1. Introduction

In this short note we give explicit ways to realize the Stiefel manifold, the Grassmannian, and certain Lie groups as minimal submanifolds in some higher dimensional spheres. The key ingredient is the following theorem.

**Theorem 1.1.** Let $M$ be a Riemannian manifold, $\Sigma$ be a submanifold of $M$, and $p \in \Sigma$. Suppose $\phi$ is an isometry from $M$ to itself such that $\phi(p) = p$ and $\phi(\Sigma) = \Sigma$. Then $d\phi_p$ fixes the mean curvature vector of $\Sigma$ at $p$.

**Proof.** Let $e_1, \ldots, e_n$ be an orthonormal frame on $\Sigma$ around $p$ and $H := \sum_{i=1}^{n}(\nabla e_i e_i)^{\perp}$ be the mean curvature vector at $p$. Then
\[
d\phi_p(H) = \sum_{i=1}^{n} d\phi_p((\nabla e_i e_i)^{\perp}) = \sum_{i=1}^{n}(d\phi_p(\nabla e_i e_i))^{\perp} = \sum_{i=1}^{n}(\nabla d\phi_p(e_i)d\phi_p(e_i))^{\perp} = H.
\]

The second equality follows from that $T_p \Sigma$ is an invariant subspace of $d\phi_p$. $\blacksquare$

If all points $p \in \Sigma$ satisfy the above condition, with $d\phi_p$ mapping every $v \in (T_p \Sigma)^{\perp}$ to $-v$, then $\Sigma$ is said to be helicoidal. Choe and Hoppe gave some examples of helicoidal hypersurfaces in $\mathbb{P}$. 

2. Stiefel Manifolds

We first introduce some notations. Let $\mathbb{F}$ denotes the reals $\mathbb{R}$, complexes $\mathbb{C}$, or quaternions $\mathbb{H}$, and let $\mathbb{F}^{n \times k}$ be the set of $n \times k$ matrices over $\mathbb{F}$. We define on $\mathbb{F}^{n \times k}$ the inner product $\langle X, Y \rangle := \text{Re}(\text{tr}(X^* Y))$, making $\mathbb{C}^{n \times k} \cong \mathbb{R}^{2nk}$ and $\mathbb{H}^{n \times k} \cong \mathbb{R}^{4nk}$ as inner product spaces. Also we let $S^N(\mathbb{R})$ be the $N$-sphere in $\mathbb{R}^{N+1}$ with radius $R$ and centered at the origin.

Moreover, let $SO(n), U(n)$, and $Sp(n)$ respectively be the special orthogonal, unitary, and sympletic group. Note that we define $Sp(n)$ to be in $\mathbb{H}^{n \times n}$ instead of $\mathbb{C}^{2n \times 2n}$. Recall that for $k \leq n$ the Stiefel manifold, denote by $V_k(\mathbb{F}^n)$, is the space of $n \times k$ matrices $X$ over $\mathbb{F}$ satisfying $X^TX = I_k$. For example $V_n(\mathbb{R}^n) = SO(n)$ and $V_1(\mathbb{R}^n) = S^{n-1}$. From $\text{tr}(X^TX) = \text{tr}(I_k) = k$, we have the isometric embeddings
\[
V_k(\mathbb{R}^n) \subset S^{nk-1}(\sqrt{k}),
\]
\[
V_k(\mathbb{C}^n) \subset S^{2nk-1}(\sqrt{k}),
\]
and
\[
V_k(\mathbb{H}^n) \subset S^{4nk-1}(\sqrt{k}).
\]

**Theorem 2.1.** The Stiefel manifolds $V_k(\mathbb{F}^n)$ is minimal in $S^{nk-1}(\sqrt{k})$. 

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Proof. We write \( \Sigma := V_k(\mathbb{R}^n) \) and \( M := S^{2nk-1}(\sqrt{k}) \) for simplicity. Since \( \Sigma \) is homogeneous, it suffices to prove that \( \Sigma \) has zero mean curvature \( H \) at the particular point \( X_0 := \begin{pmatrix} I_k \\ 0 \end{pmatrix} \). Here \( 0 \) is \((n - k) \times k \).

For every \( Q \in SO(k) \), we define a map \( \phi_Q \) from \( M \) to itself by
\[
X \mapsto \begin{pmatrix} Q & 0 \\ 0 & I_k \end{pmatrix} XQ^T.
\]
It is easy to check that \( \phi_Q \) is an isometry which fixes \( X_0 \) and maps \( \Sigma \) to itself. So we can apply Theorem 1.1, giving \( \langle d\phi \rangle_{X_0} \phi \).

By differentiating \( X(t)^T X(t) = I \) at \( t = 0 \), where \( X(t) \) is a curve on \( \Sigma \) with \( X(0) = X_0 \), we know that the tangent space \( T_{X_0} \Sigma \) is
\[
\left\{ \begin{pmatrix} A \\ B \end{pmatrix} : A \text{ is } k \times k \text{ and antisymmetric, } B \text{ is } (n - k) \times k \right\}.
\]
Since \( H \) lies in the orthogonal complement of \( T_{X_0} \Sigma \) in \( T_{X_0} M \), we can write \( H := \begin{pmatrix} A \\ 0 \end{pmatrix} \). Note that \( \langle d\phi \rangle_{X_0} = \phi_Q \) since \( \phi_Q \) is linear when extended onto \( \mathbb{R}^{nk} \), hence \( \phi_Q(H) = (d\phi_Q)_{X_0}(H) = H \). It follows that \( QAQ^T = A \). This equality holds for every \( Q \in SO(k) \), so \( A \) must be a scalar multiple of \( I_k \), thus \( H \) is a scalar multiple of \( X_0 \). But \( X_0 \notin T_{X_0} M \), so \( H = 0 \). The proof is completed.

Theorem 2.2. The Stiefel manifolds \( V_k(C^n) \) is a minimal submanifolds in \( S^{2nk-1}(\sqrt{k}) \).

Proof. Similar to proof of Theorem 2.1 we can show that the mean curvature \( H \) at \( X_0 \) is a scalar multiple of \( X_0 \). But since \( H \) is orthogonal to \( X_0 \) under the real Euclidean metric in \( \mathbb{R}^{2nk} \cong C^{n \times k} \), that scalar should be imaginary. Let \( H = \lambda X_0 \).

We next define the map \( \phi : S^{2nk-1}(\sqrt{k}) \to S^{2nk-1}(\sqrt{k}) \) that sends each matrix to its complex conjugate. Since \( V_k(C^n) \) is closed under complex conjugate, we can check that \( \phi \) satisfies all the conditions in Theorem 1.1. As a result, \( H = (d\phi)_{X_0}(H) = \phi(H) \), and thus \( \lambda X_0 = \lambda X_0 \). But \( \lambda \) is imaginary, so \( \lambda \), and hence \( H \), is zero.

An analogous result holds for \( V_k(\mathbb{H}^n) \subset S^{4nk-1}(\sqrt{k}) \). Although \( \text{tr}(XY) = \text{tr}(YX) \) does not hold in general for quaternionic matrices, \( \text{tr}(Q^*XQ) = \text{tr}(X) \) does hold for any \( Q \in Sp(k) \) by a direct computation. Therefore, similar to above, we can prove the following.

Theorem 2.3. The Stiefel manifolds \( V_k(\mathbb{H}^n) \) is a minimal submanifolds in \( S^{4nk-1}(\sqrt{k}) \).

3. Grassmannians

We define the Grassmannian \( \text{Gr}_k(\mathbb{F}^n) \) for \( k \leq n \) as the space of all \( n \times n \) orthogonal projection matrices over \( \mathbb{F} \) with rank \( k \). It is shown in [2] by Dimitrić that
\[
\text{Gr}_k(\mathbb{F}^n) = \{ X \in \mathbb{F}^{n \times n} : X^2 = X, X^* = X, \text{rank}X = k \}.
\]
Note that \( X^* = X \) implies \( \text{Gr}_k(\mathbb{F}^n) \subset \mathbb{F}^{n(n+1)/2} \).

Theorem 3.1. The Grassmannian \( \text{Gr}_k(\mathbb{R}^n) \) is a minimal submanifold in the \((\frac{n^2+n}{2} - 2)\)-dimensional sphere
\[
\{ X \in \mathbb{R}^{n \times n} : \text{tr}(X^TX) = k, X^T = X, \text{tr}X = k \},
\]
which has center \( kl_\mu/n \) and radius \( \sqrt{k}(n-k)/n \).

This result is also proven in [2], in which the analogous results for \( \mathbb{F} = \mathbb{C} \) or \( \mathbb{H} \) are also given. But here we give an alternative proof using Theorem 1.1.
Proof. Let $\Sigma := \text{Gr}_k(\mathbb{R}^n)$ and $M$ be the sphere above. Since for $X \in M$ we have
\[ \|X - kI_n/n\|^2 = \text{tr}(X^T X - k(X + X^T)/n + k^2 I_n/n^2) = k - k^2/n, \]
$M$ is a sphere with center $kI_n/n$ and radius $\sqrt{k(n-k)/n}$. And since $X^T = X$ specifies a $\frac{n^2+n}{2}$-dimensional Euclidean space, we see that $M$ is $(\frac{n^2+n}{2} - 2)$-dimensional.

Since for $X \in \Sigma$
\[ \text{tr}(X^*X) = \text{tr}(X^2) = \text{tr}(X) = \text{rank}X = k, \]
we have $\Sigma \subset M$.

Now it suffices to prove that $\text{Gr}_k(\mathbb{R}^n)$ has zero mean curvature $H$ at $X_0 := \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$.

For every $P \in SO(k)$ and $Q \in SO(n-k)$ we define a map $\phi_{P,Q} : M \to M$ by
\[ X \mapsto \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} X \begin{pmatrix} P^T & 0 \\ 0 & Q^T \end{pmatrix}. \]

It is easy to check that $\phi_{P,Q}$ is an isometry which fixes $X_0$ and maps $\text{Gr}_k(\mathbb{R}^n)$ to itself. So we can apply Theorem 1.1 giving $(d\phi_{P,Q})_{X_0}(H) = H$.

The tangent space $T_{X_0} \Sigma$ is
\[ \left\{ \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} : A is k \times (n-k) \right\}. \]

Since $H$ lies in the orthogonal complement of $T_{X_0} \Sigma$ in $T_{X_0}M$, and $H$ is symmetric by the equation $X^* = X$ defining $M$, we can write $H := \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where $B \in \mathbb{R}^{k \times k}, C \in \mathbb{R}^{(n-k) \times (n-k)}$. From $\phi_{P,Q}(H) = (d\phi_{P,Q})_{X_0}(H) = H$, it follows that $PBPT = B$ and $QCQT = C$ for every $P$ and $Q$. Thus $B = bI_k$ and $C = cI_{n-k}$ for some $b, c \in \mathbb{R}$. Then by the equation $\text{tr}(X) = 0$ defining $M$, we have $\text{tr}H = 0$ and therefore $bk + c(n-k) = 0$. This implies that $H$ is a scalar multiple of $X_0 - kI_n/n$. But $kI_n/n$ is the center of $M$. This forces $H = 0$ since $H \in T_{X_0}M$.

\section{4. Lie Groups}

\textbf{Theorem 4.1.} Let $G$ be a compact Lie group and $\rho : G \to U(n)$ be an irreducible unitary representation. Then the image $\rho(G)$ is a minimal submanifold in $S^{2n^2-1}(\sqrt{n})$.

\textbf{Proof.} We write $\Sigma := \rho(G)$ and $M := S^{2n^2-1}(\sqrt{n})$. It suffices to show that the mean curvature $H$ of $\Sigma$ at $I_n$ is zero. For each $Q \in \Sigma$, let $\phi_Q : \mathbb{C}^n \to \mathbb{C}^n$ be defined by $\phi_Q(X) := Q^TXQ$. Then it is straightforward to see that $\phi_Q$ is an isometry of $M$ which fixes $I_n$ and maps $\Sigma$ to itself. We can thus apply Theorem 1.1 and conclude that $Q^T HQ = d\phi_Q(H) = H$. So $Q$ commutes with $H$.

Suppose $H$ is not a scalar multiple of $I_n$. Then we can have an eigenvalue $\lambda$ of $H$ and an associated generalized eigenspace $E_\lambda \subset \mathbb{C}^n$ with dimension less than $n$. For each $g \in G$, since $\rho(g)H = H\rho(g)$, we can show that $E_\lambda$ is an invariant subspace of $\rho(g)$ by induction. This contradicts that $G$ is irreducible. So $H$ must be a scalar multiple of $I_n$.

Finally, as in the proof of Theorem 2.2 we consider the map on $M$ sending each matrix to its conjugate, and conclude that $H = 0$. ■
The author is not sure if the same result would hold for orthogonal representation instead of unitary representation. In this case, the proof above fails because $H$ may not have a real eigenvalue.

We now generalize to unitary representations that are not necessarily irreducible.

**Theorem 4.2.** Let $G$ be a compact Lie group and $\rho : G \to U(n)$ be a unitary representation that orthogonally decomposes into irreducible representations $\rho_i : G \to U(n_i)$ for $i = 1, \ldots, m$. We can view $\rho(G)$ as a submanifold of 

$$S^{2(n_1^2 + \cdots + n_m^2) - 1}(\sqrt{n_1 + \cdots + n_m}) \subset \mathbb{R}^{2(n_1^2 + \cdots + n_m^2)}.$$  

Then $\rho(G)$ is a minimal submanifold if and only if 

$$\dim \rho_1(G)/n_1 = \cdots = \dim \rho_m(G)/n_m.$$  

**Proof.** Let $X_i : \rho_i(G) \to \mathbb{R}^{2n_i^2}$ be the coordinate function of $\rho_i(G)$, then $X := (X_1, \ldots, X_m)$ is the coordinate function of $\rho(G)$, mapping into $\mathbb{R}^{2(n_1^2 + \cdots + n_m^2)}$. Since each $X_i$ maps $\rho_i(G)$ into $S^{2n_i^2 - 1}(\sqrt{n_i})$, $X$ maps $\rho(G)$ into 

$$S^{2(n_1^2 + \cdots + n_m^2) - 1}(\sqrt{n_1 + \cdots + n_m}).$$  

Recall the fact that for any isometric immersion $F : \Sigma \to \mathbb{R}^N$ the image $F(\Sigma)$ is minimal in a $(N - 1)$-sphere of radius $R$ if and only if $\Delta F = (k/R^2)F$.

Now by our previous theorem $\rho_i(G)$ is minimal in $S^{2n_i^2 - 1}(\sqrt{n_i})$, so 

$$\Delta X = (\Delta X_1, \ldots, \Delta X_m) = \left( \frac{\dim \rho_1(G)}{n_1}X_1, \ldots, \frac{\dim \rho_m(G)}{n_m}X_m \right).$$  

It follows that $\rho(G)$ is minimal if and only if 

$$\Delta X = \frac{\dim \rho(G)}{n_1 + \cdots + n_m}X,$$  

if and only if $\dim \rho_1(G)/n_1 = \cdots = \dim \rho_m(G)/n_m$.  

\[\blacksquare\]

**References**


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