0, id.
- $\mathrm{HH}_*(\mathbb{Q}) \cong \mathbb{Q}$.

Remark:

- For a general base field k: use $\otimes_k$ instead of $\otimes$ to define $\mathrm{HH}(R/k)$
- Remove the condition that R is flat over $\mathbb{Z}$: change $\otimes$ to the derived tensor $\otimes^L$
  - $\mathrm{HH}_*(\mathbb{F}_p) \neq \mathbb{F}_p$, since $\mathbb{F}_p$ is not flat over $\mathbb{Z}$.
  - A flat resolution of $\mathbb{F}_p$: $\mathbb{Z}[\epsilon]/\epsilon^2$, $d\epsilon = p$, $|\epsilon| = 1$.
  - $\mathrm{HH}_*(\mathbb{F}_p) \cong \mathbb{F}_p\{x, x^2/2, x^3/6 \cdots\} = \mathbb{F}_p\langle x\rangle$.

Bar complex

Distinguish Hochschild complex with Bar complex:

- Hochschild complex $\mathrm{HH}(R)$:

$$\cdots \to R \otimes R \otimes R \to R \otimes R \to R$$

  with differentials:

$$d(x_0 \otimes \cdots \otimes x_n) = \Sigma_0^{n-1}(-1)^i \cdots \otimes x_i x_{i+1} \otimes \cdots + (-1)^n x_n x_0 \otimes \cdots \otimes x_{n-1}$$

- $B_\bullet R$:

$$\cdots \to R \otimes R \otimes R \to R \otimes R$$

  with differentials

$$d'(x_0 \otimes \cdots \otimes x_{n-1}) = \Sigma_0^{n-1}(-1)^i \cdots \otimes x_i x_{i+1} \otimes \cdots$$

Tor description

- $B_\bullet R \xrightarrow{\simeq} R$ by the extra degeneracy

$$s(x_0 \otimes \cdots \otimes x_{n-1}) = 1 \otimes x_0 \otimes \cdots \otimes x_{n-1}.$$

- $B_\bullet R$ is a resolution of R by flat $R \otimes R^{op}$-modules.
- $R \otimes_{R \otimes R^{op}} B_\bullet R \cong \mathrm{HH}(R)$.
- $\mathrm{HH}(R)$ is a model for the derived tensor product $R \otimes^L_{R \otimes R^{op}} R$.
- Tor description: $\mathrm{HH}_*(R) \cong \mathrm{Tor}_*^{R \otimes R^{op}}(R, R)$.

Properties of HH

Assume flatness and commutativity:

- $\mathrm{HH}_*(R) \otimes_{\mathrm{HH}_*(T)} T \cong \mathrm{HH}(R/T)$.
- $\mathrm{HH}_*(R) \otimes T \cong \mathrm{HH}_*(R \otimes T/T)$.

Kähler differentials

Let R be a commutative ring:

- $\mathrm{HH}_0(R) = R$.
- $\mathrm{HH}_1(R) = \frac{ker(R^{\otimes 2} \to R)}{im(R^{\otimes 3} \to R^{\otimes 2})}$.

$\Omega^1_{R/k}$: Kähler differentials of R over k is a free R module with

- one generator dr for every $r \in R$.
- $d\alpha = 0$ if $\alpha \in k$.
- relations: $d(r+s) = dr + ds$, $d(rs) = r(ds) + s(dr)$.
- R commutative:

$$\mathrm{HH}_1(R) \simeq \Omega^1_{R/\mathbb{Z}} : r_1 \otimes r_2 \mapsto r_1 dr_2$$

Hochschild-Kostant-Rosenberg Theorem

- $\Omega^*_{R/\mathbb{Z}}$ Kähler forms: $\Omega^n_{R/\mathbb{Z}} \simeq \wedge_n \Omega^1_{R/\mathbb{Z}}$.
- R: commutative and smooth algebra over $\mathbb{Z}$.
- (Hochschild-Kostant-Rosenberg) The isomorphism $HH_1(R) \simeq \Omega^1_{R/\mathbb{Z}}$ extends to a natural graded ring map

$$\psi : \Omega^*_{R/\mathbb{Z}} \to \mathrm{HH}_*(R).$$

Connes operator B

Connes operator B:

- Analogous to the de Rham differential: $\Omega^n_{R/\mathbb{Z}} \to \Omega^{n+1}_{R/\mathbb{Z}}$
- Connes operator $B : R^{\otimes n+1} \to R^{\otimes n+2}$

$$B(r_0 \otimes \cdots \otimes r_n) = \Sigma_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)}((1 \otimes r_\sigma) - (-1)^n r_\sigma \otimes 1)$$

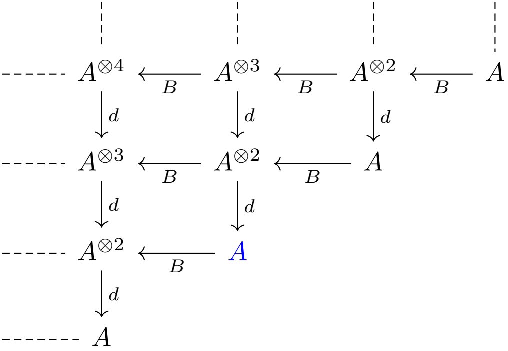
- $Bd = -dB$, i.e. B is a morphism of complexes $\mathrm{HH}(R) \to \mathrm{HH}(R)[1]$.
- $B^2 = 0$. B induces a derivation on HH_*.
- B turns HH_* into a commutative differential graded algebra.

The essential data

A complex $(A_\bullet, d, B)$ (algebraic S^1-complex):

- a chain complex $A_\bullet = \quad \cdots \overset{d}{\leftarrow} A_n \overset{d}{\leftarrow} A_{n+1} \overset{d}{\leftarrow} \cdots$
- $B : A_n \to A_{n+1}$ with $B^2 = 0$ and $Bd = -dB$.

Bicomplex



- HC: totalization of the positive subbicomplex
- HC^{-}: totalization of the negative subbicomplex
- HP: totalization of the bicomplex

Periodicity sequences

- The periodicity sequence for HC:

$$0 \to \mathrm{HH}(R) \xrightarrow{I} \mathrm{HC}(R) \to \mathrm{HC}(R)[2] \to 0.$$

- Periodicity gives a filtration and a spectral sequence:

$$E_1 = \mathbb{Z}[t] \otimes \mathrm{HH}_*(R) \implies \mathrm{HC}(R), |t| = 2$$

- Similarly for HC^-:

$$0 \to \mathrm{HC}^-(R)[-2] \to \mathrm{HC}^-(R) \xrightarrow{P} \mathrm{HH}(R) \to 0.$$

- $$0 \to \mathrm{HC}^-(R)[-2] \to \mathrm{HP}(R) \to \mathrm{HC}(R) \to 0$$

HC, HC^{-}, HP: Definition 2

Consider $A = \mathbb{Z}[b]/b^2, \quad |b| = 1$.
B equips the complex $\mathrm{HH}(R)$ with a structure of a diffential graded module over A.

- $\mathrm{HC}_*(R) := \mathrm{Tor}_*^A(\mathbb{Z}, \mathrm{HH}(R))$.
- $\mathrm{HC}_*^-(R) := \mathrm{Ext}_*^A(\mathbb{Z}, \mathrm{HH}(R))$.
- $\mathrm{HP}_*(R) := HC_*^-(R)[t^{-1}]$.
  ($\mathrm{HC}_*^-(R)$ is a module over $\mathrm{Ext}_*^A(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}[t], |t| = 2$.)

Equivalence of Def 1 and Def 2

$\mathrm{HC}_*(R) := \mathrm{Tor}_*^A(\mathbb{Z}, \mathrm{HH}(R))$.

- A free A resolution of $\mathbb{Z}$: $A[t], |t| = 2, dt = b$.

$$\cdots \to \mathbb{Z}\{t^2\} \xrightarrow{t^2 \mapsto bt} \mathbb{Z}\{bt\} \xrightarrow{0} \mathbb{Z}\{t\} \xrightarrow{t \mapsto b} \mathbb{Z}\{b\} \xrightarrow{0} \mathbb{Z}$$

- $\mathrm{Tor}_*^A(\mathbb{Z}, \mathrm{HH}(R))$ can be viewed as the same bicomplex used in Definition 1.
- Similar for HC^- and HP.

HC, HC^-, HP: Definition 3

Use $\mathbb{T}$ to denote the topological group S^1.

- There is a $\mathbb{T}$ action on $\mathrm{HH}_*(R)$.
- HC, HC^-, HP can be defined as the homotopy orbit, homotopy fixed point, Tate fixed point of HH_*.
- What is the $\mathbb{T}$ action?

A general construction

- $\mathrm{Ass}_{act}^{\otimes}$ a symmetric monoidal category.
  - objects: finite sets.
  - morphisms: $S \to T$ with a linear ordering on the preimages for each T.
  - composition: lexicographic ordering

- An associative algebra R in a symmetric monoidal category $\mathscr{C}$ $\leftrightsquigarrow$ a symmetric monoidal functor $\mathrm{Ass}_{act}^{\otimes} \to \mathscr{C} : [1] \mapsto R$.

Cyclic object

- Λ: cyclic category
  - objects: finite sets $[n]$ viewed as n dots on S^1
  - morphisms: $S^1 \to S^1$, degree 1, dots to dots, monotonous.

- Λ is self dual, by thinking of points as strings and vice versa.
- There is a functor: $\Delta^{op} \to \Lambda^{op} : [n] \to [n+1]$.
- $\Lambda \to \mathbf{Fin}$ refines to $\Lambda \to \mathrm{Ass}^{\otimes}_{act}$.

Classifying space of Λ

The classifying space $|N\Lambda| \simeq B\mathbb{T}$.

- The category Λ_∞
  - objects: $[n]$ viewed as $\frac{1}{n}\mathbb{Z}$ with $\mathbb{Z}$ action given by addition.
  - morphisms: monotonous
- Λ_∞ is contractible.
- $\Lambda \simeq \Lambda_\infty/B\mathbb{Z}$.
- $|N\Lambda| \simeq |N\Lambda_\infty|/B\mathbb{Z} \simeq BB\mathbb{Z} \simeq B\mathbb{T}$

$\mathbb{T}$-action on HH

- $\Delta^{op} \to \Lambda^{op} \simeq \Lambda \to \mathrm{Ass}^{\otimes}_{act}$.
- $\mathscr{C}$: be a symmetric monoidal ∞-category.
- $R \in Alg(\mathscr{C}) \leftrightsquigarrow$ a symmetric monoidal ∞-functor $N\mathrm{Ass}^{\otimes}_{act} \to \mathscr{C}$.
- Cyclic bar complex: $\mathrm{HH}(R/\mathscr{C})_\bullet := N\Delta^{op} \to N\mathrm{Ass}^{\otimes}_{act} \xrightarrow{R} \mathscr{C}$.
- Hochschild homology: $\mathrm{HH}(R/\mathscr{C}) := |\mathrm{HH}(R/\mathscr{C})_\bullet|$.
- $|N\Lambda| \simeq B\mathbb{T} \implies \mathrm{HH}(R/\mathscr{C}) \in \mathrm{Fun}(B\mathbb{T}, \mathscr{C})$.

HC, HC^-, HP: Definition 3

(More in talk 4.)

- $\mathrm{HC}(R) := \mathrm{HH}(R)_{h\mathbb{T}}$
- $\mathrm{HC}^-(R) := \mathrm{HH}(R)^{h\mathbb{T}}$
- $\mathrm{HP}(R) := \mathrm{HH}(R)^{t\mathbb{T}}$
- Homotopy orbit SS:

$$E_1 = H_*(B\mathbb{T};\mathbb{Z}) \otimes \mathrm{HH}_*(R) \simeq \mathbb{Z}[t] \otimes \mathrm{HH}_*(R) \implies \mathrm{HC}(R)$$

This agrees with the periodicity sequence SS:

Topological Hochschild homology

$\mathrm{HH}(R/\mathscr{C})$ in algebra:

- $\mathscr{C} = \mathcal{D}(\mathbb{Z})$, $R \in Alg(\mathscr{C})$ a differential graded algebra:
  $\implies \mathrm{HH}_*(R)$

$\mathrm{HH}(R/\mathscr{C})$ in topology:

- $\mathscr{C} = \mathbf{Sp}$, $R \in Alg(\mathscr{C})$ a E_1 ring spectrum:
  $\implies \mathrm{THH}(R)$
- $\mathrm{THH}_*(R) := \pi_*\mathrm{THH}(R)$
- Notation: for a ring R, $\mathrm{THH}(R) := \mathrm{THH}(HR)$.
  $H : \mathcal{D}(\mathbb{Z}) \to \mathbf{Sp}$, Eilenberg-Maclane spectrum functor.

THH

THH is the HH of algebras in $\mathbf{Sp}$

- Similarly can be defined using Hochschild complex or Tor:

$$\mathrm{THH}(R) = |\cdots \to R \otimes_{\mathbb{S}} R \otimes_{\mathbb{S}} R \to R \otimes_{\mathbb{S}} R \to R|$$

- Example: $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$.
- Relative version: B an E_∞ ring spectrum. $\mathscr{C} = B{-}\mathbf{Mod}$. $\implies \mathrm{THH}(R/B)$.
  - $\mathrm{THH}(A \otimes_{\mathbb{S}} B) = \mathrm{THH}(A) \otimes_{\mathbb{S}} \mathrm{THH}(B)$.
  - $\mathrm{THH}(A/B) = \mathrm{THH}(A) \otimes_{\mathrm{THH}(B)} B$.
  - $\mathrm{THH}(A) \otimes_{\mathbb{S}} B = \mathrm{THH}((A \otimes_{\mathbb{S}} B)/B)$.

THH

Compared to HH, THH has better calculation answers .

- $H : \mathcal{D}(\mathbb{Z}) \to \mathbf{Sp}$ is lax symmetric monoidal.
  $\implies$ There is a map $\mathrm{THH}_*(HR) \to \mathrm{HH}_*(R)$.
- (Bökstedt) $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$.
  Compare $\mathrm{HH}_*(\mathbb{F}_p) = \mathbb{F}_p\langle x\rangle$.
- In general, $\mathrm{THH}_*(HR) \to \mathrm{HH}_*(R)$ is 3-connected.
- $\mathrm{TC}_*^-(\mathbb{F}_p) \cong \mathbb{Z}_p[x,t]/(xt-p), |x| = 2, |t| = -2.$
  $\mathrm{TP}_*(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm}].$
  while
  $\mathrm{HC}_*^-(\mathbb{F}_p) \cong \mathbb{Z}_p[t]\langle\langle x\rangle\rangle/(xt-p), |x| = 2, |t| = -2.$
  $\mathrm{HP}_*(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm}]\langle\langle x\rangle\rangle/(xt-p).$

Thank you!