Sheet 23: Graphs

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Definition 1 *A* simple graph *is a tuple* (V, E) *where* V *is a set and* E *is a symmetric relation on* V *without loops, that is, for all* $x \in V$, $(x, x) \notin E$. *Elements of* V *are called* vertices *and (unordered) pairs* $(x, y) \in E$ *are called* edges*.*

Later we will also work with directed graphs, where the relation is not necessarily symmetric, so edges have a direction.

Definition 2 *Let* (V, E) *be a simple graph. We say that* $x \in V$ *is a* neighbour *of* $y \in V$ *if* (x, y) *is an edge. The* degree *of a vertex* $v \in V$ *is defined to be the number of neighbours of* v*.*

Reformulate the first exercise in these terms.

Exercise 3 *Is this true for infinite simple graphs?*

There are two very simple graph on every set V: the *empty graph* (where $E = \emptyset$) and the *complete graph* (where $E = V \times V \setminus \{(v, v) \mid v \in V\}$).

Exercise 4 *There is a party at Marcello's with* 19 *people attending. Is it possible that everyone knows exactly* 3 *other people?*

Exercise 5 *How about* n *people attending?*

A triangle is a set of 3 vertices such that every pair is an edge. An empty triangle is a set of 3 vertices with no edges.

Exercise 6 *Draw a graph on* 5 *vertices with no triangles or empty triangles.*

Exercise 7 *Show that every simple graph on* 6 *vertices contains a triangle or an empty triangle.*

Hint: What if there is a vertex of degree 3?

Exercise 8 *Show that in every simple graph the number of vertices of odd degree is even.*

Exercise 9 *What does a graph such that every vertex has degree* 2 *look like?*

A couple of definitions that we already had in class.

Definition 10 *Let (V, E) be a graph. A* walk *of length n from $x \in V$ to $y \in V$ is a sequence of vertices $a_0, \ldots, a_n$ where $a_0 = x$, $a_n = y$ and $(a_i, a_{i+1}) \in E$ $(0 \leq i \leq n-1)$. A walk is a* path *if $a_i \neq a_j$ $(0 \leq i < j \leq n)$. A walk is a* cycle *if $a_0 = a_n$ and $a_i \neq a_j$ $(0 \leq i < j < n)$.*

Note that we also allow paths of length 0. Sometimes it is more convenient to name the path by its edges rather than by its vertices.

Theorem 11 *If there is a walk from a to b then there is a path from a to b.*

Now we define a relation $\backsim$ on V as follows: $a \backsim b$ if there is a walk from a to b.

Theorem 12 *$\backsim$ is an equivalence relation.*

Definition 13 *The* connected components *of a graph are its $\backsim$-equivalence classes.*

Just a flashback to topological spaces and metric spaces.

Exercise 14 *Try to define the connected components of a topological space.*

Definition 15 *A graph is connected if it has one connected component.*

Let (V, E) be a connected graph. For $a, b \in V$ let $d(a, b)$ be the length of the shortest path from a to b.

Theorem 16 *d is a metric on V.*

If we dont care where the cycle starts, we better name it by its edges.

Definition 17 *A tree is a connected graph without cycles.*

Draw some trees. Try to form conjectures.

Theorem 18 *A graph is a tree if and only for all vertices a, b there is a unique simple path from a to b.*

A leaf is a vertex of degree 1.

Theorem 19 *Every finite tree on $n \geq 2$ vertices has a leaf.*

Theorem 20 *A tree on n vertices has $n - 1$ edges.*

Try to find many proofs for this.

Theorem 21 *Every finite tree on $n \geq 2$ vertices has at least 2 leaves.*

Theorem 22 *The only tree on $n \geq 2$ vertices with exactly two leaves is a path.*

Now we want to consider walks on a graph.

Definition 23 *Let* (V, E) *be a graph. A walk* $a_0, \ldots, a_n$ *is* Eulerian *if* $a_0 = a_n$ *and it uses every edge exactly once (that is, for all* $e \in E$ *there is a unique* $0 \le i < n$ *such that* $(a_i, a_{i+1}) = e$ *as an undirected pair).*

Note that edges are undirected, so the walk only uses every edge in one direction.

Theorem 24 *A simple graph has an Eulerian walk on it if and only if it is connected and every vertex has even degree.*

For the less obvious direction: start walking on the graph.