Galois Extensions in Homotopy Theory

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Henceforth G is a finite group. Rognes' theory works with topological groups that are stably dualizabile ($DG_+ \simeq G_+$) but some modifications are needed. Also, all rings are commutative and all ring spectra are E_∞.

1 Galois extensions of fields

Finite extension of fields: $E = F[a_1, ..., a_n]$ where a_i are roots of F-polynomials.

Separable extension: Irreducible F-polynomials can only have simple roots in E.

Any finite separable extension is of the form $E = F[a]$.

Normal extension: Any F polynomial with a root in E splits.

Galois extension: Separable + Normal.

Let $G = Aut_F(E)$ be the Galois group of a Galois extension $F \to E$. Write $E = F[a]$; G permutes the roots of the minimal polynomial of a, say $\mu(x) = (x - a_1) \cdots (x - a_n)$ where $a_i \neq a_j$ for $i \neq j$. Then

$$E \otimes_F E = E \otimes_F \frac{F[x]}{\mu(x)} = \frac{E[x]}{\mu(x)} = \prod_G E$$

$$e_1 \otimes e_2 \mapsto (e_1 g e_2)_{g \in G}$$

The converse is also true i.e.: Given a field extension $F \to E$ and $G \subseteq Aut_F(E)$, the extension is G Galois iff $F = E^G$ and

$$E \otimes_F E \to \prod_G E$$

$$e_1 \otimes e_2 \mapsto (e_1 g e_2)_{g \in G}$$

is an isomorphism.

2 Galois extensions of (commutative) rings

Let S be an R-algebra and $G \subseteq Aut_R(S)$ (i.e. G acts an the left on S through R-algebra maps). Then consider the maps

- $i : R \to S^G$
- $h : S \otimes_R S \to \prod_G S$

$R \to S$ is a G-Galois extension if both i, h are isomorphisms. In that case, S is invertible over R.

Proof. Define the trace $tr : S \to R$, $tr(s) = \sum_g gs$. There are $x_i, y_i \in S$ s.t. $\sum_i x_i y_i = 1$ and $\sum_i x_i g y_i = 0$ if $g \neq 1$. We have $s = \sum_i x_i tr(s y_i)$ for any $s \in S$. Thus $S \to R^n$, $s \mapsto tr(s y_i)$, is a split injection. ♠

Example: For a G-Galois extension of number fields $F \to E$, the corresponding extension of rings of integers $\mathcal{O}_F \to \mathcal{O}_E$ is G-Galois iff $F \to E$ is unramified (i.e. for any prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ the factorization of $\mathfrak{p}\mathcal{O}_E$ consists no duplicate prime ideals of $\mathcal{O}_E$).

3 Galois extensions in homotopy theory

Henceforth $\mathcal{C}$ is a stable presentable symmetric monoidal ∞-cat where $\otimes$ commutes with colimits. Examples:

- Spectra with $\wedge$.
- Modules over a fixed E_∞-ring spectrum E with $\wedge_E$.
- Spectra that are local w.r.t. a fixed spectrum E with $L_E(- \wedge -)$.

Note: If L_E is smashing then we don't need to localize the wedge and $\mathbf{Mod}(L_E S) =$ E-local spectra. That's not true for non smashing localization (eg $E = K(n)$).

Let A be a commutative algebra object with a G-action by A-algebra maps. Consider the maps

- $i : 1 \to A^{hG}$ ($(-)^{hG}$ is the right adjoint of trivial action $\mathcal{C} \to \mathcal{C}^{BG}$)
- $h : A \otimes A \to \prod_G A$ ($\prod_G$ is the right adjoint of forgetful $\mathcal{C}^{BG} \to \mathcal{C}$)

$1 \to A$ is a G-Galois extension if both maps are equivalences. When that happens, A is invertible.

The Galois extension is faithful if $A \otimes -$ reflects exact sequences. This is equivalent to $A^{tG} = *$.

For an A-algebra B, we want the maps $i : A \to B^{hG} = F(EG_+, B)^G$ and $h : B \wedge_A B \to \prod_G B = F(G_+, B)$ to be equivalences to have a Galois extension. For local A-algebras, we want them to be A_*-equivalences.

Examples:

- If $R \to S$ is a G-Galois extension of rings then $HR \to HS$ is a G-Galois extension of spectra. In fact it is faithful (this is because a Galois extension of rings is automatically faithful).

- Complexification $KO \to KU$ is a faithful C_2-Galois extension.
- $L_{K(n)}S \to E_n$ is a faithful G_n-Galois extension in the $K(n)$-local category. Here G_n is the Morava stabilizer group

$$G_n = Aut_{\mathbb{F}_p^n}(\Gamma_n) \rtimes Gal(\mathbb{F}_{p^n}, \mathbb{F}_p)$$

where Γ_n is the height n Honda FGL (this is a profinite group!). We have the Hopkins-Devinatz result

$$L_{K(n)}S = E_n^{hG_n}$$

- Define the non-ramified E_n^{nr} through deformations of the height n Honda FGL over $\overline{\mathbb{F}}_p$ i.e. $E_{n*}^{nr} = W\overline{\mathbb{F}}_p[[v_1, ..., v_{n-1}]][u^{\pm}]$. We have an action on E_n^{nr} by

$$G_n^{nr} = Aut_{\mathbb{F}_p^n}(\Gamma_n) \rtimes Gal(\overline{\mathbb{F}}_p, \mathbb{F}_p)$$

The extension $L_{K(n)}S \to E_n^{nr}$ is a faithful G_n^{nr}-Galois extension and $E_n \to E_n^{nr}$ is separably closed so G_n^{nr} is the absolute Galois group of $L_{K(n)}S$.

4 Galois Correspondence

For a faithful G-Galois extension and $K \leq G$ consider the A-algebra maps

$$F(EG_+, B)^G \to F(EG_+, B)^K \to F(EG_+, B)$$

i.e.

$$A \to B^{hK} \to B$$

Then

- $B^{hK} \to B$ is a faithful K-Galois extension
- If K is normal, $A \to B^{hK}$ is a faithful G/K-Galois extension.

So subgroups of the Galois group give to sub-extensions. To complete the correspondence we need additional hypotheses.

Assume $A \to B$ is a faithful G-Galois extension and B is connected ($\pi_0(B)$ has idempotents only $0, 1$). Then the map

$$G \to Alg_A(B, B)$$

is an equivalence.

Proof. In general, if we have a map of A-algebras $B \to C$ there is a Goerss-Hopkins SS converging to

$$\pi_* Alg_A(B, C) = \pi_* Alg_C(C \wedge_A B, C)$$

with $E_2^{0,0} = Alg_{C_*}(\pi_*(C \wedge_A B), C_*)$.

If $\pi_*(C \wedge_A B, C)$ is étale over C_*, the higher Andre-Quillen cohomology terms vanish and we are left with only $E_2^{0,0}$ and thus

$$Alg_A(B, C) = \pi_0 Alg_A(B, C) = Alg_{C_*}(\pi_*(C \wedge_A B), C_*)$$

If $A \to B$ is a faithful Galois extension and take $C = B$, then $C \wedge_A B = B \wedge_A B \simeq \prod_G B$ so the étale condition is satisfied

$$Alg_A(B, B) = Alg_{B_*}(\prod_G B_*, B_*) = G$$

since B_* has no idempotents other than $0, 1$. ♠

Now further assume we have a subextension

$$A \to C \to B$$

s.t. $A \to C$ is a separable extension ($C \wedge_A C \to C$ admits a section in the stable homotopy category) and that $C \to B$ is faithful. Then $K = \pi_0 Alg_A(C, C) \to \pi_0 Alg_A(B, B) = G$ is an injection and $C \to B$ is a faithful K-Galois extension.

5 Descent

Let $\mathcal{C}$ be a stable presentable symmetric monoidal ∞-cat where $\otimes$ commutes with colimits.

A (commutative) algebra object A is descendable if its thick $\otimes$-ideal is $\mathcal{C}$ (recall that thick= closed under fibers+retracts). A map of (commutative) algebras $A \to B$ is descendable if B is descendable in $\mathbf{Mod}(A)$.

If A is descendable then the forgetful-$\otimes A$ adjunction between $\mathcal{C}$ and $\mathbf{Mod}(A)$ is comonadic by Bar-Beck-Lurie. This gives that

$$\mathcal{C} \simeq Tot(\mathbf{Mod}(A) \rightrightarrows \mathbf{Mod}(A \otimes A) \cdots)$$

Any faithful G-Galois extension $A \to B$ is descendable. Therefore,

$$\begin{gathered}\mathbf{Mod}(A) \simeq Tot(\mathbf{Mod}(B) \rightrightarrows \mathbf{Mod}(B \wedge_A B) \cdots) \simeq \\ Tot(\mathbf{Mod}(B) \rightrightarrows \mathbf{Mod}(\prod_G B) \cdots) \simeq Tot(\mathbf{Mod}(B) \rightrightarrows \prod_G \mathbf{Mod}(B) \cdots) \simeq \\ \mathbf{Mod}(B)^{hG}\end{gathered}$$

(cobar construction computing limit indexed over BG).

Now recall that the invertible objects of $\mathcal{C}$ form an ∞-subgroupoid i.e. a space $\mathcal{P}ic(\mathcal{C})$ whose π_0 is the Picard group $\mathrm{Pic}(\mathcal{C})$. This is an E_∞ grouplike space hence deloops to a connective spectrum $pic(\mathcal{C})$. pic commutes with limits hence if $A \to B$ is a faithful G-Galois extension,

$$\mathbf{Mod}(A) \simeq \mathbf{Mod}(B)^{hG}$$

gives that

$$pic(A) \simeq \tau_{\geq 0} pic(B)^{hG}$$

The RHS can be computed by comparing the HFPSS for $pic(B)^{hG}$ and B^{hG}; it turns out that they agree in a range.

For example, recall that $KO \to KU$ is faithful C_2-Galois. This gives that

$$pic(KO) = \tau_{\geq 0} pic(KU)^{hC_2}$$

$\pi_* pic(KU)$ is $\mathbb{Z}/2$ for $* = 0, 1$ and $\mathbb{Z}$ in odd degrees $* \geq 3$. By comparing with the HFPSS for KU^{hC_2} we can see that the E_∞ of the HFPSS for $pic(KU)^{hC_2}$ in column $t - s = 0$ can have at most 3 $\mathbb{Z}/2$'s. Thus $Pic(KO)$ is a 2-group with cardinality ≤ 8. But ΣKO has order 8 hence $\mathrm{Pic}(KO) = \pi_0 pic(KO) = \mathbb{Z}/8$ generated by ΣKO.

More generally, we can use this to compute $Pic(E_n^{hG})$ for finite subgroups $G \subseteq G_n$. This is easy for $n = 1$ but after that...