Riemann mapping theorem

\[ \Omega \text{ proper simply conn } \implies \exists \text{ conformal } f : \Omega \to \mathbb{D}. \]

Need some technical lemmas.

**Lem 1:** Let \( \Omega \) open set. Let \( F \) be a

fam. of holomorphic \( f : \Omega \to \mathbb{C} \).

Suppose \( F \) is uniformly bounded on every

compact subset of \( \Omega \). Then

1. \( F \) is equicontinuous.

   **Proof:** Let \( \epsilon > 0 \) be fixed.

   \( \forall 3 \delta > 0 \exists \delta \text{ s.t. } \delta > \delta \Rightarrow (|f(x) - f(y)| < \varepsilon) \)

   so equicontinuous.

   \( \forall x \exists \delta \text{ s.t. } \forall y \in F \exists \delta \Rightarrow (|x - y| < \delta) \)
Not equicts:

\[ W \]

Given \( \epsilon \), can't find \( \delta \).

2) \( F \) is normal, i.e.,

every seq of fns \( f_1, f_2, \ldots \in F \) has subseq that w/ unifly.

O ECS of \( \Omega \).

pf:

To show: unif bdt O ECS \( \implies \) equicts.

Need holomorphic. (Note: sin example doesn't work)

Use CLF to reduce to studying easier functions.
$K \subset \Omega$, distance at least $\eta$.

Suppose given $\varepsilon > 0$. How to choose $\varepsilon$?

$$f(z) = \frac{1}{2\pi i} \oint_{\partial \mathcal{C}_{r}} \frac{f(s)}{s-z} \, ds$$

$r < \eta$

$r < \eta/2$)

so that $r \in K_2 \subset \Omega$

$$f(w) = \frac{1}{2\pi i} \oint_{\partial \mathcal{C}_r} \frac{f(s)}{s-w} \, ds$$

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \int_{\partial \mathcal{C}_{z,r}} |f(s)| \left| \frac{1}{s-z} - \frac{1}{s-w} \right|$$

$$= \frac{1}{2\pi} \int_{\partial \mathcal{C}_{z,r}} |f(s)| \frac{|z-w|}{|s-z||s-w|} \, ds$$
So \( |f(z)| \leq B \) since \( r < \frac{1}{2} \).

\[ |z-w| = r \]

\[ |z-w| > \frac{1}{2} \]

Need a bound on this.

Say \[ |z-w| < r/2 \].

Then \[ |z-w| > r/2 \].

So \[ |f(z) - f(w)| \]

\[ \leq \frac{1}{2\pi} B \cdot \frac{|z-w|}{r \cdot r/2} \cdot 2\pi r \]

Does not depend on \( f \).

So we've proved equi-iccy on \( K \)!
Now to show: equicts OECs $\implies$ normal.

$f_1, f_2, \ldots \in F$.

Idea: if $w_1 \in G$, then $\exists \epsilon > 0$ such that $f_i(w_1), f_2(w_1), \ldots$ has a subseq.

Say $g_1, g_2, \ldots$

Now $w_2 \in G$:

$g_1(w_2), g_2(w_2), \ldots$ has a subseq.

Say $h_1, h_2, \ldots$

So plug in $z = w_1$ or $z = w_2$ $\implies$ seq. $\&$ converges!
so let \( w_1, w_2 \ldots \) be a dense countable subset of \( \Omega \).

from \( f_1, f_2 \ldots \)

\[
\begin{array}{cccc}
  f_{1,1} & f_{2,1} & f_{3,1} & f_{4,1} \\
  f_{1,2} & f_{2,2} & f_{3,2} & f_{4,2} \\
  f_{1,3} & f_{2,3} & f_{3,3} & f_{4,3} \\
\end{array}
\]

\( \text{cugs at } w_1 \), \( \text{cugs at } w_2 \) (and \( w_1 \)), \( \text{cugs at } w_3 \) (and \( w_1, w_2 \))

\[ \text{take diagonal } g_n = f_{n,n} \text{.} \]

This cugs at all \( \xi \in w_1, w_2, \ldots \).

let \( K \subseteq \Omega \) compact.

Need to show \( g_n \) is uniformly Cauchy on \( K \).
idea, for any \( z \in k \), want \( w_j \) close to \( z \). Then

\[
q_n(z) - q_m(z) = q_n(z) - q_n(w_j) + q_n(w_j) - q_m(w_j) + q_m(w_j) - q_m(z)
\]

use equiuty. Suppose given \( \exists \epsilon > c \). \( \exists \delta \) s.t.

\[
\forall h, \quad |z - w| < \delta \quad \Rightarrow \quad |q_k(z) - q_k(w)| < \epsilon
\]

so good with those 2 terms. For middle terms, we want to only consider finitely many \( w_j \).
we'll eventually cover K.

Rem: true that I didn't go into complete detail, but best if you work it out on your own.

**Lemma 2:** $\Omega \subset \mathbb{C}$

$g_n$ vugs unity $\Omega ECSO \Omega 

$f_n$ injective.

$\Rightarrow f$ is inj. or const.

**Pf:** SFC. $f(\omega_1) = f(\omega_2)$.

$$g_n(\omega) = f_n(z) - f_n(\omega)$$

$$g_n^*(z) = f(z) - f(\omega)$$
study \( g_n \) near \( \omega \), \( \Rightarrow \) No zero

\[ g(z) = g_n(z) + (g(z) - g_n(z)) \]

\( g \) is not a con

by assumption \( g \neq 0 \). so

\( g \) has isolated zero at \( \omega \).

no other zeros here. (incl bdary).

so \( |g(z)| > c > 0 \) on bdary.

so for large enough \( n \)

\( |g_n(z)| > \frac{c}{2} \)

\( |g(z) - g_n(z)| < \frac{c}{2} \)

apply Rouche on one of these \( g_n \). Done!
Back to Riemann mapping.

\[ \Omega \neq \mathbb{C} \text{ simply connected} \]

\[ b/c \Omega \neq \mathbb{C} \quad (\exists \ a \neq \Omega) \]

So now suppose \( \Omega \subset \mathbb{D}, \ 0 \in \Omega \).
Can we find a conformal \( f: \Omega \to \mathbb{D} \)?

s.t. \( f(0) = 0 \)

Note if \( f(0) = \Omega = \mathbb{D} \), then one way is

\[ |f'(0)| = 1 \quad (i.e. \text{ maximize } |f'(0)|) \]

So \( F = \{ f: \Omega \to \mathbb{D} : f \text{ injective} \} \)
\[ S = \sup_{f \in F} |f'(0)| \quad \leftarrow \text{Is the sup attained? Yes!} \]

We know.

\[ \exists f_1, f_2, \ldots \quad \text{s.t.} \quad |f_n'(0)| \to S. \]

So \[ \exists \text{unifly wgt. } O_ECSO \subseteq \text{subseq} \quad \text{this seq cgy to some } f. \]

\[ f(0) = 0. \quad f \text{ is nonconst } (\text{why?}) \]

\[ f \text{ is injective.} \]

So \[ f \in F. \]

Claim: \[ f \text{ is conformal!} \]

why?
(There is a mistake on this page: g is not a D -> D map.)

\[ f : \Omega \rightarrow D. \]

Suppose \( f \) misses \( \alpha \).

\[
\begin{array}{cccc}
\Omega & \xrightarrow{f} & D & \xrightarrow{\psi_\alpha} & D & \xrightarrow{g} & D & \xrightarrow{\psi_g(\alpha)} & D \\
\end{array}
\]

\[
\begin{array}{cccc}
\Omega & \xrightarrow{f} & D & \xrightarrow{g} & D & \xrightarrow{\psi_g(\alpha)} & D \\
0 & \xrightarrow{\alpha} & 0 & \xrightarrow{g(\alpha)} & 0 \\
\end{array}
\]

\[
\Phi \circ g \circ \Phi_d
\]

\[
F = \psi_g(\alpha) \circ g \circ \psi_\alpha \circ f
\]

injective. \( \Omega \rightarrow D. \)

\[
\Phi(0) = 0,
\]

\[
\Phi : D \rightarrow D, \quad \Phi \text{ not injective} \quad \text{so} \quad |\Phi'(0)| < 1
\]