THE UNIVERSITY OF CHICAGO

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH COMPLEX COEFFICIENTS

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TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

In this paper I investigate divergence-form elliptic partial differential equations on Lipschitz domains in $\mathbf{R}^2$ whose coefficient matrices have small (but possibly nonzero) imaginary parts and depend only on one of the two coordinates.

I show that for Dirichlet boundary data in L^q for q large enough, solutions exist and are controlled by the L^q-norm of the boundary data.

Similarly, for Neumann boundary data in L^p, or for Dirichlet boundary data whose tangential derivative is in L^p ("regularity" boundary data), for p small enough, I show that solutions exist and have gradients which are controlled by the L^p-norm of the boundary data.

I prove similar results for Neumann or regularity boundary data in H^1, and for Dirichlet boundary data in L^∞ or BMO. Finally, I show some converses: if the solutions are controlled in some sense, then Dirichlet, Neumann, or regularity boundary data must exist.

ACKNOWLEDGEMENTS

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CHAPTER 1

INTRODUCTION

Let A be a uniformly elliptic matrix-valued function defined on $\mathbf{R}^n$. That is, assume that there exist constants $\Lambda > \lambda > 0$ (called the *ellipticity constants* of A) such that

$$\lambda|\eta|^2 \leq \mathbf{Re}\,\bar{\eta}\cdot A(X)\eta, \quad |\xi\cdot A(X)\eta| \leq \Lambda|\eta|\,|\xi| \tag{1.1}$$

for every $X \in \mathbf{R}^n$ and every ξ, $\eta \in \mathbf{C}^n$.

Let $1 < p \leq \infty$. We say that $(D)^A_p$, $(N)^A_p$, or $(R)^A_p$ holds in the Lipschitz domain V if, for every $f \in L^p(\partial V)$, there is a function u such that

$$(D)^A_p \begin{cases} \operatorname{div} A\nabla u = 0 & \text{in } V, \\ u = f & \text{on } \partial V, \\ \|Nu\|_{L^p} \leq C(p)\|f\|_{L^p} \end{cases}$$

$$(N)^A_p \begin{cases} \operatorname{div} A\nabla u = 0 & \text{in } V, \\ \nu\cdot A\nabla u = f & \text{on } \partial V, \\ \|N(\nabla u)\|_{L^p} \leq C(p)\|g\|_{L^p} \end{cases}$$

$$(R)^A_p \begin{cases} \operatorname{div} A\nabla u = 0 & \text{in } V, \\ \partial_\tau u = f & \text{on } \partial V, \\ \|N(\nabla u)\|_{L^p} \leq C(p)\|f\|_{L^p} \end{cases}$$

where ν, τ are the unit normal and tangent vectors to ∂V, $\nu\cdot A\nabla u$ is the conormal derivative of u on the boundary, $\partial_\tau u$ is the tangential derivative, and NF is the nontangential maximal function of F. For $(N)^A_p$, $(R)^A_p$ to hold in a domain with compact boundary, we require only that solutions exist for f such that $\int_{\partial V} f\,d\sigma = 0$; however, if V^C is compact we also require that $\lim_{|X|\to\infty} u(X)$ exist.

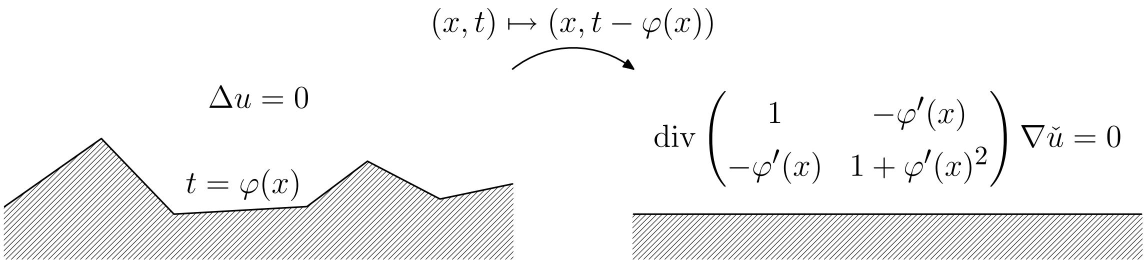

Figure 1.1: Change of variables to straighten the boundary

Furthermore, we require that solutions be unique; see Sections 12.1 and 12.2 for conditions under which uniqueness holds.

In this dissertation, we will restrict our attention to complex matrix-valued functions A defined on $\mathbf{R}^2$ which satisfy

$$\lambda|\eta|^2 \leq \mathbf{Re}\,\bar{\eta}\cdot A(x,t)\eta, \quad |\xi\cdot A(x,t)\eta| \leq \Lambda|\eta|\,|\xi|, \quad A(x,t) = A(x,s) \tag{1.2}$$

for all x, t, $s \in \mathbf{R}$ and all η, $\xi \in \mathbf{C}^2$.

We review some of the history of this problem in planar Lipschitz domains. Investigation began with the case $A \equiv I$, that is, the case where the solutions u are harmonic.

In [6], B.J. Dahlberg showed that $(D)^I_q$ holds in all bounded Lipschitz domains provided $2 \leq q < \infty$. Kenig and Fabes observed that $(N)^I_p$ holds in Lipschitz domains for $1 < p \leq 2$. (This range of p, q cannot be improved for general Lipschitz domains.)

In [30], Verchota showed that the solutions to these problems in planar Lipschitz domains could be constructed as layer potentials. This is a very useful result, as it is often easier to prove theorems about layer potentials than about arbitrary harmonic functions. Most of the theorems in this dissertation are proven using layer potentials as well.

The next step was to generalize from harmonic functions ($A \equiv I$) to other matrices. Some additional regularity is necessary; in [5], Caffarelli, Fabes and Kenig constructed a real, symmetric, elliptic 2×2 coefficient matrix A such that the L-harmonic measure associated to $L = \operatorname{div} A\nabla$ is completely singular with respect to arc length on the unit circle. (This means that solutions u to the Dirichlet problem depend only on the boundary data on a set of surface measure zero; thus, there is no hope to control the solutions in terms of L^p behavior of boundary data.)

A simple change of variables transforms functions harmonic in in the domain above a Lipschitz graph to solutions to elliptic PDE in the upper half-plane. The elliptic PDE has coefficients that are real, symmetric, and satisfy (1.2). (See Figure 1.1.) In [19], Jerison and Kenig showed that $(D)_q^A$ holds, for $2 \leq q < \infty$, for all such coefficient matrices A.

In [24], the following theorem was proven:

Theorem 1.3. *Suppose that* $A_0 : \mathbf{R}^2 \mapsto \mathbf{R}^{2\times 2}$ *is real-valued (but not necessarily symmetric) and satisfies (1.2). Let* V *be a bounded or special Lipschitz domain with Lipschitz constants at most* k_i. *Then there is some (possibly large)* $q_0 = q_0(\lambda, \Lambda, k_i)$ *and some* $C(\lambda, \Lambda, k_i)$ *such that* $(D)_{q_0}^{A_0}$ *holds in* V *with constant at most* C.

In [25] and [26], the following theorem was proven:

Theorem 1.4. *Let* A_0, V *be as in Theorem 1.4.*

Let $1/p_0 + 1/q_0 = 1$. *Then if* $(D)_{q_0}^{A_0^t}$, $(D)_{q_0}^{A_0/\det A_0}$ *hold in* V *with constants at most* $C(V, q_0)$, *then* $(N)_{p_0}^{A_0}$ *and* $(R)_{p_0}^{A_0}$ *hold in the Lipschitz domain* V, *with constants depending only on* p_0, λ, Λ, k_i, *and* $C(V, p_0)$.

In [2], the authors approached elliptic PDE (in higher dimensions) using a perturbative approach. If A_0, A are elliptic matrices (not necessarily real), A_0, A are independent of one of the coefficients, and $\|A - A_0\|_{L^\infty}$ is small enough, then if solutions to $(D)_2^{A_0}$, $(N)_2^{A_0}$, $(R)_2^{A_0}$ can be constructed using layer potentials, then solutions to $(D)_2^A$, $(N)_2^A$, $(R)_2^A$ can be constructed in the same way.

In this dissertation, I will prove a perturbative version of Theorems 1.3 and 1.4:

Theorem 1.5. *Suppose that* A_0 *and* A *satisfy (1.2). Assume that* $A_0(x)$ *is real-valued;* $A(x)$ *may complex-valued. Let* V *be a good Lipschitz domain.*

Then there is some $\epsilon > 0$, $p_0 > 1$ *depending only on* λ, Λ *and the Lipschitz constants of* V, *such that if* $\|A - A_0\|_{L^\infty} < \epsilon$ *and* $1 < p \leq p_0$, $1/p + 1/q = 1$, *then* $(N)_p^A$, $(R)_p^A$ *and* $(D)_q^A$ *hold in the Lipschitz domain* V.

Furthermore, if $f \in H^1(\partial V)$, *then there exist unique functions* v, w *such that*

$$(N)_1^A \begin{cases} \operatorname{div} A\nabla v = 0 & \text{in } \Omega, \\ \nu \cdot A\nabla v = f & \text{on } \partial\Omega, \\ \|N(\nabla v)\|_{L^1} \leq C\|f\|_{H^1}, \end{cases}$$

$$(R)_1^A \begin{cases} \operatorname{div} A\nabla w = 0 & \text{in } \Omega, \\ \partial_\tau w = f & \text{on } \partial\Omega, \\ \|N(\nabla w)\|_{L^1} \leq C\|f\|_{H^1}. \end{cases}$$

We will also prove results for boundary data in BMO and L^∞:

Theorem 1.6. *Suppose that V is a bounded Lipschitz domain. Then there is some $\epsilon_0 > 0$ such that if $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, the maximum principle holds; that is, if $f \in L^\infty(\partial V)$, then there exists a unique u with $\operatorname{div} A\nabla u = 0$ in V, $u = f$ on ∂V and $\|u\|_{L^\infty(V)} \leq C\|f\|_{L^\infty(\partial V)}$.*

Theorem 1.7. *Suppose that A and V satisfy the conditions of Theorem 1.5. Then there is some $\epsilon_0 > 0$ such that, if $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, then for every $f \in BMO(\partial V)$ there is some u such that $u = f$ on ∂V, $\operatorname{div} A\nabla u = 0$ in V, and $|\nabla u|^2 \operatorname{dist}(\cdot, \partial V)$ is a Carleson measure, that is, for any $X_0 \in \partial V$ and any $V > 0$,*

$$\frac{1}{\sigma(B(X_0, R) \cap \partial V)} \int_{B(X_0,R)\cap V} |\nabla u(X)|^2 \operatorname{dist}(X, \partial V)\, dX \leq \tilde{C} \tag{1.8}$$

where $\tilde{C} = C\|f\|_{BMO}$ and C depends only on λ, Λ, the Lipschitz constants of V, and the constants in Theorem 1.5. Furthermore, this function is unique among functions u with $\operatorname{div} A\nabla u = 0$ in V, $u = f$ on ∂V, and which satisfy (1.8) for some constant $\tilde{C}$.

Carleson-measure conditions on the gradients of solutions with BMO boundary data are a natural form of control. Such conditions bear a deep connection to the problem $(D)_p^A$: in [10], it was shown that for real coefficient matrices on bounded Lipschitz domains, this form of control is equivalent to an A_∞ condition on harmonic measure, and this A_∞ condition implies that $(D)_p^{A_0}$ holds in V for some $1 < p < \infty$.

Finally, we will prove some converses:

Theorem 1.9. *Suppose that A, V satisfy the conditions of Theorem 1.5. Suppose further that $\operatorname{div} A\nabla u = 0$ in V.*

If $N(\nabla u) \in L^1(\partial V)$, then the conormal derivative $\nu \cdot A\nabla u$ exists weakly and the boundary values $u|_{\partial V}$ exist as non-tangential limits. Furthermore, $\nu \cdot A\nabla u$ and $\partial_\tau u$ are in $H^1(\partial V)$ with H^1-norm at most $C\|N(\nabla u)\|_{L^1}$.

If $\|\,\mathbf{Im}\,A\|_{L^\infty}$ is small enough, and if $|\nabla u(X)|^2 \operatorname{dist}(X,\partial V)$ satisfies (1.8) for some $\tilde C$, then $u|_{\partial V}$ exists and is in $BMO(\partial V)$ with BMO norm at most $C\tilde C$.

Theorem 1.6 will be proven in Chapter 13. Uniqueness and converses for Theorems 1.5 and 1.7 will be proven in Chapter 12. Existence for smooth coefficients for Theorem 1.7 will be proven in Chapter 10, and we will pass to arbitrary elliptic (rough) coefficients in Theorem 11.4.

We now outline the proof of existence for Theorem 1.5; resolving the details will form the bulk of this work.

Proof of Theorem 1.5. If $f : \partial V \mapsto \mathbf{C}$ is in L^p for $1 < p < \infty$, then we can define ((2.13) and (2.12)) functions $\mathcal{D}f$, $\mathcal{S}^T f$ such that if $X \in \mathbf{R}^2 \setminus \partial V$, then $\mathcal{D}f(X)$ and $\nabla \mathcal{S}^T f(X)$ are well-defined complex numbers, and $\operatorname{div} A\nabla(\mathcal{D}f) = 0$, $\operatorname{div} A^T\nabla(\mathcal{S}^T f) = 0$ in $\mathbf{R}^2 \setminus \partial V$. (Here A^T is the matrix transpose of A. These functions are layer potentials; they were studied in [30] in the case $A \equiv I$, and used in [25] in the case where A is real.)

There exist operators $\mathcal{K}^A_\pm$, $\mathcal{L}^A$ on $L^p(\partial V)$ such that

$$\mathcal{K}^A_\pm f = \mathcal{D}f|_{\partial V_\pm}, \qquad (\mathcal{L}^A)^t f = \tau \cdot \nabla \mathcal{S}^T f|_{\partial V},$$
$$(\mathcal{K}^A_\pm)^t f = \mp \nu \cdot A^T \nabla \mathcal{S}^T f|_{\partial V_\mp}$$

in appropriate weak senses, where $V_+ = V$, $V_- = \bar V^C$, and t denotes operator transpose.

We will show that, if $\mathcal{K}^A_\pm$, $\mathcal{L}^A$ are bounded $L^p(\partial V) \mapsto L^p(\partial V)$ then

$$\|N(\mathcal{D}f)\|_{L^p(\partial V)} \leq C(p)\|f\|_{L^p(\partial V)}, \quad \|N(\nabla S^T f)\|_{L^p(\partial V)} \leq C(p)\|f\|_{L^p(\partial V)}$$

for all $1 < p < \infty$. (Theorem 5.12.) Furthermore, if $f \in H^1(\partial V)$, then $\lim_{|X|\to\infty} \mathcal{S}f(X) = 0$ and $\|N(\nabla \mathcal{S}^T f)\|_{L^1} \leq C\|f\|_{H^1}$. ((5.3), and Corollary 5.13.)

If $\mathcal{K}^A_+$ is invertible with bounded inverse on some $L^p(\partial V)$, then for every $g \in L^p(\partial V)$, we may let $u = \mathcal{D}((\mathcal{K}^A_+)^{-1} g)$. Then

$$\operatorname{div} A\nabla u = 0, \quad u|_{\partial V} = g, \quad \|Nu\|_{L^p} \leq C_p \|(\mathcal{K}^A_+) g\|_{L^p} \leq C_p \|(\mathcal{K}^A_+)^{-1}\|_{L^p \mapsto L^p} \|g\|_{L^p}$$

and so u is a solution to $(D)^A_p$.

Similarly, if $(\mathcal{K}_-^A)^t$ or $(\mathcal{L}^A)^t$ is bounded and invertible on $H^1(\partial V)$, or on (a subspace of) $L^p(\partial V)$, then $u = \mathcal{S}^T(((\mathcal{K}_-^A)^t)^{-1}g)$ or $u = \mathcal{S}^T(((\mathcal{L}^A)^t)^{-1}g)$ is a solution to $(N)_1^{A^T}$, $(R)_1^{A^T}$, $(N)_p^{A^T}$, or $(R)_p^{A^T}$.

So we need only show that $(\mathcal{K}_\pm^A)^t$, $(\mathcal{L}^A)^t$ are bounded and invertible. (If $(\mathcal{K}_\pm^A)^t$ is invertible on a reflexive Banach space, then by elementary functional analysis, $\mathcal{K}_\pm^A$ is invertible on its dual space.)

If A is smooth, and

$$V = \{X \in \mathbf{R}^2 : \varphi(X \cdot \mathbf{e}^\perp) < X \cdot \mathbf{e}\}$$

for some Lipschitz function φ (which we may assume, a priori, to be in C_0^∞) and some unit vector $\mathbf{e}$, then (working much as in [25]), we can find an $\epsilon_0 > 0$ depending only on λ, Λ, such that if A satisfies (1.2) and $\|\mathbf{Re}\,A\|_{L^\infty} < \epsilon_0$, then $(\mathcal{K}_\pm^A)^t$, $(\mathcal{L}^A)^t$ are bounded $H^1(\partial V) \mapsto H^1(\partial V)$ and $L^p(\partial V) \mapsto L^p(\partial V)$ for $1 < p < \infty$. We can then extend this to arbitrary Lipschitz domains V. (Theorems 6.1, 6.9 and 5.16.)

We will show that if $(N)_{p_0}^{A_0}$, $(R)_{p_0}^{A_0}$ hold in V and V^C for all real A_0 satisfying (1.2), and if $(\mathcal{K}_\pm^I)^t$, $(\mathcal{L}^I)^t$ are invertible on (a subspace of) $L^{p_0}(\partial V)$, then there is some $\epsilon > 0$ such that $(\mathcal{K}_\pm^A)^t$, $(\mathcal{L}^A)^t$ are invertible on that subspace, and on $H^1(\partial V)$, for all A satisfying (1.2) with $\|\mathbf{Re}\,A\|_{L^\infty} < \epsilon$. (Chapters 7 and 8.)

By Theorem 1.4 and Theorem 7.8, $(N)_{p_0}^{A_0}$ and $(R)_{p_0}^{A_0}$ hold in V and $\bar{V}^C$ for some p_0. We can easily show (Lemma 7.7) that if V is a special Lipschitz domain, then $(\mathcal{K}_\pm^I)^t$ and $(\mathcal{L}^I)^t$ are invertible on $L^{p_0}(\partial V)$. If V is bounded, by [30] $(\mathcal{K}_+^I)^t$ is invertible on $L^{p_0}(\partial V)$ for $p_0 \geq 2$; $(\mathcal{K}_-^I)^t$ and $(\mathcal{L}^I)^t$ are invertible only on the subspace $L_0^{p_0}$ of L^{p_0} consisting of functions f with $\int_{\partial V} f \, d\sigma = 0$. So we can solve $(N)_p^A$, $(R)_p^A$ only for boundary data that integrates to zero.

If p, q are conjugate exponents with $1 < p, q < \infty$, then L_0^p and L_0^q are dual spaces, so $\mathcal{K}_-$ is invertible only on L_0^q. We can still solve the Dirichlet problem in the complement of a bounded domain by taking $\mathcal{D}(\mathcal{K}_-^{-1}(f - f_V)) + f_V$ for an appropriate constant f_V.

Using standard interpolation techniques (Chapter 9), we can show that $\mathcal{K}_\pm^t$, $\mathcal{L}^t$ have bounded inverses on L^p (or L_0^p) for $1 < p < p_0$, and therefore $\mathcal{K}_\pm$ has a bounded inverse on L^q (or L_0^q) for $q_0 < q < \infty$.

Thus, we have that $(N)_p^A$, $(R)_p^A$, $(D)_q^A$ hold if A is smooth, for $p > 1$ small, $q < \infty$ large. We pass to arbitrary (rough) A in Chapter 11. □

CHAPTER 2

DEFINITIONS

If u is in the Sobolev space $W^{1,2}_{loc}(V)$, then we say that $\operatorname{div} A\nabla u = 0$ in V in the weak sense if, for all $\eta \in C_0^\infty(V)$,

$$\int A\nabla u \cdot \nabla \eta = 0. \tag{2.1}$$

Similarly, we say that $\nu \cdot A\nabla u = g$ on ∂V in the weak sense if for all $\eta \in C_0^\infty(\mathbf{R}^2)$,

$$\int_V A\nabla u \cdot \nabla \eta = \int_{\partial V} g\eta. \tag{2.2}$$

We say that $u = f$ on ∂V if f is the non-tangential limit of u a.e., that is, if

$$\lim_{\eta \to 0^+} \sup\{|u(Y) - f(X)| : |X - Y| < (1 + a)\operatorname{dist}(Y, \partial V) < \eta\}$$

holds for almost every $X \in \partial V$. We will frequently abuse notation and write $\tau \cdot \nabla u$ for the tangential derivative $\partial_\tau f$.

If the nontangential limit of u exists in $L^p(\partial V)$, and $u \in W^{1,p}(V)$ so that the trace $\mathbf{Tr}\, u$ exists, then $\mathbf{Tr}\, u$ is equal in $L^p(\partial V)$ to the nontangential limit of u. See [4, p. 6] for a proof of this result in smooth domains; we can get this result in Lipschitz domains by looking at u in small neighborhoods of the boundary and changing variables.

We fix some notation. If U is a domain, then $U = U_+$, $\bar{U}^C = U_-$.

Throughout, we will reserve the letters p and q for the exponents of L^p-spaces; recall that the norm in such spaces is defined by $\|f\|_{L^p} = \left(\int |f|^p\right)^{1/p}$. We will always let p and q be conjugate exponents given by $1/p + 1/q = 1$; if multiple such exponents are needed, we will distinguish them with subscripts.

If ∂V is bounded, we let $L^p_0(\partial V)$ be the subspace of $L^p(\partial V)$ of functions f such that $\int_{\partial V} f\, d\sigma = 0$. If ∂V is unbounded, for ease of notation we let $L^p_0(\partial V) = L^p(\partial V)$.

The inner product between L^p and L^q will be given by

$$\langle G, F\rangle = \int G(x)^t F(x)\,dx.$$

(This notation is more convenient than the usual inner product $\int \overline{G^t(x)}F(x)\,dx$.) A superscript of t will denote the transpose of a matrix or the adjoint of an operator with respect to this inner product. (So if P is an operator, then $\langle F, PG\rangle = \langle G, P^t F\rangle^t$.)

If a function or operator is defined in terms of A, then a superscript of T will denote the corresponding function or operator for A^t. (So $A^t = A^T$; however, $P^t \neq P^T$ for most operators defined in terms of A.)

If f is defined on U, we define the *non-tangential cones* γ and *non-tangential maximal function* N by

$$\gamma_{U,a}(X) = \{Y \in U : |X-Y| < (1+a)\operatorname{dist}(Y,\partial U)\}, \quad N_{U,a}f(X) = \sup_{\gamma_{U,a}(X)} |f(Y)| \tag{2.3}$$

for some number $a > 0$. When no ambiguity will arise we suppress the subscripts U or a; we let $\gamma_\pm(X) = \gamma_{U_\pm}(X)$.

Definition 2.4. We say that the domain Ω is a *special Lipschitz domain* if, for some Lipschitz function φ and unit vector $\mathbf{e}$,

$$\Omega = \{X \in \mathbf{R}^2 : \varphi(X\cdot \mathbf{e}^\perp) < X \cdot \mathbf{e}\}.$$

We refer to $\|\varphi'\|_{L^\infty}$ as the *Lipschitz constant of* Ω.

We say that U is a *Lipschitz domain* if there is some $k_1 > 0$ such that, for every $X \in \partial U$, there is some neighborhood W of X such that $U \cap W = \Omega_X \cap W$ for some special Lipschitz domain Ω_X with Lipschitz constant at most k_1.

If U is a special Lipschitz domain, then let $k_2 = k_3 = 1$. Otherwise, let k_2, $k_3 > 1$ be numbers such that ∂U may be covered by at most k_2 such neighborhoods whose size varies by at most k_3.

That is, there is a constant $r = r(U)$ such that

$$\partial U \subset \bigcup_{j=1}^{k_2} B(X_j, r_j)$$

for some $X_j \in \partial U$ and $r_j \in (r, rk_3)$. We further require that there exist unit vectors $\mathbf{e}_j$ and Lipschitz functions φ_j, with $\|\varphi_j'\|_{L^\infty} \leq k_1$, such that if

$$\begin{aligned}
\Omega_j &= \{X \in \mathbf{R}^2 : \varphi_j((X - X_j) \cdot \mathbf{e}_j^\perp) < (X - X_j) \cdot \mathbf{e}_j\},\\
R_j &= \{X \in \mathbf{R}^2 : |(X - X_j) \cdot \mathbf{e}_j^\perp| < 2r_j, |(X - X_j) \cdot \mathbf{e}_j| < (2 + 2k_1)r_j\},
\end{aligned}$$

then

$$U \cap R_j = \Omega_j \cap R_j.$$

We refer to k_1, k_2, k_3 as the *Lipschitz constants* of U. For simplicity, when we write k_i, we mean k_1, k_2, k_3.

If the k_i are finite, and ∂U is connected, then we call U a *good Lipschitz domain*.

We will reserve V for good Lipschitz domains and Ω for special Lipschitz domains. Note that if V is a good Lipschitz domain, then either V is special or ∂V is bounded.

We do not care about the value of the positive constant $r(U)$ in the definition of k_2, k_3; this is because $(D)_p^A$, $(N)_p^A$, $(R)_p^A$ are scale-invariant.

A particular parameter of U, which is at most $2k_2\sqrt{1 + k_1^2}$, is very important. Let k_4 be such that, if $X_0 \in \partial U$ and $r > 0$, then $\sigma(\partial U \cap B(X_0, r)) \leq k_4 r$. We refer to k_4 as the *Ahlfors-David constant* of U.

Remark 2.5. Usually, we will be able to work with either connected subsets of ∂U or sets of the form $B(X, r) \cap \partial U$ for $X \in \partial U$, $r > 0$, whichever is more convenient.

If $\Delta \subset \partial U$ is connected, then $\Delta \subset B \cap \partial U$ for some ball B with radius $\sigma(\Delta)/2$ such that $\sigma(B \cap \partial U) \leq \frac{1}{2} k_4 \sigma(\Delta)$.

Conversely, if $X \in \partial U$ and $r > 0$, I claim that $B(X, r) \cap \partial U$ is contained inside some connected set $\Delta \subset \partial U$. If $U = \Omega$ is a special Lipschitz domain and $X = \psi(x)$, then $B(X, r) \cap U \subset \psi((x - r, x + r))$, a connected set of surface measure at most $2r\sqrt{1 + k_1^2}$. If

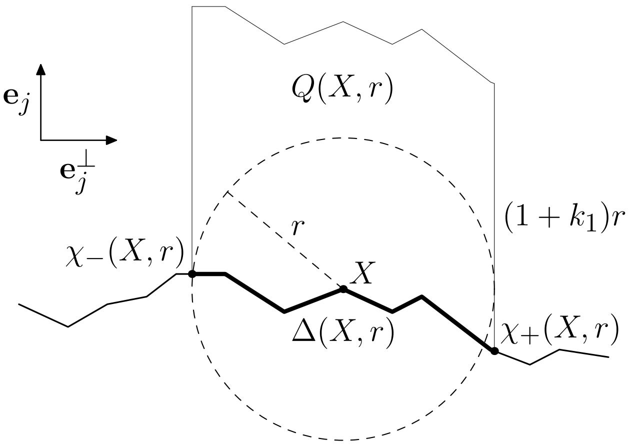

Figure 2.1: Tents on the boundary of a special Lipschitz domain

∂U is bounded, then either $r < \min_j r_j$ and so $B(X,r)\cap\partial U = B(X,r)\cap\Omega_j$ for some j, or $r > \min_j r_j \geq r(U)$. But $\sigma(\partial U) \leq \sum_{j=1}^{k_2} \sigma(B(X_j, k_3 r(U))\cap\partial\Omega_j) \leq 2k_2k_3 r(U)\sqrt{1+k_1^2}$.

So $B(X,r)\cap\partial U$ is contained in some connected subset Δ of ∂U (possibly all of ∂U) of size at most $2k_2k_3\sqrt{1+k_1^2}r(U) \leq k_2k_3\sqrt{1+k_1^2}\sigma(B(X,r)\cap\partial U)$.

In analyzing functions in the upper half-plane $\mathbf{R}^2_+$, it is often useful to consider $B(X,r)\cap\mathbf{R}^2_+$ for some $X\in\partial\mathbf{R}^2_+$. If U is a Lipschitz domain, $B(X,r)\cap U$ may be very badly behaved. To get around this, we instead work with tents $Q(X,r)$ defined as follows. If $X\in\partial U$, then $X \in B_j = B(X_j, r_j)$ for one of the balls B_j of Definition 2.4. Let $\mathbf{e}_j$, $\mathbf{e}_j^\perp$, φ_j be the unit vectors and Lipschitz function associated with the special Lipschitz domain Ω_j. Then

$$Q(X,r) = \{X\in\mathbf{R}^2 : |X\cdot\mathbf{e}_j^\perp| < r, \varphi_j(X\cdot\mathbf{e}_j^\perp) < X\cdot\mathbf{e}_j < \varphi_j(X\cdot\mathbf{e}_j^\perp) + (1+k_1)r\}. \quad (2.6)$$

(See Figure 2.1.) We let $\Delta(X,r) = \partial Q(X,r)\cap\partial U$. Then $Q(X,r)$ is a good Lipschitz domain whose Lipschitz constants depend only on k_1, which contains $U\cap B(X,r)$ and is contained in $B(X, r\sqrt{1+(1+2k_1)^2})$, and such that $\sigma(\partial Q(X,r)) \leq r(2+2k_1+4\sqrt{1+k_1^2})$.

Let $\chi_\pm(X,r) = X \pm r\mathbf{e}_j^\perp + \varphi_j(X\cdot\mathbf{e}_j^\perp \pm r)\mathbf{e}_j$ be the two endpoints of $\Delta(X,r)$. Then for a large enough (depending on k_1), $\partial Q(X,r)\setminus\partial U \subset \gamma_a(X_+)\cup\gamma_a(X_-)$.

It should be noted that $Q(X,r)$ depends on our choice of Ω_j, $\mathbf{e}_j$, and also that if U is not a special Lipschitz domain, then $Q(X,r)$ is defined only for r sufficiently small. These technicalities will not matter to our applications.

We can bound some integrals which will be needed later. If $1 < p < \infty$ then, letting $b = p^{1/(p-1)} \leq e$, we have that

$$\int_{\partial U \setminus B(X_0,r)} \frac{r^{p-1}}{|X - X_0|^p}\,d\sigma(X) \leq \sum_{j=0}^{\infty} \frac{r^{p-1}}{(b^j r)^p}\sigma(B(X_0, rb^{j+1}) \cap \partial U) \leq \frac{pek_4}{p-1} \tag{2.7}$$

In particular, if $X_0 \notin \partial U$ then

$$\int_{\partial U} \frac{1}{|X - X_0|^p}\,d\sigma(X) \leq \frac{pek_4}{p-1}\,\mathrm{dist}(X_0, \partial U)^{1-p}.$$

The letter C will always represent a positive constant, whose value may change from line to line, but which depends only on the ellipticity constants λ, Λ of A, A_0, the positive constant a in the definition of non-tangential maximal function, and the Lipschitz constants of whatever domain we are dealing with. If a particular constant depends on another parameter, it will be indicated explicitly. We will occasionally use the symbols $\lesssim$, $\gtrsim$ to indicate inequality up to a multiplicative constant (e.g., $\|\mathcal{T}f\| \lesssim \|f\|$ as shorthand for $\|\mathcal{T}f\| \leq C\|f\|$); we will use $\approx$ to mean that $\lesssim$ and $\gtrsim$ both hold.

If μ is a measure on a set E and f is a μ-measurable function defined on E, we let $\fint_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu$ be the average integral of f over E.

Recall the Hardy-Littlewood maximal function $Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f|$. If U is a domain, we may generalize this maximal function to functions defined on ∂U by either

$$M_{ball}f(X) = \sup_{r>0} \fint_{B(X,r)\cap\partial U} f\,d\sigma \quad \text{or} \quad M_{cctd}f(X) = \sup_{\Delta \ni X,\, \Delta \text{ connected}} \fint_{\Delta} f\,d\sigma$$

where σ is the surface measure on ∂U. If U is a good Lipschitz domain, then

$$\frac{1}{C} M_{cctd}f(X) \leq M_{ball}f(X) \leq C M_{cctd}f(X)$$

by Remark 2.5. In most cases, it will not matter whether we use $M_{cctd}f$ or $M_{ball}f$; we will use Mf to denote either of them.

Lemma 2.8. *Let $A : \mathbf{R}^2 \mapsto \mathbf{C}^{2\times 2}$ be an elliptic matrix-valued function. Then, for each $X \in \mathbf{R}^2$, there is a function $\Gamma_X \in W^{1,1}_{loc}(\mathbf{R}^2)$, unique up to an additive constant, such that* *for every* $Y \in \mathbf{R}^2$,

$$|\Gamma_X(Y)| \leq C + C\,|\log|X-Y||$$

and for every $\eta \in C_0^\infty(\mathbf{R}^2)$,

$$\int_{\mathbf{R}^2} A(Y)\nabla\Gamma_X(Y)\cdot\nabla\eta(Y)\,dy = -\eta(X). \tag{2.9}$$

We refer to this function as the *fundamental solution for* $\operatorname{div} A\nabla$ *with pole at* X.

This lemma will be proven in Chapter 4. By $\nabla\Gamma_X(Y)$ we mean the gradient in Y. We will sometimes wish to refer to the gradient in X; we will then write $\nabla_X\Gamma_X(Y)$.

$H^1(\mathbf{R})$ is defined to be $\{f : \mathbf{R} \mapsto \mathbf{C} \mid \|\sup_t f * \Phi_t(x)\|_{L^1(\mathbf{R})} \text{ is finite}\}$ for some Schwarz function Φ with $\int\Phi = 1$, where $\Phi_t(x) = \frac{1}{t}\Phi(x/t)$. Fixing a choice of Φ lets us define $\|f\|_{H^1}$. It can be shown ([27, Section III.2]) that if $f \in H^1$, then $f = \sum_k \lambda_k a_k$, where $\lambda_k \in \mathbf{C}$, $\int a_k = 0$, $\|a_k\|_{L^\infty} \leq 1/r_k$, $\operatorname{supp} a_k \subset B(x_k, r_k)$ for some $x_k \in \mathbf{R}$, $r_k > 0$, and $\sum_k |\lambda_k| \approx \|f\|_{H^1}$. Functions a satisfying these conditions are called *atoms*.

We may extend the definition of H^1 to $H^1(\partial U)$, where U is a Lipschitz domain. We say that $f \in H^1(\partial U)$ if $f = \sum_k \lambda_k a_k$, where the λ_k are complex numbers, and $\int_{\partial U} a_k\,d\sigma = 0$, $\operatorname{supp} a_k \subset \Delta_k$ for some $\Delta_k \subset \partial U$ connected, and $\|a_k\|_{L^\infty} \leq 1/\sigma(\Delta_k)$. The norm is the smallest $\sum_k |\lambda_k|$ among all such representations of f.

If U is a good Lipschitz domain, then this is equivalent to defining H^1 atoms to be functions a which satisfy $\int_{\partial U} a\,d\sigma = 0$, $\operatorname{supp} a \subset B(X,R)\cap\partial U$ for some $X \in \partial U$, and $\|a\|_{L^\infty} \leq 1/R$.

From [27, Section III.5.7], we have that if Δ is a bounded connected set, g is supported on Δ, $\int_\Delta g = 0$ and $1 < p \leq \infty$, then

$$\|g\|_{H^1} \leq C\|g\|_{L^p(\Delta)}\sigma(\Delta)^{1/q}. \tag{2.10}$$

Thus, if ∂V is bounded then $L^p(\partial V)\cap H^1(\partial V)$ is merely $L^p_0(\partial V)$, the set of functions in $L^p(\partial V)$ with mean zero. Conversely, if ∂V is unbounded then $H^1(\partial V)$ is dense in $L^p(\partial V)$.

We consider $BMO(\partial U)$ to be the dual of $H^1(\partial U)$. This means that

$$\|f\|_{BMO(\partial U)} = \sup_{\Delta\subset\partial U \text{ connected}} \frac{1}{\sigma(\Delta)}\int_\Delta \left|f - ⨍_\Delta f\right|\,d\sigma.$$

If U is a good Lipschitz domain, then by Remark 2.5

$$\|f\|_{BMO(\partial U)} \approx \sup_{X\in\partial U,\, R>0} \frac{1}{R}\int_{B(X,R)\cap\partial U} \left| f - \fint_{B(X,R)\cap\partial U} f \right| d\sigma. \tag{2.11}$$

Multiply connected domains are generally beyond the scope of this dissertation; Theorem 7.8 has been carefully written to holds in multiply connected domains. This theorem requires that H^1 functions integrate to 0 on each connected component of ∂V; this is one of the reasons why we define H^1 atoms supported on connected sets rather than balls.

Let V be a Lipschitz domain with unit outward normal and unit tangent vectors ν and τ. If $f : \partial V \mapsto \mathbf{C}$ is a function, we define the layer potentials by

$$\nabla\mathcal{S}f(X) = \nabla\mathcal{S}_V^A f(X) = \int_{\partial V} \nabla_X \Gamma_X^{A^T}(Y) f(Y)\, d\sigma(Y) \tag{2.12}$$

$$\mathcal{D}f(X) = \mathcal{D}_V^A f(X) = \int_{\partial V} \nu(Y)\cdot A^T(Y)\nabla\Gamma_X^{A^T}(Y) f(Y)\, d\sigma(Y). \tag{2.13}$$

This defines $\mathcal{S}f$ up to an additive constant.

We define the boundary layer potentials $\mathcal{K}$, $\mathcal{L}$ via

$$\mathcal{K}_V^A f(X) = \lim_{Z\to X,\, Z\in\gamma(X)} \int_{\partial V} \nu(Y)\cdot A^T(Y)\nabla\Gamma_Z^{A^T}(Y) f(Y)\, d\sigma(Y) \tag{2.14}$$

$$\mathcal{K}_\pm f(X) = \pm\mathcal{K}_{V_\pm}^A f(X) = \lim_{Z\to X,\, Z\in\gamma_\pm(X)} \mathcal{D}_V^A f(X), \tag{2.15}$$

$$\mathcal{L}f(X) = \mathcal{L}_V^A f(X) = \lim_{Z\to X,\, Z\in\gamma(X)} \int_{\partial V} \tau(Y)\cdot\nabla\Gamma_Z^{A^T}(Y) f(Y)\, d\sigma(Y). \tag{2.16}$$

When no confusion will arise we omit the subscripts and superscripts. We will show (5.1) that $\mathcal{D}f$ and $\nabla\mathcal{S}f$ exist for $X \notin \partial V$ and $f \in L^p(\partial V)$, $1 \leq p < \infty$. We will show that for good functions f, the limits in the definition of $\mathcal{K}$ and $\mathcal{L}$ are well-defined Lemma 5.8).

We also need the following definitions:

$$A(X) = \begin{pmatrix} a_{11}(X) & a_{12}(X) \\ a_{21}(X) & a_{22}(X) \end{pmatrix}, \quad A_0(X) = \begin{pmatrix} a_{11}^0(X) & a_{12}^0(X) \\ a_{21}^0(X) & a_{22}^0(X) \end{pmatrix} \tag{2.17}$$

$$\nabla\tilde\Gamma_X^T(Y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A(Y)\nabla\Gamma_X^T(Y) \tag{2.18}$$

$$B_6^A(X) = \begin{pmatrix} a_{11}(X) & a_{21}(X) \\ 0 & 1 \end{pmatrix} \tag{2.19}$$

$$B_7^A(X) = \Big(B_6(X)^t\Big)^{-1}\Big(A(X)\nu(X) \quad \tau(X)\Big) \tag{2.20}$$

$$K^A(X,Y) = \Big(B_6^A(Y)\nabla\Gamma_X^{A^T}(Y) \quad B_6^A(Y)\nabla\Gamma_X^{A^T}(Y)\Big)^t \tag{2.21}$$

$$\tilde{K}^A(X,Y) = B_6^A(X)\Big(\nabla_X\tilde{\Gamma}_X^{A^T}(Y) \quad \nabla_X\tilde{\Gamma}_X^{A^T}(Y)\Big) \tag{2.22}$$

$$\mathcal{T}_V^A F(X) = \lim_{Z\to X \text{ n.t.}, Z\in V} \int_{\partial V} K^A(Z,Y)F(Y)\,d\sigma(Y) \tag{2.23}$$

$$\tilde{\mathcal{T}}_V F(X) = \lim_{Z\to X \text{ n.t.}, Z\in V} \int_{\partial V} \tilde{K}(Z,Y)F(Y)\,d\sigma(Y) \tag{2.24}$$

When considering special Lipschitz domains, we will need some terminology:

$$\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \mathbf{e}^\perp = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{e} = \begin{pmatrix} e_2 \\ -e_1 \end{pmatrix} \tag{2.25}$$

$$\psi(x) = x\mathbf{e}^\perp + \varphi(x)\mathbf{e} \in \partial\Omega \tag{2.26}$$

$$\psi(x,h) = \psi(x) + h\mathbf{e} = \begin{pmatrix} xe_2 + (\varphi(x)+h)e_1 \\ -xe_1 + (\varphi(x)+h)e_2 \end{pmatrix} \tag{2.27}$$

$$\langle G, T_\pm F\rangle = \lim_{h\to 0^\pm} \int_{\mathbf{R}^2} G(x)^t K_h(x,y)F(y)\,dy\,dx \tag{2.28}$$

$$\langle G, \tilde{T}_\pm F\rangle = \lim_{h\to 0^\mp} \int_{\mathbf{R}^2} G(x)^t \tilde{K}_h(x,y)F(y)\,dy\,dx \tag{2.29}$$

$$\begin{aligned} K_h(x,y) &= \Big(B_6(\psi(y))\nabla\Gamma^T_{\psi(x,h)}(\psi(y)) \quad B_6(\psi(y))\nabla\Gamma^T_{\psi(x,h)}(\psi(y))\Big)^t \\ &= \begin{pmatrix} \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t \\ \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t \end{pmatrix} B_6(\psi(y))^t = K^A(\psi(x,h),\psi(y)) \end{aligned} \tag{2.30}$$

$$\tilde{K}_h(x,y) = B_6(\psi(x))\Big(\nabla_X\tilde{\Gamma}^T_{\psi(x)}(\psi(y,h)) \quad \nabla_X\tilde{\Gamma}^T_{\psi(x)}(\psi(y,h))\Big) \tag{2.31}$$

$$\tau(x) = \frac{1}{\sqrt{1+\varphi'(x)^2}}(\mathbf{e}^\perp + \varphi'(x)\mathbf{e}) = \frac{1}{\sqrt{1+\varphi'(x)^2}}\begin{pmatrix} e_2 + e_1\varphi'(x) \\ -e_1 + e_2\varphi'(x) \end{pmatrix} \tag{2.32}$$

$$\nu(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\tau(x) = \frac{1}{\sqrt{1+\varphi'(x)^2}}\begin{pmatrix} -e_1 + e_2\varphi'(x) \\ -e_2 - e_1\varphi'(x) \end{pmatrix} \tag{2.33}$$

$$B_1(\psi(x,h)) = \Big(B_6(\psi(x,h))^t\Big)^{-1}\sqrt{1+\varphi'(x)^2}\Big(A(\psi(x,h))\nu(x) \quad \tau(x)\Big) \tag{2.34}$$

$$B_1(x) = B_1(\psi(x)) = \sqrt{1+\varphi'(x)^2}B_7^A(\psi(x))$$

We have that $\Omega = \{X : \varphi(X\cdot \mathbf{e}^\perp) < X\cdot\mathbf{e}\} = \{\psi(x,h) : x\in\mathbf{R},\ h>0\}$ and that $(x,t) = \psi(e_2x - e_1t, e_1x + e_2t - \varphi(e_2x-e_1t))$.

We will occasionally want slightly different forms of K, T:

$$\begin{aligned}
K'_h(x,y) &= \begin{pmatrix} B_6(\psi(y,h))\nabla\Gamma^T_{\psi(x)}(\psi(y,h)) & B_6(\psi(y,h))\nabla\Gamma^T_{\psi(x)}(\psi(y,h))\end{pmatrix}^t \qquad (2.35)\\
&= \begin{pmatrix}\nabla\Gamma^T_{\psi(x)}(\psi(y,h))^t\\ \nabla\Gamma^T_{\psi(x)}(\psi(y,h))^t\end{pmatrix} B_6(\psi(y,h))^t\\
\tilde K'_h(x,y) &= B_6(\psi(x,h))\begin{pmatrix}\nabla_X\tilde\Gamma^T_{\psi(x,h)}(\psi(y)) & \nabla_X\tilde\Gamma^T_{\psi(x,h)}(\psi(y))\end{pmatrix} \qquad (2.36)\\
\langle G, T'_\pm F\rangle &= \lim_{h\to0^\pm}\int_{\mathbf{R}^2} G(x)^t K'_h(x,y)F(y)\,dy\,dx \qquad (2.37)\\
\langle G, \tilde T'_\pm F\rangle &= \lim_{h\to0^\mp}\int_{\mathbf{R}^2} G(x)^t \tilde K'_h(x,y)F(y)\,dy\,dx \qquad (2.38)
\end{aligned}$$

We will later show (Section 4.6) that if $A - I\in C_0^\infty(\mathbf{R}\mapsto\mathbf{C}^{2\times2})$, then $T_\pm = T'_\mp$ and $\tilde T_\pm = \tilde T'_\mp$ on $C_0^\infty(\mathbf{R}\mapsto\mathbf{C}^{2\times2})$. These requirements will be dealt with in Section 6.1 and Chapter 11.

If f is a function defined on $\partial\Omega$, we will often use $f(x)$ as shorthand for $f(\psi(x))$. So $\mathcal{T}_{\Omega\pm}F(\psi(x)) = T_\pm(\sqrt{1+(\varphi')^2}F\circ\psi)(x)$, and

$$\begin{aligned}
T_\pm(B_1f)(x) = \mathcal{T}_{\Omega\pm}(B_7f)(\psi(x)) &= \lim_{h\to0^\pm}\int_{\mathbf{R}} K_h(x,y)B_1(y)f(y)\,dy\\
&= \lim_{h\to0^\pm}\int_{\mathbf{R}}\begin{pmatrix}\nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t\\ \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t\end{pmatrix}\sqrt{1+\varphi'(y)^2}\begin{pmatrix}A(\psi(y))\nu(y) & \tau(y)\end{pmatrix} f(y)\,dy\\
&= \begin{pmatrix}\mathcal{K}_\pm f(\psi(x)) & \mathcal{L}f(\psi(x))\\ \mathcal{K}_\pm f(\psi(x)) & \mathcal{L}f(\psi(x))\end{pmatrix}
\end{aligned}$$

and B_1 is bounded with a bounded inverse; thus, we need only show that $T_\pm$ is bounded from $L^p(\mathbf{R}\mapsto\mathbf{C}^{2\times2})$ to itself to show that the layer potentials $\mathcal{K}_\pm$, $\mathcal{L}$ are bounded on special Lipschitz domains.

CHAPTER 3

USEFUL THEOREMS

In this chapter we collect some fairly general lemmas that will be useful throughout this dissertation.

3.1 Nontangential maximal functions

We begin with some results concerning non-tangential maximal functions.

If $NF \in L^p(\partial V)$, we can bound $F(Y)$ at any point $Y \in V$ as follows:

$$\|NF\|_{L^p}^p \geq \int_{\gamma(X)\ni Y} NF(X)^p \, d\sigma(X)$$

But if $Y \in \gamma(X)$, then $NF(X) \geq |F(Y)|$, so

$$\|NF\|_{L^p}^p \geq |F(Y)|^p \sigma\{X : Y \in \gamma(X)\}.$$

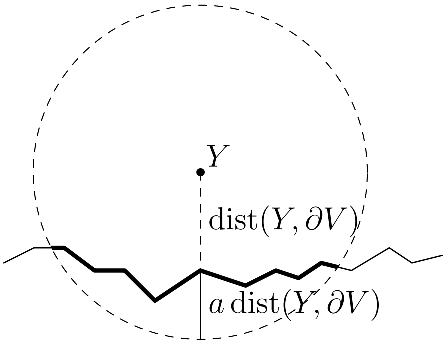

Figure 3.1: X with $\gamma(X) \ni Y$

But $Y \in \gamma(X)$ if and only if $|X - Y| < (1+a)\operatorname{dist}(Y, \partial V)$, so

$$\{X \in \partial V : Y \in \gamma(X)\} = \partial V \cap B(Y, (1+a)\operatorname{dist}(Y, \partial V))$$

which has measure at least $\min(\sigma(\partial V), 2a\operatorname{dist}(Y, \partial V))$ (see Figure 3.1); so

$$|F(Y)| \leq \frac{C\|NF\|_{L^p(\partial V)}}{\min(\sigma(\partial V), \operatorname{dist}(Y, \partial V))^{1/p}}. \tag{3.1}$$

Lemma 3.2. *Recall that* $N_aF(X) = \sup\{|F(Y)| : Y \in V,\ |X - Y| \leq (1 + a)\operatorname{dist}(Y, \partial V)\}$ *where we omit the* V *subscript. Suppose that* $0 < a < b$ *and* V *is a good Lipschitz domain. Then for all* $1 \leq p \leq \infty$,

$$\|N_bF\|_{L^p(\partial V)} \leq C\|N_aF\|_{L^p(\partial V)}$$

for some constant C *depending only on* a, b *and the Lipschitz constants of* V.

This lemma lets us assume that a is large enough that, if Q is as in (2.6), then $\partial Q(X, R) \setminus \Delta(X, R) \subset \gamma_a(\chi_+(X, R)) \cup \gamma_a(\chi_-(X, R))$.

Proof. It is obvious for $p = \infty$. Suppose that $N_aF \in L^p(\partial V)$ for $1 \leq p < \infty$. Define

$$\delta(Y) = \operatorname{dist}(Y, \partial\Omega), \quad E_a(\alpha) = \bigcup_{|F(Y)|>\alpha} B(Y, (1 + a)\delta(Y)),$$

so that $\{X : N_aF(X) > \alpha\} = E_a(\alpha) \cap \partial V$. Recall that

$$\|N_aF\|^p_{L^p(\partial V)} = \int_0^\infty p\alpha^{p-1}\sigma\{X \in \partial V : N_aF(X) > \alpha\}\, d\alpha.$$

So we need only show that $\sigma(E_b(\alpha) \cap \partial V) \leq C\sigma(E_a(\alpha) \cap \partial V)$ for all $\alpha > 0$ to complete the proof. In fact, if ∂V is bounded, we need this result only for $\alpha > 2\|N_aF\|_{L^p}\sigma(\partial V)^{-1/p}$.

If $\alpha > 2\|N_aF\|_{L^p}\sigma(\partial V)^{-1/p}$ (or if $\sigma(\partial V) = \infty$), and if $F(Y) > \alpha$, then by (3.1) we can bound $\operatorname{dist}(Y, \partial V)$. So $E_a(\alpha)$ is a union of balls with bounded radii. By the Vitali lemma there is a countable set $\{Y_i\}_{i=1}^\infty$ of points such that the $B(Y_i, (1 + a)\delta(Y))$ are pairwise disjoint and such that $E_a(\alpha) \subset \cup_i B(Y_i, C_1(1 + a)\delta(Y_i))$ where C_1 is a fixed constant (i.e., depends on nothing except the dimension of the ambient space $\mathbf{R}^2$).

If $|F(Y)| > \alpha$, then $Y \in B(Y_i, C_1(1+a)\delta(Y_i))$ for some i. Then $\delta(Y_i) \geq \delta(Y) - |Y - Y_i| \geq \delta(Y) - C_1(1 + a)\delta(Y_i)$, so $\delta(Y) \leq (C_1 + C_1a + 1)\delta(Y_i)$.

Thus, $B(Y, (1 + b)\delta(Y)) \subset B(Y_i, (1 + b)\delta(Y) + |Y - Y_i|)$ and so $E_b(\alpha) \subset \cup_i B(Y_i, C_2\delta(Y_i))$ where $C_2 > 1 + a$ depends only on a, b and C_1.

Let $Y^* \in \partial V$ with $|Y_i - Y^*| = \operatorname{dist}(Y_i, \partial V)$. Then either $\partial V \not\subset B(Y_i, (1 + a)\delta(Y_i)$, so

$$\sigma\left(B\left(Y_i, C\delta(Y_i)\right) \cap \partial V\right) \leq Ck_4\delta(Y_i) \leq C\frac{k_4}{2a}(2a\delta(Y_i)) \leq C\frac{k_4}{2a}\sigma(B(Y_i, (1 + a)\delta(Y_i)) \cap \partial V)$$

or $\partial V \subset B(Y_i, (1+a)\delta(Y_i))$, so

$$\sigma\left(B\left(Y_i, C\delta(Y_i)\right) \cap \partial V\right) \leq \sigma(\partial V) = \sigma\left(B\left(Y_i, (1+a)\delta(Y_i)\right) \cap \partial V\right)$$

Thus,

$$\begin{aligned}\sigma(E_b(\alpha) \cap \partial V) &\leq \sum_i \sigma(B(Y_i, C\delta(Y_i)) \cap \partial V) \leq \sum_i C\sigma(B(Y_i, (1+a)\delta(Y_i)) \cap \partial V) \\ &\leq C\sigma(E_a(\alpha) \cap \partial V). \qquad \square\end{aligned}$$

Lemma 3.3. *Suppose that V is a good Lipschitz domain and that $NF \in L^p(\partial V)$ for some $1 \leq p < \infty$. Then $F = F_1 + F_2$, where $\|F_1\|_{L^{2p}(V)} \leq C\|NF\|_{L^p(\partial V)}$ and $\|F_2\|_{L^\infty(V)} \leq \|NF\|_{L^p(\partial V)}/\sigma(\partial V)^{1/p}$ if ∂V is compact and $F_2 \equiv 0$ otherwise.*

Furthermore, if $F = \nabla u$ for some function u, then $u \in L^\infty_{loc}(\bar{V})$.

If V is bounded or special, this implies that $\|F\|_{L^{2p}} \leq C\|NF\|_{L^p}$.

Proof. If $NF \in L^p(\partial V)$, define

$$E(\alpha) = \{X \in V : |F| > \alpha\}, \quad e(\alpha) = \{X \in \partial V : NF > \alpha\}.$$

If $F \not\equiv 0$ then $\alpha^p \sigma(e(\alpha)) < \|NF\|^p_{L^p}$. If ∂V is compact let $\alpha_0 = \|NF\|_{L^p}/\sigma(V)^{1/p}$; otherwise let $\alpha_0 = 0$. If $\alpha \geq \alpha_0$ then there is some point in ∂V not in $e(\alpha)$.

Let $F_1 = F$ on $E(\alpha_0)$, and let $F_1 = 0$ otherwise, so $\|F_2\|_{L^\infty(V)} \leq \alpha_0$.

Let $X \in E(\alpha)$ for $\alpha \geq \alpha_0$. Let $X^* \in \partial V$ with $|X - X^*| = \operatorname{dist}(X, \partial V)$, and let $\Delta \subsetneq \partial V$ be the connected component of $e(\alpha)$ containing X^*. Then $X \notin \gamma(Y)$ for any $Y \in \partial V \setminus e(\alpha)$. So

$$\operatorname{dist}(X, \partial V) + \frac{1}{2}\sigma(\Delta) \geq \operatorname{dist}(X, \partial V \setminus e(\alpha)) \geq (1+a)\operatorname{dist}(X, \partial V)$$

and so $\operatorname{dist}(X, \partial V) \leq \frac{1}{2a}\sigma(\Delta)$. So

$$E(\alpha) \subset \bigcup_{\substack{\Delta \subset e(\alpha) \\ \text{connected}}} \bigcup_{X^* \in \Delta} B\left(X^*, \frac{1}{2a}\sigma(\Delta)\right).$$

But if Δ is a connected curve segment in the plane, then $\bigcup_{X^* \in \Delta} B\left(X^*, c\sigma(\Delta)\right)$ is of size at most $(1+2c)^2\sigma(\Delta)^2$. So $|E(\alpha)| \leq C\sigma(e(\alpha))^2$ for all $\alpha \geq \alpha_0$, where $|\cdot|$ is used for Lebesgue measure in $\mathbf{R}^2$. In particular, $|E(\alpha_0)| \leq C\sigma(\partial V)^2$ and so in general $\alpha_0^p|E(\alpha_0)| \leq C\|NF\|_{L^p(\partial V)}^{2p}$.

Now,

$$\begin{aligned}
\int_V |F_1|^{2p} &= \int_0^\infty 2p\alpha^{2p-1}|\{X : |F_1(X)| > \alpha\}|\,d\alpha \\
&\leq \alpha_0^{2p}|E(\alpha_0)| + C\int_{\alpha_0}^\infty 2p\alpha^{2p-1}\,\sigma(e(\alpha))^2\,d\alpha \\
&\leq C\|NF\|_{L^p}^{2p} + C\|NF\|_{L^p}^p \int_{\alpha_0}^\infty p\alpha^{p-1}\,\sigma(e(\alpha))\,d\alpha \leq C\|NF\|_{L^p}^{2p}.
\end{aligned}$$

We now must establish that if $N(\nabla u) \in L^p$ then $u \in L^\infty_{loc}$. By (3.1), u is continuous on compact subsets of V because ∇u is bounded; we need only look at a small neighborhood of the boundary, and so we need only consider $V = \Omega$ a special Lipschitz domain.

By Lemma 3.2, we may assume that a is large enough that $N(\nabla u)(\psi(x)) < |\nabla u(\psi(x,t))|$ for all $t > 0$. For some $X_0 = \psi(x_0) \in \partial\Omega$, $N(\nabla u)(X_0)$ is finite. Then for any $t > 0$, $|u(\psi(x_0,t)) - u(X_0)| \leq tN(\nabla u)(X_0)$ is finite.

Now, for any $x \in \mathbf{R}$ and any $t > 0$,

$$\begin{aligned}
|u(\psi(x,t)) - u(X_0)| &\leq |u(\psi(y,t)) - u(\psi(x_0,t))| + |u(\psi(x_0,t)) - u(\psi(x_0))| \\
&\leq tN(\nabla u)(X_0) + \int_{x_0}^x |\nabla u(\psi(y,s))|\,dy \\
&\leq tN(\nabla u)(X_0) + |x-x_0|^{1/q}\|N(\nabla u)\|_{L^p(\partial\Omega)}. \qquad \square
\end{aligned}$$

3.2 Bounds on elliptic PDE

We now turn to solutions to elliptic PDE. Suppose that $\operatorname{div} A\nabla u = 0$ in U where A satisfies (1.1). Let $B_r \subset \mathbf{R}^n$ be a ball of radius r, $B_{r/2}$ be the concentric ball of radius $r/2$, such that either $u = 0$ on $\partial U \cap B_r$ or $\nu \cdot A\nabla u = 0$ on $\partial U \cap B_r$.

(If $B_r \subset U$, then these conditions are both trivially true, and in fact it is this case which is used most often. Here we assume that U is a good Lipschitz domain; some of the following lemmas hold in more generality.)

In $\mathbf{R}^n$ for real coefficients, the following lemmas are well known. (See, for example, [21] and [15, Chapter 8].)

Lemma 3.4 (The Caccioppoli inequality). *For some constant C depending only on λ, Λ,*

$$\int_{U\cap B_{r/2}} |\nabla u|^2 \leq \frac{C}{r^2}\int_{U\cap B_r} |u|^2.$$

This lemma holds for complex coefficients in all dimensions, and may be proven as for real coefficients.

Lemma 3.5. *For some $C > 0$, $p > 2$ depending only on λ, Λ,*

$$\left(\frac{1}{r^n}\int_{B_{r/2}\cap U} |\nabla u|^p\right)^{1/p} \leq C\left(\frac{1}{r^n}\int_{B_r\cap U} |\nabla u|^2\right)^{1/2}.$$

This reverse Hölder inequality follows easily from [14, Theorem 1.2, Chapter V] and preceding remarks; it uses the Caccioppoli inequality.

In two dimensions, this lemma and [11, Chapter 5, Theorem 5] let us prove the following two lemmas:

Lemma 3.6. *For all $1 \leq p \leq \infty$, there is a constant $C(p)$ depending only on λ, Λ, p, such that*

$$\sup_{B_{r/2}\cap U} |u| \leq C(p)\left(\frac{1}{r^2}\int_{B_r\cap U} |u|^p\right)^{1/p}.$$

Lemma 3.7. *For some C, $\alpha > 0$ depending only on λ, Λ,*

$$\sup_{X,Y\in B_{r/2}} |u(X) - u(Y)| \leq C\frac{|X-Y|^\alpha}{r^{1+\alpha}}\|u\|_{L^2(B_r)}.$$

Now, assume that A satisfies (1.2), so $A(x,t) = A(x)$. For each $\eta \in L^1(\mathbf{R})$ with small support, if u is a solution in a ball then $u_r(x,t) = \int u(x,t-s)\frac{1}{r}\eta\left(\frac{s}{r}\right)\,ds$ is a solution in a slightly smaller ball. By letting η smooth with $\int \eta = 1$, we see that any solution may be approximated in $W^{1,p}_{loc}$, $p \leq 2$, by solutions u_r smooth in the t-variable.

But then $\nabla(\partial_t u_r) = \partial_t \nabla u_r$ exists, and

$$\int \nabla\eta(x,t) \cdot A(x)\nabla\partial_t u_r(x,t)\, dx\, dt = \frac{d}{ds}\left(\int \nabla\eta(x,t) \cdot A(x)\nabla u_r(x,t+s)\, dx\, dt\right) = 0$$

and so $\partial_t u_r$ is also a solution to $\operatorname{div} A\nabla u = 0$. So in particular, we may use Lemma 3.6:

$$\sup_{B_{r/2}} |\partial_t u_r| \leq C(p)\left(\fint_{B_r} |\partial_t u_r|^p\right)^{1/p} \leq C(p)\left(\fint_{B_r} |\nabla u_r|^p\right)^{1/p}.$$

By letting $r \to 0$, we recover that $\sup_{B_{r/2}} |\partial_t u| \leq C(p)\left(\fint_{B_r} |\nabla u_r|^p\right)^{1/p}$. Similarly, Lemma 3.7 holds for $\partial_t u$ as well as u.

In Section 4.4, we will construct a function $\tilde{u}$ such that $\operatorname{div} \frac{1}{\det A} A^T \nabla\tilde{u} = 0$, $\partial_t \tilde{u} = a_{11}\partial_x u + a_{12}\partial_t u$, and $\partial_x u = \frac{1}{a_{11}}\partial_x \tilde{u} + \frac{a_{12}}{a_{11}}\partial_t u$. The same arguments let us bound $\partial_t \tilde{u}$. Then

$$\begin{aligned}\sup_{B_{r/2}} |\nabla u| &\leq \sup_{B_{r/2}} |\partial_x u| + \sup_{B_{r/2}} |\partial_t u| \leq C \sup_{B_{r/2}} |\partial_t \tilde{u}| + C \sup_{B_{r/2}} |u_t| \\ &\leq C(p)\left(\fint_{B_r} |\partial_t \tilde{u}|^p\right)^{1/p} + C(p)\left(\fint_{B_r} |\partial_t u|^p\right)^{1/p} \leq C(p)\left(\fint_{B_r} |\nabla u|^p\right)^{1/p}\end{aligned} \tag{3.8}$$

So Lemma 3.6 holds for ∇u as well as u if $A(x,t) = A(x)$ and $B_r \subset U$.

3.3 Existence results

Finally, we prove two existence results for solutions to elliptic PDE. These lemmas do not presume smoothness of coefficients.

Recall that $W^{1,2}$ is the Sobolev space of functions with one weak derivative in L^2, with $\|f\|^2_{W^{1,2}(V)} = \|f\|^2_{L^2(V)} + \|\nabla f\|^2_{L^2(V)}$. We define the superspace $\dot{W}^{1,2}$ as the space of all (equivalence classes modulo additive constants of) functions in $W^{1,2}_{loc}$ such that the norm $\|f\|_{\dot{W}^{1,2}(V)} = \|\nabla f\|_{L^2(V)}$ is finite.

Then solutions to the Dirichlet or Neumann problems exist in this space:

Lemma 3.9. *If V is a Lipschitz domain and $f \in W^{1,2}(\partial V) \cap L^{7/6}(\partial V) \cap L^{17/6}(\partial V)$, then there is some function $u \in W^{1,2}_{loc}(V)$ such that* $\operatorname{div} A\nabla u = 0$ *in V,* $\mathbf{Tr}\, u = f$ *and*

$$\|u\|_{\dot W^{1,2}(V)} \leq C(\|f\|_{W^{1,2}(\partial V)} + \|f\|_{L^{7/6}(\partial V)} + \|f\|_{L^{17/6}(\partial V)}).$$

Lemma 3.10. *If V is a Lipschitz domain and $g \in H^1(\partial V)$, then there is some function $u \in W^{1,2}_{loc}(V)$ such that $\|\nabla u\|_{L^2} \leq C\|g\|_{H^1}$ and such that $\nu \cdot A\nabla u = g$ on ∂V in the weak sense.*

If A is real, these lemmas are proven in [25, Lemma 1.1 and 1.2] for the upper half-plane, and thus by change of variables for any special Lipschitz domain Ω. The same proof works in the complex case, except that we must use a generalization (such as the Babuška's) of the Lax-Milgram lemma to complex Hilbert spaces. We need only discuss the technicalities of domains with compact boundary.

(A similar construction is presented for bounded domains in [26], but we want these lemmas to apply in their complements as well, and want more precise control on $\|\nabla u\|_{L^2(V)}$.)

The Babuška-Lax-Milgram lemma [3, Theorem 2.1] states that, if B is a bounded bilinear form on two complex Hilbert spaces H_1 and H_2, and if B is weakly coercive in the sense that

$$\sup_{\|w\|_1=1} |B(w,v)| \geq \lambda\|v\|_2, \qquad \sup_{\|w\|_2=1} |B(u,w)| \geq \lambda\|u\|_1$$

for every $u \in H_1$, $v \in H_2$, for some fixed $\lambda > 0$, then for every linear functional T defined on H_2 there is a unique $u_T \in H_1$ such that $B(u_T, v) = \overline{T(v)}$. Furthermore, $\|u_T\|_1 \leq \frac{1}{\lambda}\|T\|$.

We apply this theorem to the bilinear form

$$B(\xi,\eta) = \int_V \nabla\bar\eta \cdot A\nabla\xi$$

on $H_1 = H_2 = \dot W^{1,2}$; then B is clearly bounded and coercive.

Proof of Lemma 3.9. Suppose that $\mathbf{Tr}\, w = f$. Then $T(\eta) = \overline{B(w,\eta)}$ is a linear functional. Suppose that $\mathbf{Tr}\, v = 0$ and $B(v,\eta) = \overline{T(\eta)}$ for all η with $\mathbf{Tr}\,\eta = 0$. Then $\mathbf{Tr}(w - v) = f$, and $\operatorname{div} A\nabla(w - v) = 0$ in V in the weak sense.

So we need only construct a w with $\mathbf{Tr}\, w = f$ and $\|w\|_{\dot W^{1,2}} \leq C|||f|||$ where $|||f||| = \|f\|_{W^{1,2}(\partial V)} + \|f\|_{L^{7/6}(\partial V)} + \|f\|_{L^{17/6}(\partial V)}$. (This will yield that T is bounded on $\dot W^{1,2}$.) If

V is the upper half-plane, then w is constructed in [25, Lemma 1.1]; by change of variables it exists for an arbitrary special Lipschitz domain Ω. In fact, for fixed $X_0 \in \partial\Omega$, this w also satisfies

$$\int_\Omega \frac{|w(X)|^2}{1+|X-X_0|^2}\,dX \leq C\|\|f\|\|^2$$

and so if η is a smooth cutoff function supported in $B(X_0, R)$ with $|\nabla\eta| \leq C/R$, we have that

$$\|\nabla(w\eta)\|_{L^2} = \|\eta\nabla w + w\nabla\eta)\|_{L^2} \leq C\|\|f\|\|.$$

Suppose that ∂V is bounded; without loss of generality $\int_{\partial V} f = 0$. Let $\sum_j \eta_j$ be a smooth partition of unity near ∂V, with η_j supported in $B(X_j, \frac{3}{2}r_j)$, where the X_js, r_js are as in Definition 2.4. Let $w_j = f\eta_j$ on $\partial\Omega_j$, $w = \sum_j \tilde\eta_j w_j$ where $\tilde\eta_j = 1$ on $\operatorname{supp}\eta_j$ and is supported in $B(X_j, 2r_j)$. We may require $|\nabla\eta_j| \leq C/r_j$, $|\nabla\tilde\eta_j| \leq C/r_j$. Then $\mathbf{Tr}\, w = f$, and

$$\|\nabla w\|_{L^2(V)} \leq \sum_j \|\nabla(\tilde\eta_j w_j)\|_{L^2(V)} \leq \sum_j C\|\|f\eta_j\|\|$$

But since $\int_{\partial V} f = 0$, we have that $\|f\|_{L^2(\partial V)} \leq C\sigma(\partial V)\|\partial_\tau f\|_{L^2}$ and so since $|\partial_\tau \eta_j| \leq C/r_j \leq C/\sigma(\partial V)$, we have that

$$\begin{aligned}\|\|f\eta_j\|\| &= \|f\eta_j\|_{W^{1,2}(\partial V)} + \|f\eta_j\|_{L^{7/6}(\partial V)} + \|f\eta_j\|_{L^{17/6}(\partial V)}\\ &\leq \|f\partial_\tau\eta_j\|_{L^2} + \|\partial_\tau f\|_{L^2} + \|f\|_{L^2} + \|f\|_{L^{7/6}} + \|f\|_{L^{17/6}}\\ &\leq C\|\partial_\tau f\|_{L^2} + \|f\|_{L^2} + \|f\|_{L^{7/6}} + \|f\|_{L^{17/6}} \leq C\|\|f\|\|.\end{aligned}$$

So $\|\nabla w\|_{L^2(V)} \leq C\|\|f\|\|$, as desired. □

Proof of Lemma 3.10. Pick some $g \in H^1(\partial V)$. Consider the map $\xi \mapsto \int_{\partial V} \bar g\, \mathbf{Tr}\,\xi$. It is clearly linear. If it is bounded, and if $u \in \dot W^{1,2}$ is such that

$$\int_{\partial V} \bar g\, \mathbf{Tr}\,\xi = \overline{B(u,\xi)} = \int_V \nabla\xi \cdot \overline{A\nabla u}$$

for all $\xi \in \dot W^{1,2}$, then $\operatorname{div} A\nabla u = 0$, $\nu \cdot A\nabla u = g$ in the weak sense, and $\|\nabla u\|_{L^2(V)} = \|u\|_{\dot W^{1,2}(V)} \leq \frac{1}{\lambda}\left\|\xi \mapsto \int_{\partial V} g\, \mathbf{Tr}\,\xi\right\|$.

We need only show that $\xi \mapsto \int_{\partial V} \bar g\, \mathbf{Tr}\,\xi$ is a bounded operator on $\dot W^{1,2}(V)$.

It suffices to show that $\mathbf{Tr}$ is bounded from $\dot{W}^{1,2}(V)$ to $BMO(\partial V)$. So if $\Delta \subset \partial V$ is connected, then we must show that $\fint_\Delta \left|\xi - \fint_\Delta \xi\right| d\sigma \leq C\|\nabla\xi\|_{L^2(V)}$. In fact, we need only do this for $\sigma(\Delta) \leq \sigma(\partial V)/C$; so we may assume that $\Delta \subset B(X_j, 2r_j) \cap \partial V$ for one of the X_js, r_js of Definition 2.4.

By the Poincaré inequality, $\int_{R_j} \left|\xi - \fint_{R_j} \xi\right|^2 \leq Cr_j^2 \int_{R_j} |\nabla\xi|^2$. We may assume that $\fint_{R_j} \xi = 0$. Multiplying ξ by a smooth cutoff function η_j as before, we see that

$$\|\nabla(\eta_j\xi)\|_{L^2(V)} \leq \|\nabla\xi\|_{L^2(R_j)} + \frac{C}{r_j}\|\xi\|_{L^2(R_j)} \leq C\|\nabla\xi\|_{L^2(R_j)}.$$

So, taking $\eta_j \equiv 1$ on Δ, we need only show $\int_\Delta \left|\eta_j\xi - \fint_\Delta \eta_j\xi\right| \, d\sigma \leq \|\nabla(\eta_j\xi)\|_{L^2(R_j)}$. So we need only show that $\mathbf{Tr} : \dot{W}^{1,2}(\Omega_j) \mapsto BMO(\partial\Omega_j)$ is bounded. This is done in the proof of [25, Lemma 1.2]. $\square$

In domains with bounded complements (which in many ways are the most annoying domains we have to deal with), we will want one other property of these solutions:

Lemma 3.11. *Suppose that ∂V is bounded and that $\operatorname{div} A\nabla u = 0$ in V, and that $\nabla u \in L^2(V)$. Then $\int_{\partial V} \nu \cdot A\nabla u \, d\sigma = 0$.*

Proof. Let R be so large that $\partial V \subset B(0, R/2)$. Let $\eta \equiv 1$ on $B(0,R)$, $\eta \in C_0^\infty(B(0,2R))$ with $|\nabla\eta| \leq C/R$; then $\|\nabla\eta\|_{L^2(\mathbf{R}^2)} \leq C$.

So by the weak definition of $\nu \cdot A\nabla u$,

$$\left|\int_{\partial V} \nu \cdot A\nabla u \, d\sigma\right| = \left|\int_V \nabla\eta \cdot A\nabla u\right| \leq C\|\nabla\eta\|_{L^2}\|\nabla u\|_{L^2(V\cap \operatorname{supp}\nabla\eta)} \leq C\|\nabla u\|_{L^2(V\setminus B(0,R))}.$$

If V is bounded then $\|\nabla u\|_{L^2(V\setminus B(0,R))} = 0$ since $V \setminus B(0,R)$ is empty. If V^C is bounded then $\|\nabla u\|_{L^2(V\setminus B(0,R))}$ may be made arbitrarily small since $\nabla u \in L^2(V)$. $\square$

CHAPTER 4

THE FUNDAMENTAL SOLUTION

4.1 A fundamental solution exists

Recall that Lemma 2.8 states that there is a function $\Gamma_X^A(Y)$, unique up to an additive constant, called the fundamental solution of the operator $L = \operatorname{div} A\nabla$, such that

$$\int_{\mathbf{R}^2} A(Y)\nabla\Gamma_X^A(Y)\cdot\nabla\eta(Y)\,dY = -\eta(X)$$

for every $\eta \in C_0^\infty(\mathbf{R}^2)$, and where $|\Gamma_X(Y)| \leq C(1 + |\log|X-Y||)$ for some constant C. We will let $\Gamma = \Gamma^A$, $\Gamma^0 = \Gamma^{A_0}$, and $\Gamma^T = \Gamma^{A^T}$. This function is defined and constructed in the real case in the appendix to [22], and in 3 or more dimensions in [18].

We provide another construction. From [1, pp. 29–31], we know that there is a function[1] $\check{K}_t(X,Y)$ such that, for all $\eta \in C_0^\infty(\mathbf{R}^2)$,

$$\int \eta(X)\partial_t\check{K}_t(X,Y)\,dX = \int A(X)\nabla_X\check{K}(X,Y)\cdot\nabla\eta(X)\,dX.$$

Furthermore, there is some β, μ, $C > 0$ depending only on λ, Λ such that

$$\begin{aligned}
|\check{K}_t(X,Y)| &\leq \frac{C}{t}\exp\left\{-\frac{\beta|X-Y|^2}{t}\right\}\\
|\check{K}_t(X,Y) - \check{K}_t(X',Y)| &\leq \frac{C}{t}\left(\frac{|X-X'|}{\sqrt{t}+|X-Y|}\right)^\mu \exp\left\{-\frac{\beta|X-Y|^2}{t}\right\}\\
|\check{K}_t(X,Y) - \check{K}_t(X,Y')| &\leq \frac{C}{t}\left(\frac{|Y-Y'|}{\sqrt{t}+|X-Y|}\right)^\mu \exp\left\{-\frac{\beta|X-Y|^2}{t}\right\}
\end{aligned}$$

whenever $|X-X'|$, $|Y-Y'| < \frac{1}{2}(\sqrt{t}+|X-Y|)$. This $\check{K}_t$ is called the Schwarz kernel of the operator e^{-tL}, where $L = \operatorname{div} A\nabla$.

---

[1] They refer to it as K; I use $\check{K}$ to differentiate it from the K in (2.14).

Formally, we wish to construct $\Gamma_X = L^{-1}\delta_X$; by the Laplace formula [1, (60), p. 52], this is given by

$$\Gamma_X = L^{-1}\delta_X = \int_0^\infty e^{-tL}\delta_X \, dt = \int_0^\infty \int \check{K}_t(\cdot, Y)\delta_X(Y)\, dY\, dt = \int_0^\infty \check{K}_t(\cdot, X)\, dt.$$

We make this construction rigorous and study the properties of Γ_X as follows. Let $J_t(Y,X) = \check{K}_t(Y,X) - \fint_{r\leq|Z-X|\leq 2r} \check{K}_t(Z,X)\, dZ$, so that if $r < |X-Y| < 2r$,

$$\begin{aligned}
|J_t(Y,X)| &= \left| \check{K}_t(Y,X) - \fint_{r\leq|Z-X|\leq 2r} \check{K}_t(Z,X)\, dZ \right| \\
&\leq \fint_{r\leq|Z-X|\leq 2r} \left|\check{K}_t(Y,X) - \check{K}_t(Z,X)\right| \, dZ \\
&\leq \fint_{r\leq|Z-X|\leq 2r} \frac{C}{t}\left(\frac{|Y-Z|}{\sqrt{t}+|X-Y|}\right)^\mu \exp\left\{-\frac{\beta|X-Y|^2}{t}\right\} dZ \\
&\leq \frac{C}{t}\left(\frac{|X-Y|}{\sqrt{t}+|X-Y|}\right)^\mu \exp\left\{-\frac{\beta|X-Y|^2}{t}\right\}
\end{aligned}$$

So for fixed X and Y, $t \mapsto J_t(X,Y) \in L^1((0,\infty))$.

From [1, p. 54], there is some $\epsilon, \beta, c > 0$, such that

$$\left(\int_{r\leq|X-Y|\leq 2r} |\nabla_Y K_t(Y,X)|^2\, dY\right)^{1/2} \leq \frac{c}{t}\left(\frac{r^2}{t}\right)^\epsilon e^{-\beta r^2/t}.$$

So, by Hölder's inequality,

$$\begin{aligned}
\int_{r\leq|X-Y|\leq 2r} |\nabla_Y K_t(Y,X)|\, dY &\leq \sqrt{3\pi} r \left(\int_{r\leq|X-Y|\leq 2r} |\nabla_Y K_t(Y,X)|^2\, dY\right)^{1/2} \\
&\leq \frac{Cr}{t}\left(\frac{r^2}{t}\right)^\epsilon e^{-\beta r^2/t}.
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^\infty \int_{r\leq|X-Y|\leq 2r} |\nabla_Y K_t(Y,X)|\, dY\, dt \\
&\qquad \leq \int_0^\infty Cr\frac{r^{2\epsilon}}{t^{1+\epsilon}} e^{-\beta r^2/t}\, dt = \int_0^\infty Cr \frac{r^{2\epsilon}}{r^{2+2\epsilon}t^{1+\epsilon}} e^{-\beta/t}\, r^2\, dt = Cr\int_0^\infty \frac{1}{t^{1+\epsilon}} e^{-\beta/t}\, dt
\end{aligned}$$

Since that integral converges, we know that

$$\int_0^\infty \nabla_Y K_t(Y,X)\,dt$$

converges almost everywhere in $B(X,2r) - B(X,r)$.

Define

$$\Gamma_X(Y) = C(r) + \int_0^\infty J_t(Y,X)\,dt$$

so

$$\nabla \Gamma_X(Y) = \int_0^\infty \nabla_Y J_t(Y,X)\,dt = \int_0^\infty \nabla_Y K_t(Y,X)\,dt$$

where $C(r)$ is chosen such that the values of Γ on different annuli agree. Then

$$\begin{aligned}\int A\nabla\Gamma_X \cdot \nabla\eta &= \int A(Y)\int_0^\infty \nabla_Y K_t(Y,X)\,dt \cdot \nabla\eta(Y)\,dY \\ &= \int_0^\infty \int A(Y)\nabla_Y K_t(Y,X)\cdot\nabla\eta(Y)\,dY\,dt = \int_0^\infty \int \partial_t K_t(Y,X)\eta(Y)\,dY\,dt \\ &= \lim_{t\to\infty}\int K_t(Y,X)\eta(Y)\,dY - \lim_{t\to 0^+}\int K_t(Y,X)\eta(Y)\,dY = 0 - \eta(X)\end{aligned}$$

whenever $\eta \in C_0^\infty$.

So we have constructed a fundamental solution.

Furthermore, we have the following bound on its gradient:

$$\int_{|X-Y|\leq r} |\nabla\Gamma_X(Y)|\,dY \leq Cr.$$

So by (3.8), we have that

$$|\nabla\Gamma_X(Y)| \leq \frac{C}{|X-Y|}. \tag{4.1}$$

This implies that $|\Gamma_X(Y)| \leq C|\log|X-Y|\,|+C$ if we choose additive constants appropriately.

4.2 Uniqueness of the fundamental solution

Let $u = \Gamma_X$, and assume that $|v(Y)| \leq C(X)(1 + |\log|Y|\,|)$, and $\int A\nabla v \cdot \nabla\eta = -\eta(X)$ for all $\eta \in C_0^\infty$. (We will need v to be this general later.) Then $w = u - v$ is a solution to $Lw = 0$ which satisfies $|w(Y)| \leq C(X) + C(X)|\log|Y|\,|$; Lemma 3.7 allows us to conclude that

$$|w(Y)-w(Z)| \leq \frac{C|Y-Z|^\alpha}{r^{1+\alpha}}\|w\|_{L^2(B(X,r))} \leq C(X)|Y-Z|^\alpha \frac{\log r}{r^\alpha}$$

for any r large enough. Letting $r \to \infty$, we see that w is a constant; thus Γ_X is unique up to an additive constant. (So $\nabla\Gamma_X$ is actually unique.)

If $A(y,s) = A(y)$, pick any $t \in \mathbf{R}$, and let $\zeta(X) = \eta(X-(0,t))$. Then

$$\begin{aligned}
-\eta(X) = -\zeta(X+(0,t)) &= \int A(y,s)\nabla\Gamma_{X+(0,t)}(y,s)\cdot\nabla\zeta(y,s)\,ds\,dy \\
&= \int A(y,s-t)\nabla\Gamma_{X+(0,t)}(y,s)\cdot\nabla\eta(y,s-t)\,ds\,dy \\
&= \int A(y,s)\nabla\Gamma_{X+(0,t)}(y,s+t)\cdot\nabla\eta(y,s)\,ds\,dy
\end{aligned}$$

and so by uniqueness,

$$\nabla\Gamma_{X+(0,t)}(Y+(0,t)) = \nabla\Gamma_X(Y). \tag{4.2}$$

4.3 Switching variables

Recall that $\Gamma_X(Y)$ is defined only up to an additive constant $C(X)$. We wish to show that there is some choice of additive constant such that, for all X and Y,

$$\Gamma_X^T(Y) = \Gamma_Y(X). \tag{4.3}$$

This was shown for the real case in [25, Lemma 2.7]; a (more detailed) proof for the complex case follows.

Let $\eta \in C_0^\infty$ be 1 on a neighborhood of $\overline{B(0,R)}$. If $R \gg |X|$, $|Y|$, then

$$\begin{aligned}
\Gamma_Y^T(X)-\Gamma_X(Y) &= \int_{\mathbf{R}^2} \nabla(\eta\Gamma_X)\cdot A^T\nabla\Gamma_Y^T - \nabla(\eta\Gamma_Y^T)A\nabla\Gamma_X \\
&= \int_{B(0,R)} \nabla\Gamma_X\cdot A^T\nabla\Gamma_Y^T - \nabla\Gamma_Y^T\cdot A\nabla\Gamma_X \\
&\qquad + \int_{B(0,R)^C} \nabla(\eta\Gamma_X)A^T\nabla\Gamma_Y^T - \nabla(\eta\Gamma_Y^T)A\nabla\Gamma_X \\
&= 0 + \int_{\partial B(0,R)^C} \Gamma_X\,\nu\cdot A^T\nabla\Gamma_Y^T - \Gamma_Y^T\,\nu\cdot A\nabla\Gamma_X\,d\sigma.
\end{aligned}$$

So we need only prove that the last integral goes to zero. This will be easier if we work with A^r and Γ^r instead of A, Γ, where $A^r = I$ on $B(0,2r)^C$ and $A^r = A$ on $B(0,r)$; we will then show that $\lim_{r\to\infty} \Gamma^r = \Gamma$.

Consider only $r > 2|X| + 2|Y|$. Then Γ^r_X is harmonic in $B(0,2r)^C$.

Think of $\mathbf{R}^2$ as the complex plane, and let $u(Z) = \Gamma^r_X(Z)$. There is some bounded harmonic function $\omega(Z)$ on $B(0,2r)^C$ such that $u(Z) = \omega(Z)$ on $\partial B(0,2r)$; let $v(Z) = \Gamma^r_X(e^Z) - \omega(e^Z)$.

Then v is harmonic in a half-plane, and $v = 0$ on the boundary of that half-plane; by the reflection principle we may extend v to an entire function. Furthermore, $|v(Z)| \leq C + C\,\mathbf{Re}\,Z$, so v is linear.

So $\Gamma^r_X(e^Z) = \omega(e^Z) + C_1 + C_2\,\mathbf{Re}\,Z$; thus, $\Gamma^r_X(Z) = \omega(Z) + C_1 + C_2 \log|Z|$ for some bounded ω and constants C_1, C_2. Using a test function which is 1 on $B(0,3r)$, we see that $C_2 = \frac{1}{2\pi}$. We thus have a standard normalization: we choose additive constants such that $\Gamma^r_X(Z) = \omega(Z) + \Gamma^I(Z)$, where $\Gamma^I(Z) = \frac{1}{2\pi}\log|Z|$ and $\lim_{|Z|\to\infty}\omega(Z) = 0$.

Since ω is bounded and harmonic on $B(0,2r)^C$, the function $f(Z) = \omega(1/Z)$ is bounded and harmonic in a disk; thus, so are its partial derivatives. By our normalization, $f(0) = \lim_{|Z|\to\infty}\omega(Z) = 0$. Then $|\omega(Z)| = |f(1/Z)| \leq C(r)/|Z|$ on $B(0,3r)^C$, and $\omega'(Z) = -f'(1/Z)/Z^2$, so $|\nabla\omega(Z)| \leq C(r)/|Z|^2$ on $B(0,3r)^C$.

Let $R > 3r$. Then on $\partial B(0,R)$,

$$\begin{aligned}\left|\Gamma^r_X\,\nu\cdot\nabla\Gamma^{r,T}_Y - \Gamma^I_X\,\nu\cdot\nabla\Gamma^I_Y\,d\sigma\right| &\leq \left|\Gamma^r_X - \Gamma^I_X\right|\left|\nabla\Gamma^{r,T}_Y\right| + \Gamma^I\left|\nabla\Gamma^{r,T}_Y - \nabla\Gamma^I_Y\right| \\ &\leq \frac{C(r)}{R}\frac{C}{R} + C\frac{\log R}{R}\frac{C}{R}.\end{aligned}$$

So since $A^r = A^{r,T} = I$ on $B(0,r)^C$,

$$\begin{aligned}|\Gamma^r_X(Y) - \Gamma^{r,T}_Y(X)| &= \left|\int_{\partial B(0,R)} \nu\cdot\left(\Gamma^r_X\nabla\Gamma^{r,T}_Y - \Gamma^{r,T}_Y\nabla\Gamma^r_X\right)d\sigma\right| \\ &= \left|\int_{\partial B(0,R)} \nu\cdot(\Gamma^r_X\nabla\Gamma^{r,T}_Y - \Gamma^I\nabla\Gamma^I) - \nu\cdot(\Gamma^{r,T}_Y\nabla\Gamma^r_X - \Gamma^I\nabla\Gamma^I)\,d\sigma\right| \\ &\leq \int_{\partial B(0,R)} \frac{C(r) + C\log R}{R^2}\,d\sigma \leq C(r)\frac{1+\log R}{R}\end{aligned}$$

which goes to 0 as $R \to \infty$. So $\Gamma_X^{r,T}(Y) = \Gamma_Y^r(X)$.

Let $u_r(Z) = \Gamma_Y^r(Z) - \Gamma_Y(Z)$. Then

$$\int A(Z)\nabla u_r(Z) \cdot \nabla \eta(Z)\, dZ = 0$$

for all $\eta \in C_0^\infty(B(0,r))$. So if $|Z|$, $|Z_0| < r/3$, then by Lemma 3.7

$$|u_r(Z) - u_r(Z_0)| \le \frac{C}{r^{1+\alpha}} \|u_r\|_{L^2(B(0,2r/3))} |Z - Z_0|^\alpha$$

and if $|Y| < r$, then

$$\int_{B(0,r)} |u_r|^2 \le \int_{B(Y,2r)} |u_r|^2 \le C \int_{B(Y,2r)} (1 + |\log|X-Y||)^2 \, dX \le C^2 r^2 \log^2 r$$

for r large enough. (Note that this is independent of Y provided $|Y| < r$.)

Thus,

$$|\Gamma_Y(Z) - \Gamma_Y^r(Z) - \Gamma_Y(Z_0) + \Gamma_Y^r(Z_0)| \le C \frac{\log r}{r^\alpha} |Z - Z_0|^\alpha$$

provided r is sufficiently large compared to $|Y|$, $|Z|$, $|Z_0|$. Fix some choice of Z_0 and of $\Gamma_{Z_0}^T(X)$. Normalize each Γ_X such that $\Gamma_X(Z_0) = \Gamma_{Z_0}^T(X)$. We now have that

$$\Gamma_Y(Z) = \lim_{r\to\infty} \left(\Gamma_Y^r(Z) - \Gamma_Y^r(Z_0) + \Gamma_{Z_0}^T(Y) \right) = \lim_{r\to\infty} \left(\Gamma_Z^{r,T}(Y) - \Gamma_{Z_0}^{r,T}(Y) \right) + \Gamma_{Z_0}^T(Y).$$

Unfortunately, we do not know that $\lim_{r\to\infty} \left(\Gamma_Z^{r,T}(Y) - \Gamma_{Z_0}^{r,T}(Y) \right) = \Gamma_Z^T(Y) - \Gamma_{Z_0}^T(Y)$; it will take more work to establish that $\Gamma_X(Y) = \Gamma_Y^T(X)$.

Fix some Y. We want to show that $f(X) = \Gamma_X(Y)$ satisfies the conditions of Γ_Y^T. First, if $\eta \in C_0^\infty(\mathbf{R}^2)$,

$$\int A^T(X)\nabla_X(\Gamma_X(Z_0)) \cdot \nabla \eta(X)\, dX = \int A^T(X) \nabla \Gamma_{Z_0}^T(X) \cdot \nabla \eta(X)\, dX = -\eta(Z_0)$$

and so, since $\Gamma^r_X(Y) - \Gamma^r_X(Z_0) \to \Gamma_X(Y) - \Gamma_X(Z_0)$ almost uniformly in X,

$$\begin{aligned}
\int A^T(X)\nabla f(X) \cdot \nabla\eta(X)\,dX& \\
&= \int \nabla_X(\Gamma_X(Y) - \Gamma_X(Z_0)) \cdot A(X)\nabla\eta(X)\,dX + \int \nabla_X\Gamma_X(Z_0) \cdot A(X)\nabla\eta(X)\,dX \\
&= \int \lim_{r\to\infty} \nabla_X(\Gamma^r_X(Y) - \Gamma^r_X(Z_0)) \cdot A(X)\nabla\eta(X)\,dX - \eta(Z_0) \\
&= \lim_{r\to\infty} \int \nabla_X(\Gamma^{r,T}_Y(X) - \Gamma^{r,T}_{Z_0}(X)) \cdot A(X)\nabla\eta(X)\,dX - \eta(Z_0) \\
&= -\eta(Y) + \eta(Z_0) - \eta(Z_0)
\end{aligned}$$

as desired.

Next,

$$|f(X)| = |\Gamma_X(Y) - \Gamma_X(Z_0) + \Gamma^T_{Z_0}(X)| \le C(1 + |\log|X|| + |\log|X-Y||).$$

This growth condition suffices to establish the inequality in Section 4.2, so by uniqueness $f(X) = \Gamma^T_Y(X)$.

4.4 Conjugates to solutions

Suppose that u satisfies $\operatorname{div} A\nabla u = 0$ in some simply connected domain $U \subset \mathbf{R}^2$. Then if Y_0, $Y \in U$, the integral

$$\int_{Y_0}^{Y} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A(Z)\nabla u(Z) \cdot dl(Z) = \int_{Y_0}^{Y} \nu(Z) \cdot A(Z)\nabla u(Z)\,dl(Z)$$

is path-independent, where ν is the unit normal to the path from Y_0 to Y.

Pick some constant C and some $Y_0 \in U$ and let

$$\tilde{u}(Y) = C + \int_{Y_0}^{Y} \nu(Z) \cdot A(Z)\nabla u(Z)\,dl(Z).$$

Then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\tilde{u} = A\nabla u. \tag{4.4}$$

We call such a function the conjugate to u in U; it is unique up to an additive constant.

It may easily be checked that $\operatorname{div} \tilde{A}\nabla\tilde{u} = 0$, where $\tilde{A} = \frac{1}{\det A} A^T$ is also an elliptic matrix.

Recall that ν is the unit outward normal to a domain. This gives us two choices for the tangential vector τ; we adopt the choice

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nu \Longrightarrow \tau \cdot \nabla\tilde{u} = \nu \cdot A\nabla u. \tag{4.5}$$

In particular, we may define the conjugate $\tilde{\Gamma}_X$ to the fundamental solution Γ_X on any simply connected domain not containing X. Note that

$$\nabla\tilde{\Gamma}_X(Y) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A(Y)\nabla\Gamma_X(Y)$$

is defined on $\mathbf{R}^2 \setminus \{X\}$, even though $\tilde{\Gamma}_X$ itself is necessarily undefined on a ray.

Letting $u = \partial_{x_i}\Gamma_X(Y)$, which still solves $\operatorname{div} A\nabla u = 0$ away from X, we have that by (4.1) $|u(Y)| \leq C/|X-Y|$ and so

$$|\partial_{x_i}\partial_{y_j}\Gamma_X(Y)| \leq |\nabla u(Y)| \leq \frac{C}{|X-Y|}\left(\fint_{(B(Y,|X-Y|/2)} |u|^2\right)^{1/2} \leq \frac{C}{|X-Y|^2}. \tag{4.6}$$

Now, since

$$\tilde{\Gamma}_X(Y) = \zeta(X) + \int_{Y_0}^{Y} \nu(Z)\cdot A(Z)\nabla\Gamma_X(Z)\,dl(Z)$$

we have that

$$\nabla_X\tilde{\Gamma}_X(Y) = \nabla\zeta(X) + \int_{Y_0}^{Y} \nu(Z)\cdot A(Z)\nabla(\nabla_X\Gamma_X(Z))\,dl(Z)$$

We can choose ζ so that

$$\nabla_X\tilde{\Gamma}_X(Y) = \int_{\infty}^{Y} \nu(Z)\cdot A(Z)\nabla(\nabla_X\Gamma_X(Z))\,dl(Z).$$

Since $|\partial_{x_i}\partial_{z_j}\Gamma_X(Z)| \leq C/|X-Z|^2$, this integral converges. Furthermore, it converges to the same value, no matter which direction we use to go off to infinity. By choosing a path which is a ray directed away from X, we see that

$$|\nabla_X \tilde{\Gamma}_X(Y)| \leq \frac{C}{|X-Y|}. \tag{4.7}$$

Now, if $\eta \in C_0^\infty(W)$ for some bounded simply connected domain W with $Y \notin W$,

$$\begin{aligned}
&\int_W A^T(X)\nabla_X \tilde{\Gamma}_X(Y) \cdot \nabla\eta(X)\,dX \\
&\qquad = \int_W A^T(X) \int_\infty^Y A(Z)\nu(Z) \cdot \nabla_Z(\nabla_X \Gamma_X(Z))\,dl(Z) \cdot \nabla\eta(X)\,dX \\
&\qquad = \int_\infty^Y A(Z)\nu(Z) \cdot \nabla_Z \int_W A^T(X)\nabla\Gamma_Z^T(X) \cdot \nabla\eta(X)\,dX\,dl(Z) \\
&\qquad = -\int_\infty^Y A(Z)\nu(Z) \cdot \nabla_Z \eta(Z)\,dl(Z) = 0.
\end{aligned}$$

Thus, if $X \neq Y$, then

$$\operatorname{div}_X A^T(X)\nabla_X \tilde{\Gamma}_X(Y) = 0. \tag{4.8}$$

4.5 Calderón-Zygmund Kernels

We would like $\nabla\Gamma_X(Y)$, $\nabla_X \tilde{\Gamma}_X(Y)$ to be Calderón-Zygmund kernels; in particular, we would like to show that they are Hölder continuous in both X and Y. Unfortunately, with rough coefficients, this is false; however, there is a bounded invertible matrix $B(Y)$ such that $B(Y)\nabla\Gamma_X(Y)$ is Hölder continuous in both X and Y.

By (4.6), we have that for any $0 \leq \alpha \leq 1$ and any matrix B with $|B(Y)| < C$, if $|X - X'| \leq |X-Y|/2$ then

$$|B(Y)\nabla\Gamma_X^T(Y) - B(Y)\nabla\Gamma_{X'}^T(Y)| \leq C\frac{|X-X'|^\alpha}{|X-Y|^{1+\alpha}}.$$

Recall that $B_6(y) = \begin{pmatrix} a_{11}(y) & a_{21}(y) \\ 0 & 1 \end{pmatrix}$. Suppose that $\operatorname{div} A\nabla u = 0$ away from the point X. Then

$$\begin{pmatrix} \partial_s \tilde{u}(y,s) \\ \partial_s u(y,s) \end{pmatrix} = B_6(y)\nabla u(y,s),$$

and because $\partial_s u(y,s)$ and $\partial_s\, \tilde{u}(y,s)$ are solutions to elliptic equations away from X, we have that for some $\alpha > 0$,

$$|B_6(Y)\nabla u(Y) - B_6(Y')\nabla u(Y')| \leq \frac{C|Y-Y'|^\alpha}{|X-Y|^{1+\alpha}} \|\nabla u\|_{L^2(B(Y,\frac{3}{4}|X-Y|))} \tag{4.9}$$

for $|Y - Y'| \leq |X - Y|/2$.

Apply (4.9) to $u(Y) = \Gamma_X^T(Y)$. Recalling that

$$K(X,Y) = \Big(B_6(Y)\nabla\Gamma_X^T(Y) \quad B_6(Y)\nabla\Gamma_X^T(Y)\Big)^t$$

we have that $K(X,Y)$ satisfies the Calderón-Zygmund kernel conditions

$$\begin{aligned} |K(X,Y)| &\leq \frac{C}{|X-Y|}, \\ |K(X,Y) - K(X',Y)| &\leq C\frac{|X-X'|^\alpha}{|X-Y|^{1+\alpha}}, \\ |K(X,Y) - K(X,Y')| &\leq C\frac{|Y-Y'|^\alpha}{|X-Y|^{1+\alpha}}, \end{aligned} \tag{4.10}$$

provided $|X - X'|$, $|Y - Y'| < \frac{|X-Y|}{2}$.

We would like to show that $\tilde{K}(X,Y) = B_6(X)\Big(\nabla_X\tilde{\Gamma}_X^T(Y) \quad \nabla_X\tilde{\Gamma}_X^T(Y)\Big)$ satisfies the same conditions. The bound on $|\tilde{K}(X,Y)|$ follows directly from (4.7). Recall that

$$\nabla_X\tilde{\Gamma}_X(Y) = \int_\infty^Y \nu(Z)\cdot A(Z)\nabla(\nabla_X\Gamma_X(Z))\,dl(Z).$$

Since $|\nabla_Z(\nabla_X\Gamma_X(Z))| \leq C/|X-Z|^2$ and the integral is path-independent, we have that

$$|\nabla_X\tilde{\Gamma}_X(Y) - \nabla_X\tilde{\Gamma}_X(Y')| \leq \int_Y^{Y'} \frac{C}{|X-Z|^2}\,dl(Z) \leq C\frac{|Y-Y'|}{|X-Y|^2}$$

whenever $|Y - Y'| \leq |X - Y|/2$.

Now, we wish to show that $\tilde{K}$ is C^α in X as well as Y. But by (4.8), $\tilde{\Gamma}_X(Y)$ solves an elliptic PDE in X away from Y, so we may apply (4.9).

So we have that, if $|X-X'|$, $|Y-Y'| < \frac{1}{2}|X-Y|$, then

$$
\begin{aligned}
|\tilde{K}(X,Y)| &\le \frac{C}{|X-Y|}, \\
|\tilde{K}(X,Y)-\tilde{K}(X',Y)| &\le C\frac{|X-X'|^\alpha}{|X-Y|^{1+\alpha}} \\
|\tilde{K}(X,Y)-\tilde{K}(X,Y')| &\le C\frac{|Y-Y'|^\alpha}{|X-Y|^{1+\alpha}}.
\end{aligned}
\tag{4.11}
$$

4.6 Analyticity

We will eventually want to compare the fundamental solutions (and related operators) for a real matrix A_0 and a nearby complex matrix A. We can explore this using analytic function theory. Let $z \mapsto A_z$ be an analytic function from $\mathbf{C}$ to $L^\infty(\mathbf{R}^2 \mapsto \mathbf{C}^{2\times 2})$. (As a particularly useful example, take $A_z = A_0 + z(\lambda/2\epsilon)(A-A_0)$, where $\epsilon = \|A-A_0\|_{L^\infty} = \sup\{|\xi\cdot(A(x)-A_0(x))\eta| : x\in\mathbf{R},\ \eta,\xi\in\mathbf{C}^2\}$.) Assume that A_z is uniformly elliptic in some neighborhood of 0, say $B(0,1)$.

Let $L_z = \operatorname{div} A_z\nabla$. Since we are working in $\mathbf{R}^2$, we know from [1] that the operator e^{-tL_z} has a Schwarz kernel $\check{K}^z_t(X,Y) = \check{K}^{A_z}_t(X,Y)$. Furthermore, by ([1, p. 57]), the map $A \mapsto \check{K}^A$ is analytic.

Fix some Y. Recall that, as in Section 4.1,

$$\nabla\Gamma^{A_z}_Y(X) = \int_0^\infty \nabla\check{K}^z_t(X,Y)\,dt.$$

Since $\nabla\check{K}^z_t \in L^1_t$, uniformly in z, we have that (e.g., by the Cauchy integral formula), $\nabla\Gamma^{A_z}_Y(X)$ is analytic in z. This lets us compute many useful inequalities.

If $|z| < \frac{1}{2}$ and ω is an appropriately chosen simple closed curve lying in $B(0,1)$, then

$$
\begin{aligned}
\nabla\Gamma^{A_z}_Y(X) - \nabla\Gamma^{A_0}_Y(X) &= \frac{1}{2\pi i}\oint_\omega \nabla\Gamma^{A_\zeta}_Y(X)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta}\right)d\zeta \\
&= \frac{1}{2\pi i}\oint_\omega \nabla\Gamma^{A_\zeta}_Y(X)\left(\frac{z}{\zeta(\zeta-z)}\right)d\zeta
\end{aligned}
$$

which has norm at most $|z|\frac{C}{|X-Y|}$. Taking $A_z = A_0 + z(\lambda/2\epsilon)(A - A_0)$ and then applying this equation to $z = 2\epsilon/\lambda$, we get that

$$|\nabla\Gamma_Y(X) - \nabla\Gamma_Y^0(X)| < \frac{C\epsilon}{|X-Y|}. \tag{4.12}$$

Similarly,

$$\begin{aligned}|\nabla\Gamma_Y(X) - \nabla\Gamma_Y^0(X) - \nabla\Gamma_Y(X') + \nabla\Gamma_Y^0(X')| &\leq \frac{C\epsilon|X-X'|^\alpha}{|X-Y|^{1+\alpha}},\\ |\nabla\Gamma_Y(X) - \nabla\Gamma_Y^0(X) - \nabla\Gamma_{Y'}(X) + \nabla\Gamma_{Y'}^0(X)| &\leq \frac{C\epsilon|Y-Y'|^\alpha}{|X-Y|^{1+\alpha}},\end{aligned}$$

provided that $|X-X'|$, $|Y-Y'|$ are less than $\frac{1}{2}|X-Y|$.

Suppose that $\mathcal{J}^z$ is a Calderón-Zygmund operator whose kernel $K^z(X,Y)$ is analytic in z, and suppose that $\mathcal{J}^z$ is uniformly bounded on L^p in some neighborhood of $z=0$. Then

$$\mathcal{J}^z f(X) - \mathcal{J}^0 f(X) = \frac{1}{2\pi i}\oint_\omega \mathcal{J}^\zeta f(X)\left(\frac{z}{\zeta(\zeta-z)}\right)d\zeta$$

and so for $|z|$ small enough,

$$\|\mathcal{J}^z f - \mathcal{J}^0 f\|_{L^p} \leq \frac{1}{2\pi}\oint_\omega \|\mathcal{J}^\zeta f\|_{L^p}\left|\frac{z}{\zeta(\zeta-z)}\right| d\zeta \leq C|z|\,\|f\|_{L^p}. \tag{4.13}$$

We now assume that A is smooth, and complete the argument (referenced in Chapter 2) that $T_\pm = T'_\mp$. Recall that $\Gamma_X^I(Y) = \frac{1}{2\pi}\ln|X-Y|$, and so $\Gamma_{X+\mathbf{e}}^I(Y) = \Gamma_X^I(Y-\mathbf{e})$ for all X, Y and $\mathbf{e}$ in $\mathbf{R}^2$. We would like a similar result for more general Γ. We know from (4.2) that $\Gamma_{X+(0,t)}(Y) = \Gamma_X(Y-(0,t))$ for any real t.

If we define $A_z(X) = A(X) + z(A(X+\xi) - A(X))$, then A_z is uniformly elliptic for all $|z|\,\|A'\|_{L^\infty}|\xi| < \lambda/2$, and so if $|\xi|$ is small enough, then

$$|\nabla\Gamma_X^{A_1}(Y) - \nabla\Gamma_X(Y)| \leq \frac{1}{|X-Y|}\frac{C\|A'\|_{L^\infty}}{\lambda}|\xi|.$$

If $A(x)$ is smooth in x and $A = I$ for large x, then $\|A'\|_{L^\infty}$ is finite (if large). But

$\Gamma_{X+\xi}(Y+\xi) = \Gamma_X^{A_1}(Y)$ by uniqueness:

$$\begin{aligned} -\eta(X) &= \int A(Y)\nabla\Gamma_{X+\xi}(Y)\nabla\eta(Y-\xi)\,dY = \int A_1(Y-\xi)\nabla\Gamma_{X+\xi}(Y)\nabla\eta(Y-\xi)\,dY \\ &= \int A_1(Y)\nabla\Gamma_{X+\xi}(Y+\xi)\nabla\eta(Y)\,dY. \end{aligned}$$

So while we do not have that $\Gamma_{Y+\xi}(X) = \Gamma_Y(X-\xi)$, we do have that for ξ small,

$$|\nabla\Gamma_{Y+\xi}(X) - \nabla\Gamma_Y(X-\xi)| \leq \frac{C\|A'\|_{L^\infty}}{|X-Y|}|\xi|. \tag{4.14}$$

This equation is not useful in most circumstances, since we do not want our estimates to depend on A'; however, it can be used to compare T and T'. Recall that

$$\begin{aligned} T_\pm F(x) &= \lim_{h\to 0^\pm}\int_{\mathbf{R}} \begin{pmatrix} \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t \\ \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t \end{pmatrix} B_6(\psi(y))^t F(y)\,dy, \\ T'_\pm F(X) &= \lim_{h\to 0^\pm}\int_{\mathbf{R}} \begin{pmatrix} \nabla\Gamma^T_{\psi(x)}(\psi(y,h))^t \\ \nabla\Gamma^T_{\psi(x)}(\psi(y,h))^t \end{pmatrix} B_6(\psi(y,h))^t F(y)\,dy, \end{aligned}$$

where $\psi(x,h) = x\mathbf{e}^\perp + (\varphi(x)+h)\mathbf{e}$.

If A is smooth, then $\lim_{h\to 0^\pm}\nabla\Gamma^T_{\psi(x)}(\psi(y,h)) = \lim_{h\to 0^\mp}\nabla\Gamma^T_{\psi(x,h)}(\psi(y))$; and so we have that $T_\pm F = T'_\mp F$ for sufficiently well-behaved F.

Similarly, if A is smooth, then $|\nabla_X(\Gamma_X(Z+\xi) - \Gamma_{X-\xi}(Z))| \leq |\xi|\frac{C\|A'\|_{L^\infty}}{|X-Y|^2}$ and so

$$\begin{aligned} |\nabla_X\tilde\Gamma_{Y+\xi}(X) - \nabla_X\tilde\Gamma_Y(X-\xi)| &= \left|\int_\infty^Y \nu(Z)\cdot A(Z)\nabla(\nabla_X(\Gamma_X(Z+\xi) - \Gamma_{X-\xi}(Z)))\,dl(Z)\right| \\ &\leq |\xi|\frac{C\|A'\|_{L^\infty}}{|X-Y|} \end{aligned}$$

and so $\tilde T'_\pm F = \tilde T_\mp F$ for well-behaved F.

CHAPTER 5
LAYER POTENTIALS

Recall that we intend to construct solutions to $(D)_q^A$, $(N)_p^A$, $(R)_p^A$ as layer potentials. We will need some properties of layer potentials. Recall that

$$\mathcal{D}f(X) = \int_{\partial V} \nu \cdot A^T(Y)\nabla\Gamma_X^T(Y)f(Y)\,d\sigma(Y), \qquad \nabla\mathcal{S}f(X) = \int_{\partial V} \nabla_X\Gamma_X^T(Y)f(Y)\,d\sigma(Y).$$

5.1 Elementary properties

(5.1) If $X \notin \partial V$, then by (2.7) and (4.1), $\mathcal{D}f(X)$ and $\nabla\mathcal{S}f(X)$ are well-defined (the integrals converge) for $f \in L^p(\partial V)$, $1 \leq p < \infty$.

(5.2) If $u = \mathcal{D}f$ or $u = \mathcal{S}f$ for such f, then by (2.1) and (4.3) $\operatorname{div} A\nabla u = 0$ in $\mathbf{R}^2\backslash\partial V$.

(5.3) If ∂V is compact and $f \in L^1(\partial V)$, then by (4.1), $\lim_{|X|\to\infty} \mathcal{D}f(X) = 0$ and for any $X_0 \notin \partial V$,

$$\lim_{|X|\to\infty} \mathcal{S}f(X) - \Gamma_X^T(X_0)\int_{\partial V} f\,d\sigma = 0.$$

So if $f \in H^1$ then $\lim_{|X|\to\infty} \mathcal{S}f = 0$.

(5.4) Suppose V is bounded, $\operatorname{div} A\nabla u = 0$ inside V, and u is continuous and ∇u is bounded on a neighborhood of $\bar{V}$. Define $f = u$, $g = \nu \cdot A\nabla u$ on ∂V. Letting $\eta \in C_0^\infty(\mathbf{R}^2)$ with $\eta \equiv 1$ near V gives us that

$$\begin{aligned} u(X) &= -\int_{\mathbf{R}^2} A^T\nabla\Gamma_X^T \cdot \nabla(u\eta) = -\int_V \nabla\Gamma_X^T \cdot A\nabla u - \int_{V^C} A^T\nabla\Gamma_X^T \cdot \nabla(u\eta) \\ &= -\int_{\partial V} \Gamma_X^T\,\nu \cdot A\nabla u\,d\sigma + \int_{\partial V} \nu \cdot A^T\nabla\Gamma_X^T u\,d\sigma = \mathcal{D}f(X) - \mathcal{S}g(X) \end{aligned}$$

for all $X \in V$. (Some of these conditions may be relaxed; however, it is obvious when they all hold.) In particular, $\int_{\partial V} \nu \cdot A^T\nabla\Gamma_X^T\,d\sigma = 1$.

(5.5) If V is bounded, then let $\eta \equiv 1$ near $\bar{V}$; if $X \notin \bar{V}$, assume that $\eta = 0$ near X. Then

$$\begin{aligned}\mathcal{D}1(X) &= \int_{\partial V} \nu \cdot A^T \nabla \Gamma_X^T \, d\sigma = \int_{\partial V} \eta \, \nu \cdot A^T \nabla \Gamma_X^T \, d\sigma = -\int_{V^C} \nabla \eta \cdot A^T \nabla \Gamma_X^T \\ &= -\int_{\mathbf{R}^2} \nabla \eta \cdot A^T \nabla \Gamma_X^T = \eta(X)\end{aligned}$$

and so $\mathcal{D}1 \equiv 1$ on V and $\mathcal{D}1 \equiv 0$ on V^C. (If V^C is bounded then $\mathcal{D}1 \equiv -1$ on $\bar{V}^C$ and $\mathcal{D}1 \equiv 0$ on V.)

If $V = \Omega$ is a special Lipschitz domain and X, $X_0 \in \Omega$, then we may define $\mathcal{D}1(X) - \mathcal{D}1(X_0)$ in the obvious way:

$$\begin{aligned}\mathcal{D}1(X) - \mathcal{D}1(X_0) &= \int_{\partial\Omega} \nu \cdot A^T (\nabla \Gamma_X^T - \nabla \Gamma_{X_0}^T) \, d\sigma \\ &= \int_{\partial(\Omega \setminus Q(X_0,R)} + \int_{\partial Q(X_0,R)}\end{aligned}$$

where $Q(X_0, R)$ are the tents of (2.6). If $R > 2|X - X_0|$, then by (4.10) the first integral is at most $C|X - X_0|^\alpha / R^\alpha$, and the second integral is equal to 0 since $Q(X_0, R)$ is bounded. Letting $R \to \infty$, we see that $\mathcal{D}1(X)$ is constant on Ω and $\bar{\Omega}^C$.

(5.6) If $f \in BMO(\partial V)$, I claim that $\mathcal{D}f$ is well-defined up to a constant. By the John-Nirenberg lemma, if ∂V is compact then $f \in L^p(\partial V)$ for any p and we are done. So we may assume $V = \Omega$ is a special Lipschitz domain. Pick some X_0, $X \in \Omega$. Let $R = 2|X - X_0| + 2 \operatorname{dist}(X_0, \partial\Omega)$, and let $X^* \in \partial\Omega$ with X, $X_0 \in Q(X^*, R/2)$. Then

$$\mathcal{D}f(X) - \mathcal{D}f(X_0) = \mathcal{D}f(X) - \mathcal{D}f_R(X) - \mathcal{D}f(X_0) + \mathcal{D}f_R(X_0),$$

where f_R is any constant. So without loss of generality $\fint_{\Delta(X^*,R)} f = 0$. By basic BMO theory, this means that

$$\fint_{\Delta(X^*,2^k R)} |f| \le C(k+1) \|f\|_{BMO}.$$

Then by (4.1),

$$\begin{aligned}
|\mathcal{D}f(X) - \mathcal{D}f(X_0)| &\leq \int_{\partial V} \left|\nu \cdot A(\nabla\Gamma_X^T - \nabla\Gamma_{X_0}^T)\right| |f| \, d\sigma \\
&\leq \int_{\Delta(X^*,R)} \left(\frac{C}{|X-Y|} + \frac{C}{|X_0 - Y|}\right) |f(Y)| \, d\sigma(Y) \\
&\quad + \int_{\partial V \setminus \Delta(X^*,R)} \left|\nu \cdot A(\nabla\Gamma_X^T - \nabla\Gamma_{X_0}^T)\right| |f| \, d\sigma(Y)
\end{aligned}$$

The first term is at most

$$CR\|f\|_{BMO} \left(\frac{C}{\operatorname{dist}(X,\partial V)} + \frac{C}{\operatorname{dist}(X_0,\partial V)}\right).$$

By (4.10), since $|X - Y| \approx |X^* - Y|$ for all $Y \in \partial V \setminus \Delta(X^*, R)$, we have that the second term is controlled by

$$\begin{aligned}
&\int_{\partial V \setminus \Delta(X^*,R)} \left|\nabla\Gamma_X^T - \nabla\Gamma_{X_0}^T\right| |f| \, d\sigma(Y) \\
&\qquad \leq \int_{\partial V \setminus \Delta(X^*,R)} \frac{|X - X_0|^\alpha}{|Y - X^*|} |f(Y)| \, d\sigma(Y) \\
&\qquad \leq \sum_{k=0}^{\infty} \int_{\Delta(X^*,2^{k+1}R) \setminus \Delta(X^*,2^k R)} \frac{|X - X_0|^\alpha}{|X^* - Y|^{1+\alpha}} |f(Y)| \, d\sigma(Y) \\
&\qquad \leq \sum_{k=0}^{\infty} \frac{C|X - X_0|^\alpha}{2^{k\alpha} R^\alpha} \fint_{\Delta(X^*,2^{k+1}R)} |f| \, d\sigma \\
&\qquad \leq \frac{C|X - X_0|^\alpha}{R^\alpha} \|f\|_{BMO} \sum_{k=0}^{\infty} \frac{k+1}{2^{k\alpha}} \leq \frac{C|X - X_0|^\alpha}{R^\alpha} \|f\|_{BMO}.
\end{aligned} \tag{5.7}$$

So the second term is also finite, so $\mathcal{D}f$ is well-defined up to a constant.

We will need a few more complicated properties of the boundary layer potentials:

Lemma 5.8. *If $f \in L^p(\partial V)$ for some $1 \leq p < \infty$, or if $f \in BMO \supset L^\infty$, and if $M(\partial_\tau f)(X)$ is finite, then the limits in the definitions of $\mathcal{K} = \mathcal{K}_V$, $\mathcal{L} = \mathcal{L}_V$ exist at X. (If $f \in BMO$, the limits exist once we have fixed our choice of additive constant for $\mathcal{D}f$.)*

Similarly, if $F \in L^p(\partial V \mapsto \mathbf{C}^{2\times 2})$ and $M(\partial_\tau B_7^{-1} F)(X)$ is finite, then $\mathcal{T}F(X)$ is well-defined.

Proof. Recall the definitions of $\mathcal{K}$, $\mathcal{L}$, $\mathcal{T}$:

$$\begin{aligned}
\mathcal{K}f(X) &= \lim_{Z\to X,\, Z\in\gamma(X)} \mathcal{D}f(Z) = \lim_{Z\to X,\, Z\in\gamma(X)} \int_{\partial V} \nu(Y)\cdot A^T(Y)\nabla\Gamma_Z^T(Y)f(Y)\,d\sigma(Y)\\
\mathcal{L}f(X) &= \lim_{Z\to X,\, Z\in\gamma(X)} \int_{\partial V} \tau(Y)\cdot\nabla\Gamma_Z^T(Y)f(Y)\,d\sigma(Y)\\
\mathcal{T}F(X) &= \lim_{Z\to X,\, Z\in\gamma(X)} \int_{\partial V} K(Z,Y)F(Y)\,d\sigma(Y)
\end{aligned}$$

If $B_7^{-1}F = \begin{pmatrix} f_1 & f_2\\ f_3 & f_4\end{pmatrix}$, then

$$\mathcal{T}F(X) = \mathcal{T}(B_7B_7^{-1}F)(X) = \begin{pmatrix}\mathcal{K}f_1(X)+\mathcal{L}f_3(X) & \mathcal{K}f_2(X)+\mathcal{L}f_4(X)\\ \mathcal{K}f_1(X)+\mathcal{L}f_3(X) & \mathcal{K}f_2(X)+\mathcal{L}f_4(X)\end{pmatrix}.$$

So our result for $\mathcal{T}$ will follow from our results for $\mathcal{K}$, $\mathcal{L}$.

Suppose $\mathbf{e}$, $\check{\mathbf{e}}$ are two small vectors such that $X+\mathbf{e}$, $X+\check{\mathbf{e}}$ are both in V, and that $|\mathbf{e}| < (1+a)\operatorname{dist}(X+\mathbf{e},\partial V)$, $|\check{\mathbf{e}}| < (1+a)\operatorname{dist}(X+\check{\mathbf{e}},\partial V)$. Then, if $\rho > 2|\mathbf{e}|+2|\check{\mathbf{e}}|$ is small enough that $Q(X,\rho)$, $\Delta(X,\rho)$ are well-defined, then

$$\begin{aligned}
|\mathcal{D}f(X+\mathbf{e}) - \mathcal{D}f(X+\check{\mathbf{e}})| &= \left|\int_{\partial V}\nu\cdot A^T(\nabla\Gamma_{X+\mathbf{e}}^T - \nabla\Gamma_{X+\check{\mathbf{e}}}^T)f\,d\sigma\right|\\
&\le \left|\int_{\Delta(X,\rho)}\nu\cdot A^T(\nabla\Gamma_{X+\mathbf{e}}^T - \nabla\Gamma_{X+\check{\mathbf{e}}}^T)(f-f(X))\,d\sigma\right|\\
&\quad + \left|f(X)\int_{\Delta(X,\rho)}\nu\cdot A^T(\nabla\Gamma_{X+\mathbf{e}}^T - \nabla\Gamma_{X+\check{\mathbf{e}}}^T)\,d\sigma\right|\\
&\quad + \left|\int_{\partial V\setminus\Delta(X,\rho)}\nu\cdot A^T(\nabla\Gamma_{X+\mathbf{e}}^T - \nabla\Gamma_{X+\check{\mathbf{e}}}^T)f\,d\sigma\right|.
\end{aligned}$$

If V is a good Lipschitz domain and $f\in L^p$ for $p<\infty$, then by (2.7), the third term is at most

$$\begin{aligned}
&\int_{\partial V\setminus\Delta(X,\rho)} \frac{C|\mathbf{e}-\check{\mathbf{e}}|^\alpha}{|Y-X|^{1+\alpha}}|f(Y)|\,d\sigma(Y)\\
&\qquad\le \|f\|_{L^p}\left\|\frac{C|\mathbf{e}-\check{\mathbf{e}}|^\alpha}{|\cdot - X|^{1+\alpha}}\right\|_{L^q(\partial V\setminus\Delta(X,\rho))} \le \|f\|_{L^p}\frac{C|\mathbf{e}-\check{\mathbf{e}}|^\alpha}{\rho^{\alpha+1/p}}.
\end{aligned}$$

If $f \in BMO(\partial V)$ and ∂V is not compact, so f might not be in $L^2(\partial V)$, then since $\mathcal{D}1$ is constant on V, we may assume that $\fint_{\Delta(X,\rho)} f\,d\sigma = 0$, and so we may use (5.7) to see that the third term is at most

$$C\frac{|\mathbf{e}-\check{\mathbf{e}}|^\alpha}{\rho^\alpha}\|f\|_{BMO}.$$

To control the first term, recall that $M(\partial_\tau f)(X)$ is finite, so there is some $C(X)$ such that, if $|X-Y|$ is small, then $|f(X)-f(Y)| < C(X)|X-Y|$. So, provided ρ is small enough,

$$\left|\int_{\Delta(X,\rho)} \nu\cdot A^T\nabla\Gamma^T_{X+\mathbf{e}}(f-f(X))\,d\sigma\right| \leq \int_{\Delta(X,\rho)} \frac{C}{|X+\mathbf{e}-Y|}C(X)|X-Y|\,d\sigma(Y)$$

Since $|\mathbf{e}| \leq (1+a)\operatorname{dist}(X+\mathbf{e},\partial V) \leq (1+a)|X+\mathbf{e}-Y|$, we have that $|X-Y| \leq C|X+\mathbf{e}-Y|$ and so

$$\left|\int_{\Delta(X,\rho)} \nu\cdot A^T\nabla\Gamma^T_{X+\mathbf{e}}(f-f(X))\,d\sigma\right| \leq \int_{\Delta(X,\rho)} \frac{C}{|X-Y|}C(X)|X-Y|\,d\sigma(Y) \leq C(X)\rho.$$

To control the middle term, note that

$$\int_{\partial Q(X,\rho)} \nu\cdot A^T\nabla\Gamma^T_{X+\mathbf{e}}\,d\sigma = 1 = \int_{\partial Q(X,\rho)} \nu\cdot A^T\nabla\Gamma^T_{X+\check{\mathbf{e}}}\,d\sigma$$

and so

$$\begin{aligned}\left|f(X)\int_{\Delta(X,\rho)} \nu\cdot A^T(\nabla\Gamma^T_{X+\mathbf{e}}-\nabla\Gamma^T_{X+\check{\mathbf{e}}})\,d\sigma\right| \\ &= \left|f(X)\int_{\partial Q(X,\rho)\setminus\Delta(X,\rho)} \nu\cdot A^T(\nabla\Gamma^T_{X+\mathbf{e}}-\nabla\Gamma^T_{X+\check{\mathbf{e}}})\,d\sigma\right| \\ &\leq |f(X)|\int_{\partial Q(X,\rho)\setminus\Delta(X,\rho)} C\frac{|\mathbf{e}-\check{\mathbf{e}}|^\alpha}{|X-Y|^{1+\alpha}}\,d\sigma \leq C|f(X)|\frac{\|\mathbf{e}-\check{\mathbf{e}}\|^\alpha}{\rho^\alpha}.\end{aligned}$$

We may control all three terms of $\mathcal{D}f(X+\mathbf{e})-\mathcal{D}f(X+\check{\mathbf{e}})$ by first making ρ small and then making $|\mathbf{e}|$ and $|\check{\mathbf{e}}|$ small.

Similarly,

$$\int_{\partial Q(X,\rho)} \tau\cdot\nabla\Gamma^T_{X+\mathbf{e}}\,d\sigma = \int_{\partial Q(X,\rho)} \tau\cdot\nabla\Gamma^T_{X+\check{\mathbf{e}}}\,d\sigma = 0,$$

and the other terms of $\int_{\partial V} \tau \cdot (\nabla \Gamma^T_{X+\mathbf{e}} - \nabla \Gamma^T_{X+\check{\mathbf{e}}}) f \, d\sigma$ may be controlled as for the terms of $\mathcal{D}f(X+\mathbf{e}) - \mathcal{D}f(X+\check{\mathbf{e}})$, and so

$$\left| \int_{\partial V} \tau \cdot \nabla \Gamma^T_{X+\mathbf{e}} f \, d\sigma - \int_{\partial V} \tau \cdot \nabla \Gamma^T_{X+\check{\mathbf{e}}} f \, d\sigma \right|$$

goes to 0 as $|\mathbf{e}|$, $|\check{\mathbf{e}}| \to 0$. $\square$

Lemma 5.9. *If $f \in L^p$, $1 < p < \infty$, then we have the following equations for the transposes of $\mathcal{K}$ and $\mathcal{L}$:*

$$\mathcal{K}^t_\pm f = \mp \nu \cdot A^T \nabla \mathcal{S}^T f|_{\partial V_\mp}, \qquad \mathcal{L}^t f(X) = \partial_\tau \mathcal{S}^T f(Z).$$

In our equation for $\mathcal{K}^t_\pm$, ν is taken to be the outward unit normal to $V_\mp$, so that we may easily use the weak definition

$$\int_{\partial V} \eta \, \mathcal{K}^t_\pm f = \mp \int_{V_\mp} \nabla \eta \cdot A^T \nabla \mathcal{S}^T f.$$

If A, $\mathcal{S}^T f$ are smooth enough to allow for pointwise definitions of $\nu \cdot A^T \nabla \mathcal{S}^T f$, and we prefer to let $\nu = \nu_V$ be the outward unit normal to V in both cases, then $\mathcal{K}^t_\pm f = \nu \cdot A^T \nabla \mathcal{S}^T f|_{\partial V_\mp}$.

Proof. Consider $\mathcal{K}^t_+$ first. Pick some $\eta \in C^\infty_0(\mathbf{R}^2)$. We need to show that $\int_{\partial V} \eta \mathcal{K}^t_+ f \, d\sigma = -\int_{V^C} \nabla \eta \cdot A^T \nabla \mathcal{S}^T f$.

If $f \in L^p$, $1 < p < \infty$, then by (2.7) and (4.1),

$$|\nabla \mathcal{S}^T f(X)| \leq \int_{\partial V} |\nabla_X \Gamma_X f| \, d\sigma \leq \|f\|_{L^p} \left\| \frac{C}{\cdot - X} \right\|_{L^q(\partial V)} \leq C_p \|f\|_{L^p} \operatorname{dist}(X, \partial V)^{-1/p}$$

which is in $L^1_{loc}(\mathbf{R}^2)$, so since $\eta \in C^\infty_0$, the right-hand integral converges.

By definition of $\mathcal{K}$, and since $\operatorname{div} A^T \nabla \Gamma^T_Z = 0$ in V_- provided $Z \in V$,

$$\begin{aligned}
\int_{\partial V} \eta \mathcal{K}^t_+ f \, d\sigma &= \int_{\partial V} \mathcal{K}_+ \eta(Y) f(Y) \, d\sigma(Y) \\
&= \int_{\partial V} \lim_{Z \to Y, \, Z \in \gamma_+} \int_{\partial V} \nu \cdot A^T \nabla \Gamma^T_Z(X) \eta(X) \, d\sigma(X) \, f(Y) \, d\sigma(Y) \\
&= -\int_{\partial V} \lim_{Z \to Y, \, Z \in \gamma_+} \int_{V^C} \nabla \eta(X) \cdot A^T \nabla \Gamma^T_Z(X) \, dX \, f(Y) \, d\sigma(Y).
\end{aligned}$$

Now,

$$\begin{aligned}\int_{\partial V_-} \eta\, \nu \cdot A^T \nabla \mathcal{S}^T f\, d\sigma &= \int_{V^C} \nabla \eta \cdot A^T \nabla \mathcal{S}^T f \\ &= \int_{V^C} \nabla \eta(X) \cdot A^T(X) \int_{\partial V} \nabla_X \Gamma_X(Y) f(Y)\, d\sigma(Y)\, dX \\ &= \int_{\partial V} \int_{V^C} \nabla \eta(X) \cdot A^T(X) \nabla \Gamma_Y^T(X)\, dX\, f(Y)\, d\sigma(Y).\end{aligned}$$

To show that $-\nu \cdot A^T \nabla \mathcal{S}^T f|_{\partial V_-} = \mathcal{K}_+^t f$, it suffices to prove that

$$\lim_{Z \to Y,\, Z \in \gamma_+} \int_{V^C} \nabla \eta(X) \cdot A^T(X) \nabla \Gamma_Z^T(X)\, dX = \int_{V^C} \nabla \eta(X) \cdot A^T(X) \nabla \Gamma_Y^T(X)\, dX.$$

Let $\delta = 2|Z - Y|$, and let $R > 0$ large enough that $\operatorname{supp} \eta \subset B(Y, R)$. Then

$$\begin{aligned}&\left| \int_{V^C} \nabla \eta(X) \cdot A^T \nabla \Gamma_Z^T(X)\, dX - \int_{V^C} \nabla \eta(X) \cdot A^T \nabla \Gamma_Y^T(X)\, dX \right| \\ &\qquad = \left| \int_{V^C} \nabla \eta(X) \cdot A^T (\nabla \Gamma_Z^T(X) - \nabla \Gamma_Y^T(X))\, dX \right| \\ &\qquad \leq C(\eta) \int_{|X-Y| \leq R} |\nabla \Gamma_Z^T(X) - \nabla \Gamma_Y^T(X)|\, dX \\ &\qquad \leq C(\eta) \int_{\delta < |X-Y| \leq R} \frac{|Z-Y|^\alpha}{|X-Y|^{1+\alpha}}\, dX + C(\eta) \int_{|X-Y| \leq \delta} \frac{C}{|X-Y|} + \frac{C}{|X-Z|}\, dX \\ &\qquad \leq C(\eta) |Z-Y|^\alpha R^{1-\alpha} + C(\eta)\delta = C(\eta)|Z-Y|^\alpha R^{1-\alpha} + C(\eta)|Z-Y|.\end{aligned}$$

This clearly goes to 0 as $Z \to Y$; thus $\mathcal{K}_+^t f = -\nu \cdot A^T \nabla \mathcal{S}^T f|_{\partial V_-}$. Similarly, $\mathcal{K}_-^t f = \nu \cdot A^T \nabla \mathcal{S}^T f|_{\partial V_+}$.

Now we come to $\mathcal{L} f$. Let $\eta \in C_0^\infty(\mathbf{R}^2)$ again. Let R be such that $\operatorname{supp} \eta \subset B(0, R)$, and let f_1, f_2 be such that $f_1 + f_2 = f$, $f_1 \equiv 0$ on $B(0, 2R) \cap \partial V$, $f_2 \equiv 0$ on $\partial V \setminus B(0, 3R)$. Then

$$\int_{\partial V} \eta\, \mathcal{L}^t f\, d\sigma = \int_{\partial V} \eta\, \mathcal{L}^t f_1\, d\sigma + \int_{\partial V} \eta\, \mathcal{L}^t f_2\, d\sigma = \int_{\partial V} \mathcal{L}\eta\, f_1\, d\sigma + \int_{\partial V} \mathcal{L}\eta\, f_2\, d\sigma.$$

But

$$\int_{\partial V} \mathcal{L}\eta\, f_1\, d\sigma = \int_{\partial V} f_1(Y) \lim_{Z \to Y} \int_{\partial V} \tau(X) \cdot \nabla \Gamma_Z^T(X)\, \eta(X)\, d\sigma(X)\, d\sigma(Y).$$

Since $f_1(Y)$ is zero for all Y near $\operatorname{supp}\eta$, the limit reduces to

$$\begin{aligned}\int_{\partial V} f_1(Y) \int_{\partial V} \tau(X)\cdot\nabla\Gamma_Y^T(X)\,\eta(X)\,d\sigma(X)\,d\sigma(Y)&\\ &= \int_{\partial V} \eta(X)\,\tau(X)\cdot\int_{\partial V} f_1(Y)\nabla_X\Gamma_X(Y)\,d\sigma(Y)\,d\sigma(X)\\ &= \int_{\partial V} \eta(X)\,\tau(X)\cdot\nabla\mathcal{S}^T f_1(X)\,d\sigma(X).\end{aligned}$$

Conversely,

$$\int_{\partial V} \mathcal{L}\eta\, f_2\,d\sigma = \int_{\partial V} f_2(Y)\lim_{Z\to Y}\int_{\partial V} \tau(X)\cdot\nabla\Gamma_Z^T(X)\,\eta(X)\,d\sigma(X)\,d\sigma(Y).$$

If ∂V is bounded, let $W = V$. If V is a special Lipschitz domain, then let $W = Q(0, 4R)$; if $f_2(Y)\neq 0$ then $Z \in W$ for Z sufficiently close to Y. Then since η is supported in $B(0, R)$,

$$\int_{\partial V} \mathcal{L}\eta\, f_2\,d\sigma = \int_{\partial V} f_2(Y)\lim_{Z\to Y}\int_{\partial W} \tau(X)\cdot\nabla\Gamma_Z^T(X)\,\eta(X)\,d\sigma(X)\,d\sigma(Y).$$

But since $\int_{\partial W}\tau\cdot g\,d\sigma = 0$ for any bounded domain W and well-behaved function g, we have that

$$\int_{\partial V} \mathcal{L}\eta\, f_2\,d\sigma = \int_{\partial V} f_2(Y)\lim_{Z\to Y}\int_{\partial W} \tau(X)\cdot\nabla\Gamma_Z^T(X)\,(\eta(X)-\eta(Y))\,d\sigma(X)\,d\sigma(Y).$$

But if $Z \in \gamma_\pm(Y)$ and if we let $\delta = 2|Z-Y|$, $\rho > 2\delta$,

$$\begin{aligned}\int_{\partial W} |\tau(X)\cdot(\nabla\Gamma_Z^T(X) - \nabla\Gamma_Y^T(X))(\eta(X)-\eta(Y))|\,d\sigma(X)&\\ &\leq \int_{\partial W\cap B(Y,\rho)} \left(\frac{C}{|Z-X|}+\frac{C}{|Y-X|}\right)\|\eta'\|_{L^\infty}|X-Y|\,d\sigma(X)\\ &\quad+\int_{\partial W\setminus B(Y,\rho)} \frac{C|Z-Y|^\alpha}{|X-Y|^{1+\alpha}}\|\eta\|_{L^\infty},\,d\sigma(X)\\ &\leq C\|\eta'\|_{L^\infty}\rho^2/\delta + C(\delta/\rho)^\alpha\left\|\frac{\rho^\alpha}{|X-\cdot|^{1+\alpha}}\right\|_{L^1(\partial W\setminus B(Y,\rho))}\|\eta\|_{L^\infty}.\end{aligned}$$

Choosing $\rho = \delta^{2/3}$, we see that this goes to 0 as $|X-Z|\to 0$.

Thus

$$\begin{aligned}
\int_{\partial V} \eta \mathcal{L}^t f_2 \, d\sigma &= \int_{\partial V} f_2(Y) \lim_{Z\to Y} \int_{\partial W} \tau(X) \cdot \nabla \Gamma_Z^T(X) \, (\eta(X) - \eta(Y)) \, d\sigma(X) \, d\sigma(Y) \\
&= \int_{\partial V} f_2(Y) \int_{\partial W} \tau(X) \cdot \nabla \Gamma_Y^T(X)(\eta(X) - \eta(Y)) \, d\sigma(X) \, d\sigma(Y) \\
&= \int_{\partial W} (\eta(X) - \eta(Y)) \, \tau(X) \cdot \int_{\partial V} f_2(Y) \nabla_X \Gamma_X(Y) \, d\sigma(Y) \, d\sigma(X) \\
&= \int_{\partial W} (\eta(X) - \eta(Y)) \, \tau(X) \cdot \nabla \mathcal{S}^T f_2(X) \, d\sigma(X) \\
&= \int_{\partial W} \eta(X) \, \tau(X) \cdot \nabla \mathcal{S}^T f_2(X) \, d\sigma(X) = \int_{\partial V} \eta(X) \, \tau(X) \cdot \nabla \mathcal{S}^T f_2(X) \, d\sigma(X).
\end{aligned}$$

This concludes the proof. □

5.2 Nontangential maximal functions of layer potentials

In this section, we assume that the boundary layer potential $\mathcal{T}_V$ of (2.23) is bounded $L^p(\partial V) \mapsto L^p(\partial V)$. (We will prove boundedness in Chapter 6.) If $X \notin \partial V$, let

$$\mathcal{R}_V F(X) = \int_{\partial V} K^A(X, Y) F(Y) \, d\sigma(Y)$$

so that $\mathcal{T}_V$ is the nontangential limit of $\mathcal{R}_V$, as $\mathcal{K}$ is the nontangential limit of $\mathcal{D}$. Then

$$\mathcal{R}_V(B_7 f)(X) = \begin{pmatrix} \mathcal{D}f(X) & \mathcal{S}(\partial_\tau f)(X) \\ \mathcal{D}f(X) & \mathcal{S}(\partial_\tau f)(X) \end{pmatrix}. \tag{5.10}$$

We begin with a lemma. (We will need to use it for operators other than $\mathcal{T}_V$, and so it is presented in slightly more generality.)

Lemma 5.11. *Assume that $K(X, Y)$ satisfies the Calderón-Zygmund kernel conditions*

$$\begin{aligned}
|K(X, Y)| &\leq \frac{\beta}{|X - Y|}, \\
|K(X, Y) - K(X', Y)| &\leq \frac{\beta |X - X'|^\alpha}{\min(|X - Y|, |X' - Y|)^{1+\alpha}}, \\
|K(X, Y) - K(X, Y')| &\leq \frac{\beta |Y - Y'|^\alpha}{\min(|X - Y|, |X - Y'|)^{1+\alpha}}
\end{aligned}$$

for some β*,* $\alpha > 0$*.*

Then if $\mathcal{R}F(X) = \int_{\partial V} K(X,Y)F(Y)\, d\sigma(Y)$*, then for any* $X \in V$*,* $X^* \in \partial V$ *with* $|X - X^*| \leq (1+a)\operatorname{dist}(X, \partial V)$*,*

$$|\mathcal{R}F(X)| \leq C\beta MF(X^*) + \mathcal{T}_* F(X^*).$$

Here if $\mathcal{T}$ is an operator on $L^2(\partial V)$ with Calderón-Zygmund kernel $K(X,Y)$, then

$$\mathcal{T}_* f(X) = \sup_{\epsilon > 0} \left| \int_{Y \in \partial V, |X-Y| > \epsilon} K(X,Y) f(Y)\, d\sigma(Y) \right|$$

is the standard truncated maximal operator associated with $\mathcal{T}$.

It is well-known (see, for example, [16, Remark 8.1.12] and [27, p. 34]) that if $\partial V = \mathbf{R}$, then L^2-boundedness of $\mathcal{T}$ implies L^2-boundedness of $\mathcal{T}_*$. It is easy to see that this remains true if V is any good Lipschitz domain.

Proof. Define

$$\mathcal{T}_h F(X^*) = \int_{|Y-X^*|>h,\, Y \in \partial V} K(X^*, Y) F(Y)\, d\sigma(Y).$$

If $h = \operatorname{dist}(X, \partial\Omega)$, and if $|X - X^*| < (1+a)h$, then

$$\begin{aligned}
|\mathcal{R}F(X) - \mathcal{T}_h F(X^*)| &\leq \left| \int_{|Y-X^*|>h} (K(X,Y) - K(X^*,Y)) F(Y)\, d\sigma(Y) \right| \\
&\quad + \left| \int_{|Y-X^*|<h} K(X,Y) F(Y)\, d\sigma(Y) \right| \\
&\leq \int_{|Y-X^*|>h} \frac{C\beta h^\alpha |F(Y)|}{|Y-X^*|^{1+\alpha}}\, d\sigma(Y) + \int_{|Y-X^*|<h} \frac{C\beta}{h} |F(Y)| d\sigma(Y) \\
&\leq C\beta MF(X^*)
\end{aligned}$$

where M is the standard maximal function and our constants depend on the Lipschitz constants of V.

But $|\mathcal{T}_h F(X)| \leq \mathcal{T}_* F(X)$, and so we are done. □

Theorem 5.12. *Let V be a good Lipschitz domain. If $\mathcal{T}_V$ is bounded from $L^p(\partial V)$ to itself for all $1 < p < \infty$, then for any such p,*

$$\|N_{V_\pm}(\mathcal{D}f)\|_{L^p} \leq C(p)\|f\|_{L^p}, \quad \|N_{V_\pm}(\nabla \mathcal{S}^T g)\|_{L^p} \leq C(p)\|g\|_{L^p}.$$

If $\tilde{\mathcal{T}}_V(\partial V)$ is bounded on L^p for all $1 < p < \infty$, then

$$\|N_{V_\pm}(\nabla \mathcal{D}f)\|_{L^p} \leq C(p)\|\partial_\tau f\|_{L^p}.$$

Proof. Since $\mathcal{T} = \mathcal{T}_V$ is bounded on $L^p(\partial V)$, so is $\mathcal{T}_*$; if $1 < p < \infty$, then M is bounded on $L^p(\partial V)$. Therefore, by (4.10) and Lemma 5.11, we have that $F \mapsto N(\mathcal{R}_V F)$ is bounded $L^p \mapsto L^p$. By (5.10), this completes the proof for $N(\mathcal{D}f)$.

Note that

$$\begin{aligned} \nabla \mathcal{D}f(X) &= \nabla_X \int_{\partial V} \nu \cdot A^T \nabla \Gamma_X^T f \, d\sigma = \nabla_X \int_{\partial V} \tau \cdot \nabla \tilde{\Gamma}_X^T f \, d\sigma \\ &= -\nabla_X \int_{\partial V} \tilde{\Gamma}_X^T \partial_\tau f \, d\sigma = -\int_{\partial V} \nabla_X \tilde{\Gamma}_X^T \partial_\tau f \, d\sigma \end{aligned}$$

and so, recalling that the kernel of $\tilde{\mathcal{T}}_V$ is $\tilde{K}^A(X,Y) = B_6^A(X) \begin{pmatrix} \nabla_X \tilde{\Gamma}_X^T(Y) & \nabla_X \tilde{\Gamma}_X^T(Y) \end{pmatrix}$, we may use (4.11) and Lemma 5.11 to bound $\|N(\nabla \mathcal{D}f)\|_{L^p}$.

We now turn to $N(\nabla \mathcal{S}^T g)$. Recall that

$$\nabla \mathcal{S}^T g(Z) = \int_{\partial V} \nabla_Z \Gamma_Z(Y) g(Y) \, d\sigma(Y).$$

Define $\mathcal{T}_h$ as before, by

$$\begin{aligned} \mathcal{T}_h F(X) &= \int_{|X-Y|>h} K^A(X,Y) F(Y) \, d\sigma(Y) \\ &= \int_{|X-Y|>h} \begin{pmatrix} \nabla \Gamma_X^T(Y) & \nabla \Gamma_X^T(Y) \end{pmatrix}^t B_6(Y)^t F(Y) \, d\sigma(Y). \end{aligned}$$

Then

$$\begin{aligned}\mathcal{T}_h^t F(X) &= \int_{|X-Y|>h} K^A(Y,X)^t F(Y)\, d\sigma(Y) \\ &= B_6(X) \int_{|X-Y|>h} \begin{pmatrix} \nabla_X \Gamma_X(Y) & \nabla_X \Gamma_X(Y) \end{pmatrix} F(Y)\, d\sigma(Y)\end{aligned}$$

If $\mathcal{T}$ is bounded on all $L^p(\partial V)$, then $\mathcal{T}^t$ is bounded on all $L^q(\partial V)$, and so $\mathcal{T}_*^t$ is as well. But by Lemma 5.11, $N(\nabla \mathcal{S}^T g)(X) \le CMg(X) + C\mathcal{T}_*^t g(X)$. This completes the proof. □

This theorem has two corollaries:

Corollary 5.13. *If $\mathcal{T}_V$ is bounded on $L^2(\partial V)$, then there exists a $\beta > 0$ such that if g is a H^1 atom supported in $B(X_0, R)$, then*

$$\int_{\partial V} N(\nabla \mathcal{S} g)(X)(1 + |X - X_0|/R)^\beta \, d\sigma(X) \le C.$$

As an immediate consequence we have that $\|N(\nabla \mathcal{S} g)\|_{L^1(\partial V)} \le C\|g\|_{H^1(\partial V)}$ for all $g \in H^1(\partial V)$.

Proof. Suppose that g is a H^1 atom supported in some connected set $\Delta \subset \partial V$ with $\sigma(\Delta) = R$ and $X_0 \in \Delta$. Then

$$\|g\|_{L^\infty} \le \frac{1}{\sigma(\Delta)} = \frac{1}{R},$$

and so $\|g\|_{L^2} \le 1/\sqrt{R}$.

Therefore, letting $b > 0$ be a constant, we have that by Theorem 5.12 and Hölder's inequality,

$$\begin{aligned}\int_{B(X_0,bR)\cap\partial V} |N(\nabla \mathcal{S} g)| &\le bk_4 R \left(⨍_{B(X_0,2R)\cap\partial V} |N(\nabla \mathcal{S} g)|^2 \right)^{1/2} \\ &\le bk_4 \sqrt{R} C \|g\|_{L^2} \le bC.\end{aligned}$$

We need to bound $\int_{\partial V \setminus B(X_0,bR)} |N(\nabla \mathcal{S} g)|(1 + |X - X_0|/R)^\beta \, d\sigma$. If $Y \in \gamma(X)$ for some $X \in \partial V$, then either $|X - Y| \le |X_0 - X|/2$ and so $|X_0 - Y| \ge |X_0 - X|/2$, or

$|X-Y| > |X_0 - X|/2$ and so

$$|X_0 - Y| \geq \operatorname{dist}(Y, \partial V) \geq \frac{1}{1+a}|Y-X| > \frac{|X_0 - X|}{2+2a}.$$

In any case,

$$N(\nabla \mathcal{S}g)(X) \leq \sup\left\{|\nabla \mathcal{S}g(Y)| : |Y - X_0| > \frac{|X_0 - X|}{2+2a}\right\}.$$

If $|Y - X_0| > R$, then

$$\begin{aligned}
|\nabla \mathcal{S}g(Y)| &\leq \left|\int_{B(X_0,R)\cap V} \nabla_Y \Gamma_Y(Z) g(Z)\, d\sigma(Z)\right| \\
&\leq \left|\int_{B(X_0,R)\cap V} (\nabla_Y \Gamma_Y(Z) - \nabla_Y \Gamma_Y(X_0)) g(Z)\, d\sigma(Z)\right| \\
&\leq \int_{B(X_0,R)\cap V} \frac{|Z-X_0|^\alpha}{(|Y-X_0|-R)^{1+\alpha}} \frac{C}{R}\, d\sigma(Z) \\
&\leq C \frac{R^\alpha}{(|Y-X_0|-R)^{1+\alpha}}.
\end{aligned}$$

Therefore, if $|X - X_0| > (2+2a)R$, then

$$N(\nabla \mathcal{S}g)(X) \leq C \frac{R^\alpha}{(|X-X_0| - R(2+2a))^{1+\alpha}}.$$

and so if we choose $b = 4+4a$, then

$$\begin{aligned}
&\int_{\partial V \setminus B(X_0,bR)} N(\nabla \mathcal{S}g)(X)(1+|X-X_0|/R)^{\alpha/2}\, d\sigma(X) \\
&\qquad\qquad \leq \int_{\partial V \setminus B(X_0,bR)} C \frac{R^{\alpha/2}}{|X-X_0|^{1+\alpha/2}}\, d\sigma(X) \leq C.
\end{aligned}$$

This completes the proof. ☐

Corollary 5.14. *If $\mathcal{T}_V$ and $\mathcal{T}_V^t$ are bounded on L^p for all $1 < p < \infty$, so Theorem 5.12 holds, then the limits in the definition of $\mathcal{K}f$, $\mathcal{L}f$ exist pointwise a.e. for $f \in L^p$, even if f is not smooth.*

Proof. We work with $\mathcal{K}$ only; the proof for $\mathcal{L}$ is identical. Let $f_n \in L^p$ be smooth and such that $\|f_n - f\|_{L^p} \leq 4^{-n}$. Then for each $X \in \partial V$, $\lim_{Y\to X \text{ n.t.}} \mathcal{D}f_n(Y)$ exists. Since $\mathcal{K}$ is bounded on L^p, $\lim_{n\to\infty} \mathcal{K} f_n = \mathcal{K} f$ exists in L^p.

Let $E_n = \{X \in \partial V : |\mathcal{K} f(X) - \mathcal{K} f_n(X)| > 2^{-n}$ or $N(\mathcal{D}(f_n - f))(X) > 2^{-n}\}$; then $\sigma(E_n) \le C2^{-n}$. Thus, $\sigma\left(\cup_{m=n}^{\infty} E_m\right) \le C2^{-n}$, so

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$$

has measure 0.

Suppose $X \in \partial V$, $X \notin E$. So there is some $N > 0$ such that, if $n > N$, then

$$\begin{aligned}
|\mathcal{D} f(Y) - \mathcal{K} f(X)| &\le |\mathcal{D} f(Y) - \mathcal{D} f_n(Y)| + |\mathcal{D} f_n(Y) - \mathcal{K} f_n(X)| + |\mathcal{K} f_n(X) - \mathcal{K} f(X)| \\
&\le N(\mathcal{D}(f - f_n))(X) + |\mathcal{D} f_n(Y) - \mathcal{K} f_n(X)| + |\mathcal{K} f_n(X) - \mathcal{K} f(X)| \\
&\le C2^{-n} + |\mathcal{D} f_n(Y) - \mathcal{K} f_n(X)|.
\end{aligned}$$

So for every $\epsilon > 0$, there is some $n > N$ such that $C2^{-n} < \epsilon/2$, and some $\delta > 0$ such that $|\mathcal{D} f_n(Y) - \mathcal{K} f_n(X)| < \epsilon/2$ provided $Y \in \gamma(X)$ and $|X - Y| < \delta$; thus, $|\mathcal{D} f(Y) - \mathcal{K} f(X)| < \epsilon$ if $|X - Y| < \delta$, and so the non-tangential limit exists at X, as desired. ☐

5.3 Potentials on H^1

We wish to investigate the behavior of layer potentials on $H^1(\partial V)$. We begin by proving a lemma:

Lemma 5.15. *Suppose that $f \in L^1(\partial V)$, V is good Lipschitz domain, and for some $p > 1$ and some c_1, c_p, R, $\alpha > 0$,*

$$\int_{\partial V} f \, d\sigma = 0, \quad \|f\|_{L^p(\partial V)} \le \frac{c_p}{R^{1-1/p}}, \quad \int_{\partial V} |f(X)|(1 + |X|/R)^{\alpha} \, d\sigma(X) \le c_1.$$

Then f is in $H^1(\partial V)$ with H^1 norm depending only on c_1, c_p, p, α and the Lipschitz constants of V. (Specifically, not on R.)

Proof. Since we can parameterize ∂V by arc length, it suffices to prove this in the case where $\partial V = \mathbf{R}$. If $g(x) = Rf(Rx)$, then g satisfies the conditions of the lemma with $R = 1$, and $\|g\|_{H^1} = \|f\|_{H^1}$; so we may assume $R = 1$.

Let ϕ be a Schwarz function with $\int \phi = 1$. Recall that $f \in H^1$ if

$$\int \sup_t \left| \int f(y) \frac{1}{t} \phi\left(\frac{x-y}{t}\right) \, dy \right| \, dx$$

is finite, and that its H^1 norm is comparable to the value of this integral (with comparability constants depending only on ϕ). Choose ϕ nonnegative with Schwarz norm 1.

So the inner integral is at most $CMf(x)$. So

$$\begin{aligned}\int_{|x|<1} \sup_t \left| \int f(y) \frac{1}{t} \phi\left(\frac{x-y}{t}\right) \, dy \right| \, dx \leq C \int_{|x|<1} Mf(x) \, dx &\leq C\|Mf\|_{L^p} \\ &\leq C(p)\|f\|_{L^p} \leq C(p)c_p.\end{aligned}$$

For any $x \in \mathbf{R}$, we have that

$$\int f(y) \frac{1}{t} \phi\left(\frac{x-y}{t}\right) \, dy = \int f(y) \frac{1}{t} \left(\phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x}{t}\right)\right) \, dy.$$

Now, assume $|x| > 1$. Then

$$\begin{aligned}\left| \int_{|y|<|x|/2} f(y) \frac{1}{t} \left(\phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x}{t}\right)\right) \, dy \right| &\leq \int_{|y|<|x|/2} |f(y)| \frac{1}{t} \frac{|y|}{t} \frac{C}{(|x|/t)^2} \, dy \\ = C \int_{|y|<|x|/2} |f(y)| \, |y|^\alpha \frac{|y|^{1-\alpha}}{|x|^2} \, dy &\leq \frac{C}{|x|^{1+\alpha}} \int |f(y)| \, |y|^\alpha \, dy \leq \frac{Cc_1}{|x|^{1+\alpha}}\end{aligned}$$

and

$$\begin{aligned}&\left| \int_{2|x|<|y|} f(y) \frac{1}{t} \left(\phi\left(\frac{x-y}{t}\right) - \phi\left(\frac{x}{t}\right)\right) \, dy \right| \\ &\qquad \leq \int_{2|x|<|y|} |f(y)| \frac{1}{t} \left(\frac{Ct}{|y-x|} + \frac{Ct}{|x|}\right) \, dy \leq \int_{2|x|<|y|} |f(y)| \, |y|^\alpha \frac{C}{|x|^{1+\alpha}} \, dy \leq \frac{Cc_1}{|x|^{1+\alpha}}\end{aligned}$$

and similarly

$$\left| \int_{|x|/2<|y|<2|x|} f(y) \frac{1}{t} \phi\left(\frac{x}{t}\right) \, dy \right| \leq \frac{Cc_1}{|x|^{1+\alpha}}.$$

But $\int_{|x|>1} \frac{1}{|x|^{1+\alpha}}\,dx = \frac{2}{\alpha}$. So to complete the proof, we need only bound

$$\int_{|x|>1} \sup_t \left| \int_{|x|/2<|y|<2|x|} f(y)\frac{1}{t}\phi\left(\frac{x-y}{t}\right)\,dy \right|\,dx.$$

This is equal to

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_{2^{k-1}<|x|<2^k} \sup_t \left| \int_{|x|/2<|y|<2|x|} f(y)\frac{1}{t}\phi\left(\frac{x-y}{t}\right)\,dy \right|\,dx \\
&\qquad\qquad \leq \sum_{k=1}^{\infty} \int_{2^{k-1}<|x|<2^k} \sup_t \int_{2^{k-2}<|y|<2^{k+1}} |f(y)|\frac{1}{t}\phi\left(\frac{x-y}{t}\right)\,dy\,dx \\
&\qquad\qquad \leq \sum_{k=1}^{\infty} \int_{2^{k-1}<|x|<2^k} CMf_k(x)\,dx
\end{aligned}$$

where $f_k(x) = f(x)$ if $2^{k-2} < |x| < 2^{k+1}$ and $f_k(x) = 0$ otherwise.

Now,

$$\int_{2^{k-1}<|x|<2^k} Mf_k(x)\,dx = \int_0^{\infty} \lambda(\beta)\,d\beta$$

where $\lambda(\beta) = |\{x : 2^{k-1} < |x| < 2^k, Mf_k(x) < \beta\}|$. We have three upper bounds on $\lambda(\beta)$:

- $\lambda(\beta) \leq 2^k$.
- Since $f \mapsto Mf$ is weak $(1,1)$-bounded, $\lambda(\beta) \leq \frac{C\|f_k\|_{L^1}}{\beta} \leq \frac{Cc_1}{2^{k\alpha}\beta}$.
- Since $f \mapsto Mf$ is strong (p,p)-bounded, $\lambda(\beta) \leq \frac{C\|f_k\|_{L^p}^p}{\beta^p} \leq \frac{Cc_p}{\beta^p}$.

So

$$\begin{aligned}
\int_0^{\infty} \lambda(\beta)\,d\beta &= \int_0^{2^{-k-k\alpha}} \lambda(\beta)\,d\beta + \int_{2^{-k-k\alpha}}^{2^{k\alpha/(p-1)}} \lambda(\beta)\,d\beta + \int_{2^{k\alpha/(p-1)}}^{\infty} \lambda(\beta)\,d\beta \\
&\leq \int_0^{2^{-k-k\alpha}} 2^k\,d\beta + \int_{2^{-k-k\alpha}}^{2^{k\alpha/(p-1)}} \frac{Cc_1}{2^{k\alpha}\beta}\,d\beta + \int_{2^{k\alpha/(p-1)}}^{\infty} \frac{Cc_p}{\beta^p}\,d\beta \\
&= \frac{1}{2^{k\alpha}} + \frac{Cc_1}{2^{k\alpha}} \ln(2^{k+k\alpha+k\alpha/(p-1)}) + \frac{Cc_p}{p-1}2^{-k\alpha} \leq C(p)(c_p+c_1)(1+k)2^{-k\alpha}
\end{aligned}$$

Summing from $k = 1$ to infinity gives a finite number depending only on p and α, as desired. □

Theorem 5.16. *If V is a good Lipschitz domain, and $\mathcal{K}^t$ and $\mathcal{L}^t$ are bounded on $L^p(\partial V)$ for some $1 < p < \infty$, then they are bounded $H^1(\partial V) \mapsto H^1(\partial V)$ as well, with bounds depending only on λ, Λ, p, and the Lipschitz constants of V.*

Proof. Suppose that f is an atom in $H^1(\partial V)$.

Then $\int f\,d\sigma = 0$, $\operatorname{supp} f \subset B(X_0, R)$, $\sigma(\operatorname{supp} f) < 2R$, $\|f\|_{L^\infty} \leq 1/R$ for some $X_0 \in \mathbf{R}^2$ and $R > 0$; without loss of generality we let $X_0 = 0$.

Since $\mathcal{K}^t$ and $\mathcal{L}^t$ are bounded on $L^p(\partial V)$,

$$\|\mathcal{K}^t f\|_{L^p(V)},\ \|\mathcal{L}^t f\|_{L^p(V)} \leq C(p)\|f\|_{L^p} \leq C(p)R^{1/p-1}.$$

Next, by Corollary 5.13 and Lemma 5.9, there exists some $\alpha > 0$ so that

$$\int_{\partial V} |\mathcal{K}^t f(X)|(1 + |X|/R)^\alpha \,d\sigma(X) \leq C, \qquad \int_{\partial V} |\mathcal{L}^t f(X)|(1 + |X|/R)^\alpha \,d\sigma(X) \leq C.$$

We need only show that $\int \mathcal{K}^t f\,d\sigma = \int \mathcal{L}^t f\,d\sigma = 0$. For bounded domains, this is simple:

$$\int_{\partial V} \mathcal{L}^t_V f(X)\,d\sigma(X) = \int_{\partial V} \tau \cdot \nabla \mathcal{S}^T f(X)\,d\sigma(X).$$

In Lemma 7.11, we will show that $\mathcal{S}^T f$ is continuous on $\mathbf{R}^2$; thus, this must be zero.

Also, by Lemma 5.9, if $\eta \in C_0^\infty(\mathbf{R}^2)$ with $\eta \equiv 1$ on $\bar{V}$, then

$$\int_{\partial V} \mathcal{K}^t_- f(X)\,d\sigma(X) = \int_{\partial V} \eta(X)\,\mathcal{K}^t_- f(X)\,d\sigma(X) = \int_V \nabla\eta \cdot A^T \nabla \mathcal{S}^T f(X)\,d\sigma(X) = 0.$$

In Lemma 7.10, we will show that $\mathcal{K}^t_+ f - \mathcal{K}^t_- f = f$, so $\int_{\partial V} \mathcal{K}^t_+ f\,d\sigma = 0$ if $f \in H^1(\partial V)$. A similar argument holds if V^C is bounded.

Finally, consider the case where $V = \Omega$ is a special Lipschitz domain. Then

$$\mathcal{S}^T f(X) = \int_{\partial\Omega} \Gamma_X(Y) f(Y)\,d\sigma(Y) = \int_{\partial\Omega} (\Gamma_X(Y) - \Gamma_X(0)) f(Y)\,d\sigma(Y)$$

and so if $|X|$ is large enough, then

$$|\mathcal{S}^T f(X)| \leq \int_{\partial\Omega} |\Gamma_X(Y) - \Gamma_X(0)|\,|f(Y)|\,d\sigma(Y) \leq \int_{\operatorname{supp} f} \frac{C|Y|}{|X|}\frac{1}{R}\,d\sigma(Y) \leq \frac{CR}{|X|}$$

and so $\int_{\partial\Omega} \mathcal{L}^t f(X)\,d\sigma(X) = \int_{\partial\Omega} \tau \cdot \nabla \mathcal{S}^T f(X)\,d\sigma(X) = 0$.

Now, pick some $r > 2R$, and let $\eta \in C_0^\infty(B(0,2r))$ with $\eta \equiv 1$ on $B(0,r)$ and $|\nabla\eta| < C/r$. Then

$$\begin{aligned}
\int_{\partial\Omega} \eta(X)\mathcal{K}_\pm^t f(X)\,d\sigma(X) &= \mp \int_{\partial\Omega_\mp} \eta(X)\,\nu \cdot A^T \nabla \mathcal{S}^T f(X)\,d\sigma(X) \\
&= \mp \int_{\Omega_\mp} \nabla\eta(X) \cdot A^T \nabla \mathcal{S}^T f(X)\,dX \\
&= \mp \int_{\Omega_\mp \cap B(0,2r)\setminus B(0,r)} \nabla\eta(X) \cdot A^T \nabla \mathcal{S}^T f(X)\,dX
\end{aligned}$$

We have that for $|X| > 2R$,

$$|\nabla \mathcal{S}^T f(X)| \leq \int_{\partial V} |\nabla\Gamma_X(Y) - \nabla\Gamma_X(0)|\,|f(Y)|\,d\sigma(Y) \leq \frac{CR^\alpha}{|X|^{1+\alpha}}$$

and so

$$\left|\int_{\partial\Omega} \eta(X)\mathcal{K}_\pm^t f(X)\,d\sigma(X)\right| \leq \int_{B(0,2r)\setminus B(0,r)} \frac{CR^\alpha}{r|X|^{1+\alpha}}\,dX \leq \frac{CR^\alpha}{r^\alpha}.$$

Since this goes to 0 as $r \to \infty$, we know $\int_{\partial\Omega} \mathcal{K}^t f\,d\sigma = 0$.

So, applying Lemma 5.15, we are done. ☐

5.4 Nearness to the real case

Theorem 5.17. *Let V be a good Lipschitz domain and let A_0 be real and satisfy (1.2). Assume that there is some $\epsilon_0 > 0$ such that, for all A satisfying (1.2) with $\|A - A_0\|_{L^\infty} < \epsilon_0$, we have that $\mathcal{T} = \mathcal{T}_V^A$ is bounded on $L^p(\partial V)$ for all $1 < p < \infty$, with bound depending only on p (not on A).*

If $\|A - A_0\|_{L^\infty}$ is small enough (compared to ϵ_0), then for all $1 < p < \infty$,

$$\begin{aligned}
\|N(\mathcal{D}^A f - \mathcal{D}^{A_0} f)\|_{L^p(\partial V)} &\leq C(p)\|f\|_{L^p(\partial V)}\|A - A_0\|_{L^\infty}, \\
\|N(\nabla\mathcal{S}^A f - \nabla\mathcal{S}^{A_0} f)\|_{L^p(\partial V)} &\leq C(p)\|f\|_{L^p(\partial V)}\|A - A_0\|_{L^\infty}.
\end{aligned}$$

Furthermore, if f is an H^1 atom, then

$$\int_{\partial V} N(\nabla \mathcal{S}^A f - \nabla \mathcal{S}^{A_0} f)(X)(1 + |X|/R)^\alpha \, d\sigma(X) \leq C\|A - A_0\|_{L^\infty}. \tag{5.18}$$

As an immediate corollary, we have that for such A,

$$\|\mathcal{K} - \mathcal{K}_0\|_{L^p \mapsto L^p}, \|\mathcal{L} - \mathcal{L}_0\|_{L^p \mapsto L^p} \leq C(p)\|A - A_0\|_{L^\infty},$$
$$\|\mathcal{K}^t - \mathcal{K}_0^t\|_{H^1 \mapsto H^1}, \|\mathcal{L}^t - \mathcal{L}_0^t\|_{H^1 \mapsto H^1} \leq C\|A - A_0\|_{L^\infty}.$$

Proof. Consider $\mathcal{R} = \mathcal{R}^A - \mathcal{R}^{A_0}$. Let $\|A - A_0\|_{L^\infty} = \epsilon$. Then by (4.12) and Lemma 5.11, we know that

$$|\mathcal{R}f(Z)| \leq C\epsilon M f(X) + C(\mathcal{T}^A - \mathcal{T}^{A_0})_* f(X)$$

for any $X \in \partial V$ with $|X - Z| < (1 + a) \operatorname{dist}(Z, \partial V)$.

But by (4.13), $\|\mathcal{T}^A - \mathcal{T}^{A_0}\|_{L^p(\partial V) \mapsto L^p(\partial V)} \leq C_p \epsilon$. So as in the proof of Theorem 5.12, $\|N(\mathcal{D}^A f - \mathcal{D}^{A_0} f)\|_{L^p(\partial V)} \leq C\epsilon \|f\|_{L^p(\partial V)}$ and $\|N(\nabla \mathcal{S}^A f - \nabla \mathcal{S}^{A_0} f)\|_{L^p(\partial V)} \leq C\epsilon \|f\|_{L^p(\partial V)}$, as desired.

If f is an H^1 atom, then (5.18) follows from the L^2-boundedness of $f \mapsto N(\nabla \mathcal{S}^A f - \nabla \mathcal{S}^{A_0} f)$ and from (4.12), as in the proof of Corollary 5.13. $\square$

CHAPTER 6

BOUNDEDNESS OF LAYER POTENTIALS

Theorem 6.1. *Let A_0, A be matrices defined on $\mathbf{R}^2$ which are independent of the t-variable, that is, $A(x,t) = A(x,s) = A(x)$, $A_0(x,t) = A_0(x,s) = A_0(x)$ for all $x, t, s \in \mathbf{R}$.*

Assume that A_0 is uniformly elliptic (that is, satisfies (1.2)) and $A_0(x) \in \mathbf{R}^{2\times 2}$ for all x. We assume that A, A_0 are smooth.

Then there exists an $\epsilon_0 = \epsilon_0(\lambda, \Lambda) > 0$ such that, if $\|A - A_0\|_{L^\infty} \leq \epsilon_0$, then if V is a bounded or special Lipschitz domain, the layer potentials $\mathcal{T}$ and $\tilde{\mathcal{T}}$ defined by (2.23) and (2.24) are L^p-bounded for any $1 < p < \infty$, with bounds depending only on λ, Λ, p, and the Lipschitz constants of V.

We assume $\epsilon_0 < \lambda/2$; this will ensure that A is elliptic as well.

If $\mathcal{T}$ is bounded $L^2 \mapsto L^2$, then by basic Calderón-Zygmund theory (e.g. [27, I.7]), $\mathcal{T}$ is bounded $L^p \mapsto L^p$ for $1 < p < \infty$.

We will start with special cases:

Theorem 6.2. *Suppose that we are working in a special Lipschitz domain Ω (so that we may consider T instead of $\mathcal{T}$). Then Theorem 6.1 holds.*

We will pass from Theorem 6.2 to Theorem 6.1 in Section 6.2.

We will want to start with an even more specialized case:

Theorem 6.3. *Suppose that we are working over a special Lipschitz domain Ω; we may write that $\Omega = \{X : \varphi(X \cdot \mathbf{e}^\perp) < X \cdot \mathbf{e}\}$ for some $\mathbf{e}$, $\mathbf{e}^\perp$ and some Lipschitz function φ. Suppose that φ is smooth and compactly supported. Further assume that*

$$A_0(x) = \begin{pmatrix} 1 & a^0_{12}(x) \\ 0 & a^0_{22}(x) \end{pmatrix}$$

for some functions a_{12}^0, a_{22}^0 *which leave* A_0 *uniformly elliptic, and that for some* R_0 *large,*

$$A(x) = A_0(x) = I \text{ for } |x| > R_0, \quad \fint_{-R_0}^{R_0} \frac{a_{21}(y)}{a_{11}(y)}\,dy = 0, \text{ and } \fint_{-R_0}^{R_0} \frac{1}{a_{11}(y)}\,dy = 1. \tag{6.4}$$

Then there is an $\delta_0 = \delta_0(\lambda, \Lambda) > 0$ *such that if* $\|\varphi'\|_{L^\infty} < \delta_0$*, then* T *and* $\tilde{T}$ *are* L^2*-bounded.*

We will pass from this theorem to Theorem 6.2 in Section 6.1, and will prove this theorem in Sections 6.3–6.7.

We let $O(\lambda, \Lambda)$ denote a term which, while not a constant, may be bounded by a constant depending only on λ, Λ; for example, since we know $|\nabla\Gamma_X(Y)| \leq C/|X-Y|$, we may write $\nabla\Gamma_X(Y) = O(\lambda,\Lambda)/|X-Y|$.

6.1 Buildup to arbitrary special Lipschitz domains

We will prove Theorem 6.3 in Sections 6.4–6.7; in this section, we will assume it. We wish to prove Theorem 6.2. We work only with T; the proof for $\tilde{T}$ is identical.

Theorem 6.5. *Theorem 6.3 holds if we relax the condition (6.4) on* A *and the requirement that* $\varphi \in C_0^\infty$*, and replace the requirement that* $\|\varphi'\|_{L^\infty} < \delta_0$ *with the requirement that* $\|\varphi' - \gamma\|_{L^\infty} < \delta_0$ *for some* $\gamma \in \mathbf{R}$*, and permit* δ_0 *to depend on* γ *as well as* λ, Λ*.*

Proof. We first look at the requirements (6.4) and $\varphi \in C_0^\infty$. Recall that $T_\pm F(x) = \lim_{h\to 0^\pm} T_h F(x)$, where

$$T_h F(x) = \int_{\mathbf{R}} K(x\mathbf{e}^\perp + (\varphi(x)+h)\mathbf{e}, y\mathbf{e}^\perp + \varphi(y)\mathbf{e})F(y)\,dy.$$

We know that the limits exist; see Lemma 5.8. We need only show that the T_h are uniformly bounded on $L^2(\mathbf{R})$, so we may consider some fixed choice of h.

Pick some $F \in L^2(\mathbf{R})$. Then for each $\mu > 0$, there is some positive number $R > 0$ with $\|F\|_{L^2(\mathbf{R}\setminus(-R,R))} < \mu$. If T_h^δ is an operator similar to T, but based on a different elliptic matrix A or Lipschitz function φ, then

$$|T_h^\delta F(x) - T_h F(x)| \leq \left| \int_{-R}^{R} F(y)(K_h^\delta(x,y) - K_h(x,y)), dy \right| + \int_{|y|>R} |F(y)| \frac{C}{|x-y|+|h|}\,dy$$

Then

$$\begin{aligned}|T_h^\delta F(x) - T_h F(x)| &\leq \|F\|_{L^2(\mathbf{R}\setminus(-R,R))} \left\|\frac{C}{\cdot}\right\|_{L^2(\mathbf{R}\setminus(-R,R))} + \left|(T_h^\delta - T_h)(F\mathbf{1}_{(-R,R)})\right| \\ &\leq \mu\frac{C}{\sqrt{R}} + \left|(T_h^\delta - T_h)(F\mathbf{1}_{(-R,R)})\right|\end{aligned}$$

provided that $|x| < R$.

If we choose T^δ to satisfy the requirements of Theorem 6.3, then by Lemma 5.11, $\|T_h^\delta\|_{L^2\mapsto L^2} \leq C + \|T_*^\delta\|_{L^2\mapsto L^2} \leq C + C\|T^\delta\|_{L^2\mapsto L^2} \leq C$ for all h. If we can further choose T^δ such that

$$\|T_h^\delta F - T_h F\|_{L^2((-R,R))} < C\|F\|_{L^2}$$

for all $F \in L^2$ supported in $(-R, R)$, then we will have that

$$\begin{aligned}\|T_h F\|_{L^2(-R,R)} &\leq \|T_h^\delta F\|_{L^2((-R,R))} + \|(T_h^\delta - T_h)F\|_{L^2((-R,R))} \\ &\leq C\|F\|_{L^2} + C\mu + \left\|(T_h^\delta - T_h)(F\mathbf{1}_{(-R,R)})\right\|_{L^2((-R,R))} \leq C\|F\|_{L^2} + C\mu.\end{aligned}$$

By letting $R \to \infty$ and δ, $\mu \to 0$, we will achieve our desired result.

We first remove the requirement (6.4). Assume that A is smooth and satisfies (1.2). For $R > 1$, let $A_\delta(x) = A(x)$ on $(-R^2, R^2)$, $A_\delta(x) = I$ if $|x| > 2R^2$, such that A_δ is smooth and satisfies (1.2) and (6.4). Define Γ^δ, K_h^δ, T_h^δ in the obvious way.

Then $\operatorname{div} A\nabla(\Gamma_X(Y) - \Gamma_X^\delta(Y)) = 0$ for $Y = (y, s)$ with $|y| < R^2$; therefore, since $|\Gamma_X(Y) - \Gamma_X^\delta(Y)| \leq C \log R$ if $|X|$, $|Y| < R^2/2$, we have that if $|X|$, $|Y| < R^2/4$, then

$$|\nabla\Gamma_X(Y) - \nabla\Gamma_X^\delta(Y)| < \frac{C \log R}{R^2}.$$

Taking $R > 4$, we have that $K_h^\delta(x, y) - K_h(x, y) < C\frac{\log R}{R^2}$ if $|x|$, $|y| < R$ and h is small. Therefore, for such x,

$$\|K_h^\delta(x, \cdot) - K_h(x, \cdot)\|_{L^2((-R,R))} \leq \frac{C \log R}{R^{3/2}}.$$

So if $\operatorname{supp} F \subset (-R,R)$, then $|T_h F(x) - T_h^\delta F(x)| \leq \|F\|_{L^2} \frac{C\log R}{R^{3/2}}$ and so

$$\|T_h F(x) - T_h^\delta F(x)\|_{L^2((-R,R))} \leq \|F\|_{L^2} \frac{C\log R}{R}.$$

But $\frac{C\log R}{R}$ is bounded (and in fact goes to zero) for R large; thus we are done.

We next remove the requirement that $\varphi \in C_0^\infty$. Assume instead that $\|\varphi'\|_{L^\infty} < \delta_0$. Choose φ_δ compactly supported and smooth with $\|\varphi_\delta - \varphi\|_{L^\infty} < \delta$ on $(-R,R)$. Then

$$\left|K_h(x,y) - K_h^\delta(x,y)\right| \leq \frac{|\varphi(x)-\varphi_\delta(x)|^\alpha + |\varphi(y)-\varphi_\delta(y)|^\alpha}{|x-y|^{1+\alpha} + |h|^{1+\alpha}}$$

As before, this gives that

$$\|T_h F(x) - T_h^\delta F(x)\|_{L^2((-R,R))} \leq C\|F\|_{L^2} \frac{\delta^\alpha \sqrt{R}}{h^{\alpha+1/2}}$$

if F is supported in $(-R,R)$; thus by forcing δ very small in comparison with h, R, we are done.

Finally, we relax from $\|\varphi'\|_{L^\infty} < \delta_0$ to $\|\varphi' - \gamma\|_{L^\infty} < \delta_0(\gamma)$ for some real number γ. Fix some choice of $\mathbf{e}$, φ and γ. Then $\Omega = \{X \in \mathbf{R}^2 : \varphi(X \cdot \mathbf{e}^\perp) < X \cdot \mathbf{e}\}$.

Define $\check{\mathbf{e}} = \frac{\mathbf{e} - \gamma \mathbf{e}^\perp}{\sqrt{1+\gamma^2}}$. If $\|\varphi' - \gamma\|_{L^\infty}$ is small enough, relative to $|\gamma|$, then there is some function $\check\varphi : \mathbf{R} \mapsto \mathbf{R}$ such that

$$\varphi(X \cdot \mathbf{e}^\perp) < X \cdot \mathbf{e} \text{ if and only if } \check\varphi(X \cdot \check{\mathbf{e}}^\perp) < X \cdot \check{\mathbf{e}}.$$

Applying Theorem 6.3 to $\check\varphi$, we see that the layer potentials $\check{T}_\pm$ are bounded $L^2 \mapsto L^2$. By definition, so are the potentials $\check{\mathcal{T}}_{\Omega_\pm}$. But $\mathcal{T}_{\Omega_\pm} = \check{\mathcal{T}}_{\Omega_\pm}$, and therefore $T_\pm$ is bounded $L^2 \mapsto L^2$. $\square$

Now we wish to remove the assumption that $\varphi' - \gamma$ must be small. This may be done exactly as in [25, Section 5], using the buildup scheme of David; for convenience, this construction is summarized here. Let

$$\Lambda^k(\delta_0) = \left\{\varphi : \mathbf{R} \mapsto \mathbf{R} \middle| \text{there is a constant } \gamma \in (-k,k) \text{ such that } \|\varphi' - \gamma\|_{L^\infty} < \delta_0\right\}.$$

We require $0 < \delta_0 \leq k$.

Lemma 6.6. *Suppose that for some fixed choice of* $\mathbf{e}$ *and* $\mathbf{e}^\perp$, *we have that for every* $k > 0$ *there is a* $\delta_0(k) > 0$ *such that* $T_\pm$ *is bounded for every* $\varphi \in \Lambda^k(\delta_0)$ *with bounds depending only on* λ, Λ *and* k.

Then $T_\pm$ *is bounded on* L^2 *for any Lipschitz function* φ *with bounds depending on* λ, Λ *and* $\|\varphi'\|_{L^\infty}$.

Proof. We have the following useful theorems:

Theorem 6.7. *[8, Proposition 10] Suppose that* $\delta,\ k > 0$, $\varphi \in \Lambda^k(\delta)$, *and* $I \subset \mathbf{R}$ *is an interval. Then there is a compact subset* $E \subset I$ *and a function* $\psi \in \Lambda^{k+\frac{\delta}{10}}\left(\frac{9}{10}\delta\right)$ *such that*

- $|E| \geq \frac{1}{3\sqrt{1+(k+\delta)^2}}|I|$,
- $\varphi(x) = \psi(x)$ *for all* $x \in E$, *and*
- *Either* $-\frac{4}{5}\delta \leq \psi'(x) \leq \delta$ *a.e., or* $-\delta \leq \psi'(x) \leq \frac{4}{5}\delta$ *a.e.*

Theorem 6.8. *[20, p. 110] Suppose that* $K : \mathbf{R}^2 \mapsto \mathbf{C}^{2\times 2}$ *is a Calderón-Zygmund kernel with constants no more than* C_6. *Suppose that there is a constant* $\theta \in (0, 1]$ *such that for any interval* $I \subset \mathbf{R}$, *there is a Calderón-Zygmund kernel* K_I *and a compact subset* $E \subset I$ *such that*

- $|E| > \theta|I|$,
- *For all* $x, y \in E$ *we have that* $K_I(x, y) = K(x, y)$, *and*
- $\|T_I^*\|_{L^2\mapsto L^2} \leq C_6$

where T_I^* *is the maximal singular integral operator associated to* K_I. *Then* $\|T_*\|_{L^2\mapsto L^2} \leq C(\theta)C_6$.

From Lemma 5.11 and following remarks, T is bounded from L^2 to L^2 if and only if T_* is.

Pick some $\varphi \in \Lambda^{k-\delta/9}\left(\frac{10}{9}\delta\right)$. If $I \subset \mathbf{R}$ is an interval, then by Theorem 6.7 there is an E and a $\varphi_I \in \Lambda^k(\delta)$ such that if T_I is as in (2.28) with φ replaced by φ_I, then

- $|E| \geq \frac{1}{3\sqrt{1+(k+\delta)^2}}|I|$,
- $K_I(x,y) = K(x,y)$ for all $x, y \in E$.

So by Theorem 6.8, if T_I^* is bounded on L^2 for all I, then so is T_*. Thus, if Theorem 6.2 holds for all functions $\varphi \in \Lambda^k(\delta)$, then it holds for all $\varphi \in \Lambda^{k-\delta/9}\left(\frac{10}{9}\delta\right)$.

Let $a_n = \frac{1}{9}(1 + (10/9) + \ldots + (10/9)^{n-1}) = (10/9)^n - 1$. So if Theorem 6.2 holds for all $\varphi \in \Lambda^{k-\delta a_n}((10/9)^n\delta)$, it also holds for all $\varphi \in \Lambda^{k-\delta a_{n+1}}((10/9)^{n+1}\delta)$, provided that $k - \delta a_{n+1} \geq (10/9)^{n+1}\delta$.

Pick any $k > 0$. Let $\delta(k)$ be the δ_0 given by Theorem 6.5, so Theorem 6.2 holds for all $\varphi \in \Lambda^k(\delta_0)$. Let $n(k)$ be the positive integer such that $k \geq 2\delta(k)(10/9)^{n(k)} > (9/10)k$. Then Theorem 6.2 holds for all

$$\varphi \in \Lambda^{k-\delta a_{n(k)}}((10/9)^{n(k)}\delta) \supset \Lambda^{k/2}\left(\frac{9}{20}k\right).$$

Thus, for *any* k, Theorem 6.2 holds for any $\|\varphi'\|_{L^\infty} \leq \frac{9}{20}k$ (with constants depending on k). Thus, it holds for any Lipschitz function φ, as desired. ☐

Finally, we deal with the assumption that A_0 is upper triangular with $a_{11}^0 \equiv 1$.

Consider the mapping $J : (x,t) \mapsto (f(x), t + g(x))$ where $\lambda < f' < \Lambda$. Then $\check{\Gamma}_X(Y) = \Gamma_{J(X)}(J(Y))$ is the fundamental solution with pole at X associated with the elliptic matrix

$$\check{A}(y) = \frac{1}{f'(y)} \begin{pmatrix} 1 & 0 \\ -g'(y) & f'(y) \end{pmatrix} A(f(y)) \begin{pmatrix} 1 & -g'(y) \\ 0 & f'(y) \end{pmatrix}.$$

If we choose $f'(y) = \frac{1}{a_{11}^0(f(y))}$ (or, put another way, choose $f^{-1}(y) = \int_0^y a_{11}^0$), then we will have $\check{a}_{11}^0 = 1$, and if we choose $g'(y) = \frac{f(y)a_{21}^0(f(y))}{a_{11}^0(f(y))}$, then we will have $\check{a}_{21}^0 = 0$.

But if $\varphi \in \Lambda^k(\delta_0)$ for δ_0 sufficiently small (depending on λ, Λ and k), where φ is the Lipschitz function in the definition of Ω, then

$$J(\partial\Omega) = \{(f(xe_2 + \varphi(x)e_1), -xe_1 + \varphi(x)e_2 + g(xe_2 + \varphi(x)e_1)) : x \in \mathbf{R}\}.$$

Recall $e_1^2 + e_2^2 = 1$. If $\|\varphi'\|_{L^\infty}$ and $|\mathbf{e}_2|$ are small enough, then the function $x \mapsto -xe_1 + \varphi(x)e_2 + g(xe_2 + \varphi(x)e_1)$ is invertible on $\mathbf{R}$, and both it and its inverse have bounded derivatives; thus $J(\Omega)$ is a special Lipschitz domain with coordinate vectors $\mathbf{e} = \begin{pmatrix}\pm 1 & 0\end{pmatrix}$ and $\mathbf{e}^\perp = \begin{pmatrix}0 & \mp 1\end{pmatrix}$. If $|\mathbf{e}_2|$ is not small but $\|\varphi'\|_{L^\infty}$ is, then $x \mapsto f(xe_2+\varphi(x)e_1)$ is invertible on $\mathbf{R}$ and both it and its inverse have bounded derivatives; thus, $J(\Omega)$ is a special Lipschitz domain with coordinate vectors $\mathbf{e} = \begin{pmatrix}0 & \pm 1\end{pmatrix}$ and $\mathbf{e}^\perp = \begin{pmatrix}\pm 1 & 0\end{pmatrix}$.

In any case, if $\|\varphi'\|_{L^\infty}$ is small enough, then $J(\Omega)$ is a special Lipschitz domain and so $\check{T}_\pm$ is bounded on $L^2(\partial\Omega)$; thus, $T_\pm$ is bounded on $L^2(\partial\Omega)$, and so by Lemma 6.6 we may build up to arbitrary Lipschitz domains.

6.2 Patching: $\mathcal{T}$ is bounded on good Lipschitz domains

Recall that we wish to prove Theorem 6.1. In this section, we reduce to the case of special Lipschitz domains:

Theorem 6.9. *If $\mathcal{T}_\Omega$ and $\tilde{\mathcal{T}}_\Omega$ are bounded $L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$ for all special Lipschitz domains Ω, then for any good Lipschitz domain V the operators $\mathcal{T}_V$ and $\tilde{\mathcal{T}}_V$ are bounded $L^p(\partial V) \mapsto L^p(\partial V)$, with bounds depending only on λ, Λ, p, the Lipschitz constants of V, and the operator norms of the $\mathcal{T}_\Omega$s and $\tilde{\mathcal{T}}_\Omega$s.*

As in Section 6.1, we work with $\mathcal{T}$; the proof for $\tilde{\mathcal{T}}$ is identical.

Proof. For any domains U, V and any function F defined on $\partial U \cap \partial V$, if we extend F to ∂U and ∂V by zero, then $\mathcal{T}_U F = \mathcal{T}_V F$.

From Definition 2.4, we may partition ∂V as follows: there are k_2 points $X_i \in \partial V$ with associated numbers $r_i > 0$, such that $\partial V \subset \cup_i B(X_i, r_i)$ and $B(X_i, 2r_i) \cap V = B(X_i, 2r_i) \cap \Omega_i$ for some special Lipschitz domains Ω_i. Let $\sum_i \eta_i$ be a partition of unity with $\operatorname{supp} \eta_i \subset \partial V \cap B(X_i, r_i)$, and let $F_i = F\eta_i$.

Then

$$\|\mathcal{T}_V F\|_{L^p} \leq \sum_{i=1}^{k_2} \|\mathcal{T}_V F_i\|_{L^p}.$$

But

$$\|\mathcal{T}_V F_i\|_{L^p(\partial V)}^p = \|\mathcal{T}_V F_i\|_{L^p(\partial V \cap B(X_i, 2r_i))}^p + \|\mathcal{T}_V F_i\|_{L^p(\partial V \setminus B(X_i, 2r_i))}^p$$

and

$$\|\mathcal{T}_V F_i\|_{L^p(\partial V \cap B(X_i, 2r_i))} = \|\mathcal{T}_{\Omega_i} F_i\|_{L^p(\partial V \cap B(X_i, 2r_i))} \leq \|\mathcal{T}_{\Omega_i} F_i\|_{L^p(\partial \Omega_i)} \leq C\|F_i\|_{L^p}.$$

But if $|Y - X_i| > 2r_i$, then

$$|\mathcal{T} F_i(Y)| = \left|\int_{\partial V} K(Y,Z) F_i(Z)\, d\sigma(Z)\right| \leq \frac{C}{|X_i - Y|}\int_{\partial V} |F_i(Z)|\, d\sigma(Z) \leq \frac{C r_i^{1/q}}{|X_i - Y|}\|F_i\|_{L^p}.$$

But then

$$\|\mathcal{T} F_i\|_{L^p}^p \leq \int_{|X_i - Y| > 2r_i} \frac{C r_i^{p-1}}{|X_i - Y|^p}\|F_i\|_{L^p}^p\, d\sigma + C\|F_i\|_{L^p}^p \leq C_p \|F_i\|_{L^p}^p,$$

where C_p depends on the Lipschitz constants of V.

Therefore,

$$\|\mathcal{T}_V F_i\|_{L^p}^p \leq C_p \|F_i\|_{L^p}^p$$

and so

$$\|\mathcal{T}_V F\|_{L^p}^p \leq C_{p,k_2} \sum_{i=1}^{k_2} \|\mathcal{T}_V F_i\|_{L^p}^p \leq \sum_i C_{p,k_2}\|F_i\|_{L^p}^p \leq C_{p,k_2}\|F\|_{L^p}^p. \qquad \square$$

Note that C_{p,k_2} does *not* depend on $\operatorname{diam} V$.

6.3 Proving Theorem 6.3: preliminary remarks

From [9, p. 42], we have the following useful theorem:

Theorem 6.10. *Suppose that B_1, $B_2 : \mathbf{R} \mapsto \mathbf{C}^{2\times 2}$ are invertible matrices at all points, and suppose that $\|B_1\|_{L^\infty}$, $\|B_2\|_{L^\infty} \leq C_1$.*

Assume that there exist nonnegative real smooth functions v_i with $\operatorname{supp} v_i \subset [-1,1]$, $\int v_i = 1$, and $\|v_i\|_{L^\infty}$, $\|v_i'\|_{L^\infty} \leq C_2$, and such that for all $x \in \mathbf{R}$ and all $t > 0$,

$$\left|\left(\int \frac{1}{t} v_i\left(\frac{x-y}{t}\right) B_i(y)\, dy\right)^{-1}\right| \leq C_3. \tag{6.11}$$

Suppose that the operator $T : B_1\mathcal{S} \mapsto (B_2\mathcal{S})'$ *has a Calderón-Zygmund kernel, the operator* $f \mapsto B_2^t T(B_1 f)$ *is weakly bounded, and that the constants in the definition of weak boundedness and Calderón-Zygmund kernel are no more than* C_4.

Suppose finally that $T(B_1)$ *and* $T^t(B_2)$ *have BMO norm no more than* C_5.

Then T *has a continuous extension to* L^2*, and its norm may be bounded by a constant depending only on* C_1, C_2, C_3, C_4 *and* C_5.

Note that if $B(x) = \beta(x)I$ for some scalar-valued function β, and if $\mu \le \mathbf{Re}\,\beta(x) \le |\beta(x)| \le M$ for some constants μ, $M > 0$, then B satisfies (6.11).

$TB \in BMO$ is defined as follows: if M_0 is a smooth H^1 atom (that is, $\operatorname{supp} M_0 \subseteq \overline{B(x_0, R)} = [x_0 - R, x_0 + R]$, $\|M_0\|_{L^\infty} \le \frac{1}{R}$, and $\int M_0 = 0$), then

$$\langle M_0, TB\rangle = \langle M_0, T(\eta B)\rangle + \int_{\mathbf{R}}\int_{\mathbf{R}} M_0(x)(K_0(x,y) - K_0(x_0,y))\,dx\,(1-\eta(y))B(y)\,dy$$

whenever $\eta \in C_0^\infty$, $0 \le \eta \le 1$, and $\eta \equiv 1$ on $[x_0 - 2R, x_0 + 2R]$.

The proof of this theorem is involved. However, it is easy to establish a converse, namely, that if T is bounded $L^2 \mapsto L^2$ then it must be bounded $L^\infty \mapsto BMO$. This is a classic result of Calderón-Zygmund theory (see, for example, [16, Theorem 8.2.7]) which is established as follows:

If $\eta \equiv 1$ on $B(x_0, 2R)$, $\operatorname{supp}\eta \subset B(x_0, 3R)$, then

$$\begin{aligned}\left|\int_{\mathbf{R}}\int_{\mathbf{R}} M_0(x)(K_0(x,y) - K_0(x_0,y))\,dx\,(1-\eta(y))B(y)\,dy\right|& \\ \le \int_{\mathbf{R}} |M_0(x)| \int_{|y-x_0|>2R} \frac{|x-x_0|^\alpha}{|y-x_0|^{1+\alpha}}\,dx\,\|B\|_{L^\infty}\,dy \le C\|B\|_{L^\infty}& \qquad (6.12)\end{aligned}$$

If T is bounded $L^2 \mapsto L^2$, then since $\|M_0\|_{L^2} \le \sqrt{2/R}$, $\|\eta B\|_{L^2} \le \|B\|_{L^\infty}\sqrt{6R}$, we may bound $\langle M_0, T(\eta B)\rangle$, and so

$$|\langle M_0, TB\rangle| \le (C + C\|T\|_{L^2 \mapsto L^2})\|B\|_{L^\infty}.$$

So $\|TB\|_{BMO} \le (C + C\|T\|_{L^2 \mapsto L^2})\|B\|_{L^\infty}$.

We now outline the proof of Theorem 6.3; the remainder of this chapter will be devoted to establishing the facts used in this proof. We will consider only T_+; the case for T_- is similar.

Proof of Theorem 6.3. In Section 6.4, we will find a matrix B_1 which is bounded, invertible, and satisfies (6.11). In Lemmas 6.16 and 6.17, we will show that $f \mapsto B_0^t T(B_1 f)$ and $f \mapsto B_0^t \tilde{T}^t(B_1 f)$ are weakly bounded for *any* bounded matrix B_0. In Corollary 6.18, we will show that $\|TB_1\|_{BMO}$ and $\|\tilde{T}^t B_1\|_{BMO}$ are bounded.

By Theorem 6.10, this tells us that if B is bounded and satisfies (6.11), then

$$\begin{aligned}
\|T\|_{L^2 \mapsto L^2} &\leq C + C\|TB_1\|_{BMO} + C\|T^t B\|_{BMO} \leq C + C\|T^t B\|_{BMO},\\
\|\tilde{T}\|_{L^2 \mapsto L^2} &\leq C + C\|\tilde{T}^t B_1\|_{BMO} + C\|\tilde{T} B\|_{BMO} \leq C + C\|\tilde{T} B\|_{BMO}.
\end{aligned}$$

In Section 6.7, we will find a B_8 such that $\|T^t(B_8)\|_{BMO}$ is finite, and so we will know that $\|T\|_{L^2 \mapsto L^2}$ is finite. Unfortunately, our bound on $T^t(B_8)$ will depend on nasty quantities such as $\|A'\|_{L^\infty}$; therefore, we will seek a better bound on $\|T\|_{L^2 \mapsto L^2}$.

In Section 6.6, we will find bounded matrices $B_3, \ldots, B_6$ such that $\|\tilde{T} B_3\|_{BMO} \leq C + C\|T^t B_4\|_{BMO}$ and $\|T^t B_6\|_{BMO} \leq C + C\|\tilde{T} B_5\|_{BMO}$, where B_3, B_6 satisfy 6.11 and B_5 is small (depending on $\|A - A_0\|_{L^\infty} = \epsilon_0$, $\|\varphi'\|_{L^\infty} = \delta_0$). This implies that

$$\begin{aligned}
\|T\|_{L^2 \mapsto L^2} &\leq C + C\|T^t B_6\|_{BMO} \leq C + C\|\tilde{T} B_5\|_{BMO} \leq C + C\|\tilde{T}\|_{L^2 \mapsto L^2} \|B_5\|_{L^\infty}\\
&\leq C + \Big(C + C\|\tilde{T} B_3\|_{BMO}\Big) \|B_5\|_{L^\infty} \leq C + \Big(C + C\|T^t B_4\|_{BMO}\Big) \|B_5\|_{L^\infty}\\
&\leq C + \left(C + C\|T\|_{L^2 \mapsto L^2} \|B_4\|_{L^\infty}\right) \|B_5\|_{L^\infty}\\
&\leq C + C\|T\|_{L^2 \mapsto L^2} \|B_4\|_{L^\infty} \|B_5\|_{L^\infty}.
\end{aligned}$$

Since $\|T\|_{L^2 \mapsto L^2}$ is finite, if $\|B_5\|$ is small enough then $\|T\|_{L^2 \mapsto L^2} \leq C$.

Furthermore,

$$\|\tilde{T}\|_{L^2 \mapsto L^2} \leq C + C\|\tilde{T} B_3\|_{BMO} \leq C + C\|T^t\|_{L^2 \mapsto L^2} \|B_4\|_{L^\infty} \leq C$$

as desired. $\square$

6.4 A B for our TB theorem

Recall (2.34):

$$B_1(y) = \begin{pmatrix} a_{11}(\psi(y)) & 0 \\ a_{21}(\psi(y)) & 1 \end{pmatrix}^{-1} \begin{pmatrix} A(\psi(y))\nu(y) & \tau(y) \end{pmatrix} \sqrt{1+\varphi'(y)^2}.$$

In this section, we will show that B_1 satisfies (6.11). Let

$$B_0(y) = \begin{pmatrix} a^0_{11}(\psi(y)) & 0 \\ a^0_{21}(\psi(y)) & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_0(\psi(y))\nu(y) & \tau(y) \end{pmatrix} \sqrt{1+\varphi'(y)^2}.$$

Then B_0 is near B_1. We first show that, for any interval I, $⨍_I B_0$ is invertible with bounded inverse. If $\|A - A_0\|$ is sufficiently small, this will imply that $⨍_I B_1$ is as well.

We will use this result in Section 10.4, when we will not be making the change of variables which forces $a^0_{11} \equiv 1$, $a^0_{21} \equiv 0$; thus, we do not use this change of variables here.

Let

$$E(y) = \begin{pmatrix} 1/a^0_{11} & -a^0_{21}/a^0_{11} \\ 0 & 1 \end{pmatrix}, \quad \check{A}_0(y) = a^0_{11} E(y)^T A_0(\psi(y)) E(y) = \begin{pmatrix} 1 & a^0_{12} - a^0_{21} \\ 0 & \det A_0 \end{pmatrix}$$

where all the a^0_{ij}s are evaluated at $\psi(y)$.

Since E is a real invertible matrix and $a^0_{11} > \lambda$, we have that $\check{A}_0(y)$ is elliptic, with $\bar\eta \cdot \check{A}_0(y)\eta \geq \lambda^2 |E^{-1}(y)|^{-2} |\eta|^2$. Let $\rho = \sup_y \lambda^2 |E^{-1}(y)|^{-2} \geq 1/C(\lambda, \Lambda)$.

Now,

$$\begin{aligned} ⨍_I B_0 &= ⨍_I E(y)^T \begin{pmatrix} A_0(\psi(y))\nu(y) & \tau(y) \end{pmatrix} \sqrt{1+\varphi'(y)^2}\, dy \\ &= ⨍_I \begin{pmatrix} E(y)^T A_0(\psi(y)) E(y) (E(y)^{-1}\nu(y)) & E(y)^T \tau(y) \end{pmatrix} \sqrt{1+\varphi'(y)^2}\, dy \\ &= ⨍_I \begin{pmatrix} \frac{1}{a^0_{11}} \check{A}_0(y) (E(y)^{-1}\nu(y)) & E(y)^T \tau(y) \end{pmatrix} \sqrt{1+\varphi'(y)^2}\, dy. \end{aligned}$$

Recalling the definitions of $\nu(y)$ and $\tau(y)$, and letting $\psi(x) = \begin{pmatrix}\psi_1(x) & \psi_2(x)\end{pmatrix}$, we have that

$$\fint_I B_0 = \fint_I \left(\frac{1}{a^0_{11}} \check{A}_0(y) \begin{pmatrix} a^0_{11} & a^0_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_2'(y) \\ -\psi_1'(y) \end{pmatrix} \quad \begin{pmatrix} 1/a^0_{11} & 0 \\ -a^0_{21}/a^0_{11} & 1 \end{pmatrix} \begin{pmatrix} \psi_1'(y) \\ \psi_2'(y) \end{pmatrix} \right) dy.$$

Let $\alpha(y) = \psi_1'(y)/a^0_{11}(\psi(y))$, $\beta(y) = \psi_2'(y) - \psi_1'(y) a^0_{21}(\psi(y))/a^0_{11}(\psi(y))$. Then

$$\fint_I B_0 = \fint_I \left(\check{A}_0(y) \begin{pmatrix} \beta(y) \\ -\alpha(y) \end{pmatrix} \quad \begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix} \right) dy = \fint_I \left(\begin{pmatrix} \beta(y) - \alpha(y)\check{a}^0_{12} \\ -\check{a}^0_{22}\alpha(y) \end{pmatrix} \quad \begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix} \right) dy.$$

Since $\fint_I B_0$ is bounded, to bound its inverse we need only bound its determinant from below. But for any number $\theta > 0$,

$$\begin{aligned} \det \fint_I B_0 &= \left(\fint_I \beta(y)\,dy \right)^2 - \fint_I \beta(y)\,dy \fint_I \check{a}^0_{12}(y)\alpha(y)\,dy + \fint_I \alpha(y)\,dy \fint_I \check{a}^0_{22}(y)\alpha(y)\,dy \\ &\geq \left(\fint_I \beta(y)\,dy \right)^2 + \fint_I \alpha(y)\,dy \fint_I \check{a}^0_{22}(y)\alpha(y)\,dy \\ &\quad - \frac{\theta}{2} \left(\fint_I \beta(y)\,dy \right)^2 - \frac{1}{2\theta} \left(\fint_I \check{a}^0_{12}(y)\alpha(y)\,dy \right)^2. \end{aligned}$$

If $A \in C^{2\times 2}$ satisfies (1.1), then, by either letting η be a coordinate vector, or letting $\eta = \begin{pmatrix}\sqrt{\mathbf{Re}\, a_{22} - \lambda} & \zeta\sqrt{\mathbf{Re}\, a_{11} - \lambda}\end{pmatrix}$ for an appropriate $\zeta \in \mathbf{C}$ with $|\zeta| = 1$, we have that

$$\mathbf{Re}\, a_{11} \geq \lambda, \quad \mathbf{Re}\, a_{22} \geq \lambda, \quad |a_{12} + \overline{a_{21}}| \leq 2\sqrt{(\mathbf{Re}\, a_{11} - \lambda)(\mathbf{Re}\, a_{22} - \lambda)}.$$

So in particular, since $\check{a}^0_{22}$ is real, $\check{a}^0_{21} = 0$ and $\check{a}^0_{11} = 1$,

$$|\check{a}^0_{11}(y)| \leq 2\sqrt{(1-\rho)(\check{a}^0_{22}(y) - \rho)}.$$

So

$$\begin{aligned}\det\fint_I B_0 &\geq \left(\fint_I \beta(y)\,dy\right)^2\left(1-\frac{\theta}{2}\right) + \fint_I \alpha(y)\,dy \fint_I \check{a}^0_{22}(y)\alpha(y)\,dy \\ &\quad - \frac{2}{\theta}\left(\fint_I \alpha(y)\sqrt{(1-\rho)(\check{a}^0_{22}(y)-\rho)}\,dy\right)^2 \\ &\geq \left(\fint_I \beta(y)\,dy\right)^2\left(1-\frac{\theta}{2}\right) + \fint_I \alpha(y)\,dy \fint_I \check{a}^0_{22}(y)\alpha(y)\,dy \\ &\quad - \frac{2(1-\rho)}{\theta}\fint_I \alpha(y)\,dy \fint_I \alpha(y)(\check{a}^0_{22}(y)-\rho)\,dy.\end{aligned}$$

Choosing $\theta = 2-2\rho$, we see that

$$\det\fint_I B_0 \geq \rho\left(\fint_I \beta(y)\,dy\right)^2 + \rho\left(\fint_I \alpha(y)\,dy\right)^2.$$

But if $I=(a,b)$, then

$$\fint_I \alpha(y)\,dy = \frac{1}{b-a}\int_a^b \frac{\psi_1'(y)}{a^0_{11}(\psi_1(y))}\,dy = \frac{1}{b-a}\int_{\psi_1(a)}^{\psi_1(b)} \frac{1}{a^0_{11}(s)}\,ds \geq \lambda\frac{\psi_1(b)-\psi_1(a)}{b-a}$$

and

$$\fint_I \beta(y)\,dy = \frac{1}{b-a}\int_a^b \psi_2'(y) - \frac{\psi_1'(y)a^0_{21}(\psi_1(y))}{a^0_{11}(\psi_1(y))}\,dy \geq \frac{\psi_2(b)-\psi_2(a)}{b-a} - \frac{1}{b-a}\int_{\psi_1(a)}^{\psi_1(b)} \frac{a_{21}(s)}{a^0_{11}(s)}\,ds$$

so

$$\left(\fint_I \beta(y)\,dy\right)^2 + \left(\fint_I \alpha(y)\,dy\right)^2 \geq \frac{1}{C}\frac{|\psi(b)-\psi(a)|}{b-a}$$

which, recalling (2.27), is at least $1/C$. Let $2\mu = \rho/C$.

Note that $|\det(M+N) - \det M - \det N| \leq C(|M| * |N|)$. So if $\|A-A_0\|_{L^\infty} < \epsilon_0$, then $\mathbf{Re}\det\fint_I B_1 \geq 2\mu - C\epsilon_0$, so for ϵ_0 small enough, $\mathbf{Re}\det\fint_I B_1 \geq \mu$.

But (6.11) requires a *smooth* truncation of B_1.

Lemma 6.13. *Suppose that* $B : \mathbf{R} \mapsto \mathbf{C}^{2\times 2}$ *satisfies* $|\xi\cdot B(x)\eta| \leq M|\eta|\,|\xi|$ *for all* $x\in\mathbf{R}^2$, η, $\xi \in C^2$. *Assume further that for some number* $\mu > 0$,

$$\left|\det\fint_I B(x)\,dx\right| \geq \mu \tag{6.14}$$

for all intervals $I \subset \mathbf{R}$.

Then there is a smooth real function v*, with* $0 \le v \le 1$, $\int v = 1$, $\operatorname{supp} v \subset [-1, 1]$*, and* $\|v'\| \le C(\mu, M)$*, such that if* $v_t(x) = \frac{1}{t} v\left(\frac{x}{t}\right)$*, then* $|\det B * v_t(x)| \ge \mu/2$ *for all* $t > 0$ *and all* $x \in \mathbf{R}$.

Proof. Choose some v such that $v \equiv 1$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $\operatorname{supp} v \subset \left[-\frac{1}{2} - \rho, \frac{1}{2} + \rho\right]$, and $\int v = 1$, $0 \le v \le 1$, and $\|v'\|_{L^\infty} \le \frac{2}{\rho}$ for some positive real number $\rho < 1/2$ to be determined. Then

$$\begin{aligned} v_t * B(x) &= \int v_t(x-y) B(y)\, dy \\ &= \frac{1}{t} \int_{x-t/2}^{x+t/2} B(y)\, dy + \int_{t/2}^{t/2+t\rho} B(y) v_t(x-y)\, dy + \int_{-t/2-t\rho}^{-t/2} B(y) v_t(x-y)\, dy \end{aligned}$$

and so since $|v_t(x-y)| \le 1/t$ for all x, y,

$$\left| v_t * B(x) - \fint_{x-t/2}^{x+t/2} B(y)\, dy \right| \le 2\rho \|B\|_{L^\infty}.$$

Therefore,

$$|\det v_t * B(x)| \ge \mu - C\rho \|B\|_{L^\infty}^2.$$

So if $\rho < \mu/CM^2$, then $\mathbf{Re} \det v_t * B(x) \ge \mu/2$ for all x. □

6.5 T and $\tilde{T}^t$ are continuous and weakly bounded on $B_1\mathcal{S}$

For real A, this is shown in [25], in Lemmas 4.3, 4.7, 4.8, and 4.10. We now prove weak boundedness in the complex case.

Definition 6.15. A function F is called a *normalized bump function* if $\operatorname{supp} F \subset B(X_0, 10)$ for some X_0, and

$$|\partial^\alpha F(X)| \le 1$$

for any multiindex α with $|\alpha| \le 2$. For such a function F, let $F_R(X) = \frac{1}{R} F(X/R)$.

If there is a constant C such that, for any $R > 0$ and any normalized bump functions F and G, the equation

$$|\langle G_R, P F_R \rangle| \le \frac{C}{R}$$

holds, then we say the operator P is *weakly bounded.*

Lemma 6.16. *The operator T is a continuous linear operator from $B_1\mathcal{S}$ to $(B_0\mathcal{S})'$ for any bounded B_0. The operator $F \mapsto B_0^t T(B_1F)$ is weakly bounded; in fact, $\|T(B_1F_R)\|_{L^\infty} \leq C/R$ for any normalized bump function F.*

Proof. We know from Lemma 5.8 that if $f : \partial\Omega \mapsto \mathbf{C}$ is well-behaved, then $T_\pm(B_1f)(x)$ is pointwise well-defined and that the limits in the definition converge uniformly. Recall that

$$B_1(y) = (B_6(\psi(y))^t)^{-1}\begin{pmatrix} A(\psi(y))\nu(y) & \tau(y)\end{pmatrix}\sqrt{1+\varphi'(y)^2}.$$

Choose some $F, G \in \mathcal{S}^{2\times 2}$ (that is, matrix-valued functions whose components are all in $\mathcal{S}$). Recall that

$$\langle B_0G, T(B_1F)\rangle = \lim_{h\to 0^+}\int_{\mathbf{R}}\int_{\mathbf{R}} G(x)^t B_0(x)^t K_h(x,y)B_1(y)F(y)\,dy\,dx$$

Now, the components of

$$K_h(x,y)B_1(y) = \begin{pmatrix}\nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t\\ \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t\end{pmatrix}\begin{pmatrix} A(\psi(y))\nu(y) & \tau(y)\end{pmatrix}\sqrt{1+\varphi'(y)^2}$$

are

$$\frac{d}{dy}\Gamma^T_{\psi(x,h)}(\psi(y)) \text{ and } \frac{d}{dy}\tilde{\Gamma}^T_{\psi(x,h)}(\psi(y)).$$

Let f be a component of F. We need only show that

$$\lim_{h\to 0^+}\int_{\mathbf{R}} f(y)\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\,dy, \quad \lim_{h\to 0^+}\int_{\mathbf{R}} f(y)\frac{d}{dy}\tilde{\Gamma}_{\psi(x,h)}(\psi(y))\,dy$$

are bounded by $C\|F\|_{\mathcal{S}}$ and converge uniformly in x, and that if $F = \check{F}_R$ for some normalized bump function $\check{F}$, then those integrals are at most C/R.

For any number $R > 0$, we have that

$$\begin{aligned}
&\left|\int_{\mathbf{R}} f(y)\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\,dy\right| \\
&\qquad \le \left|\int_{|x-y|>2R} f(y)\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\,dy\right| + \left|\int_{x-2R}^{x+2R} f(y)\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\,dy\right| \\
&\qquad \le \frac{C}{R}\|f\|_{L^1} + \left|f(x)\int_{x-2R}^{x+2R}\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\,dy\right| \\
&\qquad\quad + \left|\int_{x-2R}^{x+2R}(f(y)-f(x))\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\,dy\right| \\
&\qquad \le \frac{C}{R}\|f\|_{L^1} + \|f\|_{L^\infty}\left|\Gamma_{\psi(x,h)}(\psi(x+2R)) - \Gamma_{\psi(x,h)}(\psi(x-2R))\right| \\
&\qquad\quad + \|f'\|_{L^\infty}\int_{x-2R}^{x+2R}\left|\frac{d}{dy}\Gamma_{\psi(x,h)}(\psi(y))\right||x-y|\,dy \\
&\qquad \le \frac{C}{R}\|f\|_{L^1} + C\|f\|_{L^\infty} + CR\|f'\|_{L^\infty}
\end{aligned}$$

Picking $R = 1$, we see that this is bounded by the Schwarz norm of f, and if $F = \check{F}_R$ where $\check{F}$ is a normalized bump function, this is clearly at most C/R.

Similarly,

$$\lim_{h\to 0^+}\int_{\mathbf{R}}\frac{d}{dy}\tilde{\Gamma}_{\psi(x,h)}(\psi(y))f(y)\,dy$$

exists, converges uniformly in x, and is at most C/R provided $F = \check{F}_R$ for some normalized bump function $\check{F}$. □

Lemma 6.17. *The operator $\tilde{T}^t$ is a continuous linear operator from $B_1\mathcal{S}$ to $(B_0\mathcal{S})'$ for any bounded B_0. Also, $F \mapsto B_0^t\tilde{T}^tB_1F$ is weakly bounded; in fact, $\|\tilde{T}^t(B_1F_R)\|_{L^\infty} \le C/R$ for any normalized bump function F.*

Proof. Fix F, $G \in \mathcal{S}^{2\times 2}$. Again, we wish to show that $\langle B_0G, \tilde{T}^t(B_1F)\rangle$ exists and is bounded, and that if $F = \check{F}_R, G = \check{G}_R$ for some normalized bump functions $\check{F}$, $\check{G}$, then $|\langle B_0G, \tilde{T}^t(B_1F)\rangle| \le 1/R$.

Now,

$$\langle B_0G, \tilde{T}^t_\pm(B_1F)\rangle = \lim_{h\to 0^\pm}\int_{\mathbf{R}} G(x)B_0(x)\int_{\mathbf{R}}\tilde{K}_h(y,x)^tB_1(y)F(y)\,dy\,dx$$

But

$$\tilde{K}_h(y,x)^t B_1(y) = \begin{pmatrix} \nabla_Y \tilde{\Gamma}^T_{\psi(y)}(\psi(x,h))^t \\ \nabla_Y \tilde{\Gamma}^T_{\psi(y)}(\psi(x,h))^t \end{pmatrix} \begin{pmatrix} A(\psi(y))\nu(y) & \tau(y) \end{pmatrix} \sqrt{1+\varphi'(y)^2}.$$

We may deal with the $\sqrt{1+\varphi'(y)^2}\nabla_Y \tilde{\Gamma}^T_{\psi(y)}(\psi(x,h)) \cdot \tau(y) = \frac{d}{dy}\tilde{\Gamma}^T_{\psi(y)}(\psi(x,h))$ as before. We are left trying to show that

$$\lim_{h\to 0^\pm} \int_{\mathbf{R}} \nu \cdot A^T(\psi(y)) \nabla_Y \tilde{\Gamma}^T_{\psi(y)}(\psi(x,h)) f(y) \sqrt{1+\varphi'(y)^2}\, dy$$

is bounded and converges uniformly in x, for any f a component of F.

But this integral is

$$\int_{\partial\Omega} \nu(Y) \cdot A^T(Y) \nabla_Y \tilde{\Gamma}^T_Y(\psi(x,h)) f(\psi^{-1}(Y)) d\sigma(Y).$$

Let $m \in C_0^\infty(\mathbf{R})$ with $m \equiv 1$ on $(-R - R\|\varphi'\|_{L^\infty}, R + R\|\varphi'\|_{L^\infty})$, and $0 \le m \le 1$, $\operatorname{supp} m \subset (-CR, CR)$, and $|m'| < C/R$. Let $u(\psi(y,t)) = f(y)m(t+\varphi(y))$, so $u(y\mathbf{e}^\perp + t\mathbf{e}) = f(y)m(t)$ and $|\nabla u(y\mathbf{e}^\perp + t\mathbf{e})| \le |f(y)|\,|m'(t)| + |f'(y)|\,|m(t)| \le C\|f'\|_{L^\infty} + C\|f\|_{L^\infty}/R$. Then, provided $\pm h > 0$, by (4.8) and the weak definition of conormal derivative we have that

$$\begin{aligned}\int_{\partial\Omega} \nu(Y) \cdot A^T(Y) \nabla_Y \tilde{\Gamma}^T_Y(\psi(x,h)) f(\psi^{-1}(Y)) d\sigma(Y) \\ = \mp \int_{\Omega_\mp} A^T(Y) \nabla_Y \tilde{\Gamma}^T_Y(\psi(x,h)) \cdot \nabla u(Y)\, dY.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left| \int_{\partial\Omega} \nu(Y) \cdot A^T(Y) \nabla_Y \tilde{\Gamma}_Y(\psi(x,h)) f(\psi^{-1}(Y)) d\sigma(Y) \right| \\ &\quad = \left| \int_{\Omega_\pm} A^T(Y) \nabla_Y \tilde{\Gamma}_Y(\psi(x,h)) \cdot \nabla u(Y)\, dY \right| \le \int_{\Omega_\pm} \frac{C}{|Y - \psi(x,h)|} |\nabla u(Y)|\, dY.\end{aligned}$$

Since $|\nabla u|$ is bounded and in $L^1(\mathbf{R}^2)$, and since the other term in the integral is in $L^1(\mathbf{R}^2) + L^\infty(\mathbf{R}^2)$, this integral is also bounded, and the limit as $h \to 0^+$ converges uniformly in x.

Thus, we have shown that $\tilde{T}^t$ is continuous on $B_1\mathcal{S}$.

Furthermore, if f is a component of a normalized bump function, then $|\operatorname{supp} f| \leq CR$, $|f| \leq C/R$, $|f'| \leq C/R^2$, so $|\nabla u| \leq C/R^2$ and is supported in some ball of radius CR. So if $F = \check{F}_R$ where $\check{F}$ is a normalized bump function, then $|\tilde{T}^t(B_1 F)(x)| \leq C/R$.

So $F \mapsto B_0^t \tilde{T}^t(B_1 F)$ is weakly bounded as well. ☐

Corollary 6.18. $\|T(B_1)\|_{BMO} \leq C$, $\|\tilde{T}^t(B_1)\|_{BMO} \leq C$.

Proof. Recall that if M_0 is a smooth H^1 atom and $\eta \in C_0^\infty$ is 1 in a neighborhood of its support, then

$$\langle M_0, TB\rangle = \langle M_0, T(\eta B)\rangle + \int_{\mathbf{R}}\int_{\mathbf{R}} M_0(x)(K_0(x,y) - K_0(x_0,y))\,dx\,(1-\eta(y))B(y)\,dy.$$

By (6.12), the second term on the right-hand side is bounded in terms of λ and Λ, so we need only bound $\langle M_0, T(\eta B_1)\rangle$ and $\langle M_0, \tilde{T}^t(\eta B_1)\rangle$.

In fact, we need only bound $\|T(\eta B_1)\|_{L^\infty}$ and $\|\tilde{T}^t(\eta B_1)\|_{L^\infty}$. If $\operatorname{supp}\eta \subset B(x_0,R)$, then $\eta = RF_R$ for some normalized bump function F, and so this follows immediately from Lemma 6.16 and Lemma 6.17. ☐

6.6 The transpose inequalities: controlling $\|T^t_\pm(B_2)\|_{BMO}$

In the proof of Theorem 6.3 in Section 6.3, we promised to produce functions B_3, B_4, B_5, B_6 such that B_3, B_6 satisfy (6.11), B_4 is bounded, and B_5 is small provided ϵ_0, δ_0 are, and such that

$$\|\tilde{T}B_3\|_{BMO} \leq C\|T^t B_4\|_{BMO}, \quad \|T^t B_6\|_{BMO} \leq C\|\tilde{T}B_5\|_{BMO}.$$

Recall that

$$T_\pm F(x) = \lim_{h\to 0^\pm} \int \begin{pmatrix} \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t \\ \nabla\Gamma^T_{\psi(x,h)}(\psi(y))^t \end{pmatrix} B_6(\psi(y))^t F(y)\,dy,$$

$$T'_\pm F(x) = \lim_{h\to 0^\pm} \int \begin{pmatrix} \nabla\Gamma^T_{\psi(x)}(\psi(y,h))^t \\ \nabla\Gamma^T_{\psi(x)}(\psi(y,h))^t \end{pmatrix} B_6(\psi(y,h))^t F(y)\,dy,$$

By (4.14), $T_\pm = T'_\mp$ and $\tilde{T}_\pm = \tilde{T}'_\mp$; it will be easier to work with $(T')^t$ and $\tilde{T}'$, not T^t.

Suppose that $B_2(x) = \beta(x)I$ for some scalar-valued function β with $1/C < |\beta| < C$. Then to show that $\|(T')^t(B_2)\|_{BMO} \leq C$, we need only show that for each $R > 0$, there is some $\eta \in C_0^\infty(\mathbf{R})$ with $\eta \equiv 1$ on $(-R, R)$, such that

$$\int K'_{-h}(y,x)^t \beta(y)\eta(y)\,dy$$

is bounded for arbitrary $h \neq 0$ and $|x| < R/2$.

But $K'_h(y,x)^t = B_6(\psi(x,h))\begin{pmatrix}\nabla\Gamma^T_{\psi(y)}(\psi(x,h)) & \nabla\Gamma^T_{\psi(y)}(\psi(x,h))\end{pmatrix}$ and so we need only bound

$$\int \nabla\Gamma^T_{\psi(y)}(\psi(x,-h))\beta(y)\eta(y)\,dy = \int \nabla_X\Gamma_{\psi(x,-h)}(\psi(y))\beta(y)\eta(y)\,dy.$$

Similarly, to bound $\|\tilde{T}'(\beta I)\|_{BMO}$, we need only bound

$$\int \nabla_X\tilde{\Gamma}^T_{\psi(x,h)}(\psi(y))\beta(y)\eta(y)\,dy.$$

If $X \in U$ and $f \in C_0^\infty(\mathbf{R}^2)$, then the divergence theorem tells us that for any $U \subset \mathbf{R}^2$,

$$\begin{aligned}-f(X) + \int_U \Gamma_X \operatorname{div} A^T\nabla f &= \int_U \Gamma_X \operatorname{div} A^T\nabla f + \int_U \nabla f\cdot A\nabla\Gamma_X + \int_{U^C}\nabla f\cdot A\nabla\Gamma_X\\ &= \int_U \operatorname{div}(\Gamma_X A^T\nabla f) + \int_{U^C}\nabla f\cdot A\nabla\Gamma_X\end{aligned}$$

and so since $\operatorname{div} A\nabla\Gamma_X = 0$ in U^C,

$$-f(X) + \int_U \Gamma_X \operatorname{div} A^T\nabla f = \int_{\partial U}\Gamma_X\,\nu\cdot A^T\nabla f\,d\sigma - \int_{\partial U} f\,\nu\cdot A\nabla\Gamma_X\,d\sigma. \tag{6.19}$$

Let $X = \psi(x,-h)$, with $|x| < R/2$. Then, by taking the gradient in X of (6.19), we have that

$$\begin{aligned}&\nabla_X\int_{\Omega^C}\Gamma_X(Y)\operatorname{div}(A^T(Y)\nabla f(Y))\,dY - \nabla f(X)\\ &= \nabla_X\int_{\partial\Omega}\Gamma_X(Y)\,\nu(Y)\cdot A^T(Y)\nabla f(Y)\,d\sigma(Y) - \nabla_X\int_{\partial\Omega} f(Y)\,\nu(Y)\cdot A(Y)\nabla\Gamma_X(Y)\,d\sigma(Y)\end{aligned}$$

We may simplify this:

$$\begin{aligned}\nabla_X \int_{\partial\Omega} f\,\nu \cdot A\nabla\Gamma_X\,d\sigma &= \nabla_X \int_{\mathbf{R}} f(\psi(y))\,\nu(y)\cdot(A(\psi(y))\nabla\Gamma_X(\psi(y))\sqrt{1+\varphi'(y)^2}\,dy\\ &= \nabla_X \int_{\mathbf{R}} f(\psi(y))\tau(y)\cdot\nabla\tilde{\Gamma}_X(\psi(y))\sqrt{1+\varphi'(y)^2}\,dy\\ &= -\int_{\mathbf{R}} \nabla_X\tilde{\Gamma}_X(\psi(y))\frac{d}{dy}f(\psi(y))\,dy\end{aligned}$$

and

$$\int_{\partial\Omega} \nabla_X\Gamma_X\,\nu\cdot A^T\nabla f\,d\sigma = \int_{\mathbf{R}} \nabla_X\Gamma_X(\psi(y))\,\nu(y)\cdot A^T(\psi(y))\nabla f(\psi(y))\sqrt{1+\varphi'(y)^2}\,dy.$$

So

$$\begin{aligned}&\int_{\Omega^C} \nabla_X\Gamma_X(Y)\,\mathrm{div}(A^T(Y)\nabla f(Y))\,dY - \nabla f(X)\\ &\qquad= \int_{\mathbf{R}} \nabla_X\Gamma_X(\psi(y))\,\nu(y)\cdot A^T(\psi(y))\nabla f(\psi(y))\sqrt{1+\varphi'(y)^2}\,dy\\ &\qquad\quad+ \int_{\mathbf{R}} \nabla_X\tilde{\Gamma}_X(\psi(y))\frac{d}{dy}f(\psi(y))\,dy\end{aligned}\tag{6.20}$$

Now, suppose that $f(y,s)=\rho(y,s)g(y,s)$. We require that $\mathrm{div}\,A^T(y)\nabla g(y,s)=0$ outside of some $B(0,R_0)$, $|\nabla g(y,s)|$ is bounded, and $g(0,0)=0$, so that $|g(X)|\le\|\nabla g\|_{L^\infty}|X|$. We assume that $\rho\equiv 1$ on $B(0,R)$, $\rho\in C_0^\infty(B(0,R+1))$, and $|\rho'|<C$, $|\rho''|<C$.

Then

$$\nabla f(y,s)=\nabla g(y,s)\rho(y,s)+g(y,s)\nabla\rho(y,s)$$

and so

$$A^T(y)\nabla f(y,s)=\rho(y,s)A^T(y)\nabla g(y,s)+g(y,s)A^T(y)\nabla\rho(y,s).$$

Then

$$\begin{aligned}\mathrm{div}\,A^T(y)\nabla f(y,s)&=\rho(y,s)\ \mathrm{div}\,A^T(y)\nabla g(y,s)+\nabla\rho(y,s)\cdot A^T(y)\nabla g(y,s)\\ &\quad+\nabla g(y,s)\cdot A^T(y)\nabla\rho(y,s)+g(y,s)\ \mathrm{div}\,A^T(y)\nabla\rho(y,s).\end{aligned}$$

Then

$$\begin{aligned}\int_{\Omega^C} \nabla_X \Gamma_X \,\operatorname{div} A^T \nabla f = \int_{\Omega^C} \nabla_X \Gamma_X \rho \,\operatorname{div} A^T \nabla g + \int_{\Omega^C} \nabla_X \Gamma_X \,\nabla \rho \cdot A^T \nabla g \\ + \int_{\Omega^C} \nabla_X \Gamma_X \,\nabla g \cdot A^T \nabla \rho + \int_{\Omega^C} \nabla_X \Gamma_X g \,\operatorname{div} A^T \nabla \rho.\end{aligned}$$

Assume that $|X| < R/2$. The second and third integrands are zero away from $\operatorname{supp} \nabla\rho = B(0, R+1) \setminus B(0, R)$, where they are at most $C\|\nabla g\|_{L^\infty}/R$; thus, the second and third integrals are $O(\lambda, \Lambda)\|\nabla g\|_{L^\infty}$.

Letting $L^T g = \operatorname{div} A^T \nabla g$, the first integral is $O(\lambda, \Lambda) R_0 \|L^T g\|_{L^\infty}$; note that it is zero if $\operatorname{div} A^T \nabla g \equiv 0$.

So

$$\begin{aligned}&\int_{\Omega^C} \partial_{x_i} \Gamma_X(Y) \operatorname{div} A^T(Y) \nabla f(Y)\, dY \\ &\qquad = O(\lambda, \Lambda)(\|\nabla g\|_{L^\infty} + R_0 \|L^T g\|_{L^\infty}) + \int_{\Omega^C} \partial_{x_i} \Gamma_X(Y) g(Y) \operatorname{div} A^T(Y) \nabla \rho(Y)\, dY \\ &\qquad = O(\lambda, \Lambda)(\|\nabla g\|_{L^\infty} + R_0 \|L^T g\|_{L^\infty}) + \int_{\Omega^C} \operatorname{div} \Big(\partial_{x_i} \Gamma_X(Y) g(Y) A^T(Y) \nabla \rho(Y) \Big)\, dY \\ &\qquad\qquad - \int_{\Omega^C} \nabla \left(\partial_{x_i} \Gamma_X(Y) g(Y) \right) \cdot A^T(Y) \nabla \rho(Y)\, dY.\end{aligned}$$

The second integrand zero away from $\operatorname{supp} \nabla\rho$, where it is at most $C\|\nabla g\|_{L^\infty}/R$, so as before the second integral is $O(\lambda, \Lambda)\|\nabla g\|_{L^\infty}$. By the divergence theorem the first integral is equal to

$$-\int_{\partial\Omega} \nu \cdot \Big(\partial_{x_i} \Gamma_X(Y) g(Y) A^T(Y) \nabla \rho(Y) \Big)\, dY$$

which is $O(\lambda, \Lambda)\|\nabla g\|_{L^\infty}$.

Now,

$$\begin{aligned}&\int_{\partial\Omega} \nabla_X \Gamma_X(Y)\, \nu(Y) \cdot A^T(Y) \nabla f(Y)\, d\sigma(Y) \\ &\qquad = \int_{\partial\Omega} \nabla_X \Gamma_X(Y) \Big(\rho(Y)\, \nu(Y) \cdot A^T(Y) \nabla g(Y) + g(Y)\, \nu(Y) \cdot A^T(Y) \nabla \rho(Y) \Big)\, d\sigma(Y) \\ &\qquad = \int_{\partial\Omega} \nabla_X \Gamma_X(Y) \rho(Y)\, \nu(Y) \cdot A^T(Y) \nabla g(Y)\, d\sigma(Y) + O(\lambda, \Lambda)\|\nabla g\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y)) \frac{d}{dy} f(\psi(y))\, dy \\
&\qquad = \int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y)) \left(\rho(\psi(y)) \frac{d}{dy} g(\psi(y)) + g(\psi(y)) \frac{d}{dy} \rho(\psi(y)) \right) dy \\
&\qquad = \int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y)) \rho(\psi(y)) \frac{d}{dy} g(\psi(y))\, dy + O(\lambda, \Lambda) \|\nabla g\|_{L^\infty}
\end{aligned}$$

So, letting $\eta(y) = \rho(\psi(y))$, we have that by (6.20),

$$\begin{aligned}
&O(\lambda, \Lambda)(\|\nabla g\|_{L^\infty} + R_0 \|L^T g\|_{L^\infty}) \\
&\qquad = \int_{\mathbf{R}} \nabla_X \Gamma_X(\psi(y))\, \eta(y)\, \nu(y) \cdot A^T(\psi(y)) \nabla g(\psi(y)) \sqrt{1 + \varphi'(y)^2}\, dy \\
&\qquad\quad + \int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y))\, \eta(y)\, \frac{d}{dy} g(\psi(y))\, dy.
\end{aligned} \tag{6.21}$$

Now, choose $g(y, s) = \xi s + m(y)$, where $m'(y) = \frac{\zeta - \xi a_{21}(y)}{a_{11}(y)}$, for some constants ξ, ζ. Then

$$\nabla g(y, s) = \begin{pmatrix} \frac{\zeta - \xi a_{21}(y)}{a_{11}(y)} \\ \xi \end{pmatrix}$$

and so

$$A^T(y) \nabla g(y, s) = \begin{pmatrix} \zeta \\ a_{12}(y) \frac{\zeta - \xi a_{21}(y)}{a_{11}(y)} + \xi a_{22}(y) \end{pmatrix} = \begin{pmatrix} \zeta \\ \frac{\zeta a_{12}(y) + \xi \det A(y)}{a_{11}(y)} \end{pmatrix}, \quad \operatorname{div} A^T \nabla g \equiv 0,$$

$$g(\psi(y)) = g(ye_2 + \varphi(y)e_1, -ye_1 + \varphi(y)e_2) = \xi \varphi(y) e_2 - \xi y e_1 + m(ye_2 + \varphi(y)e_1).$$

So

$$\begin{aligned}
O(\lambda, \Lambda)(|\zeta| + |\xi|) &= \int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y))\, \eta(y)\, \frac{d}{dy} g(\psi(y))\, dy \\
&\quad + \int_{\mathbf{R}} \nabla_X \Gamma_X(\psi(y))\, \eta(y)\, \nu(y) \cdot A^T(\psi(y)) \nabla g(\psi(y)) \sqrt{1 + \varphi'(y)^2}\, dy \\
&= \int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y)) \beta_1(y, \zeta, \xi) \eta(y)\, dy - \int_{\mathbf{R}} \nabla_X \Gamma_X(\psi(y)) \beta_2(y, \zeta, \xi) \eta(y)\, dy
\end{aligned}$$

where

$$\beta_1(y,\zeta,\xi) = \xi\varphi'(y)e_2 - \xi e_1 + \frac{\zeta - \xi a_{21}(\psi(y))}{a_{11}(\psi(y))}(\varphi'(y)e_1 + e_2),$$
$$\beta_2(y,\zeta,\xi) = \zeta(e_1 - \varphi'(y)e_2) + \frac{\zeta a_{12}(\psi(y)) + \xi \det A(\psi(y))}{a_{11}(\psi(y))}(\varphi'(y)e_1 + e_2)$$

Define $\beta_3, \dots, \beta_6$ as follows:

$$\begin{aligned}
\beta_3(y) &= \beta_1\left(y, \frac{\Lambda^4}{\lambda^3}e_2, -e_1\right) = (e_1)^2 - \varphi'(y)e_1e_2 + \frac{\frac{\Lambda^4}{\lambda^3}e_2 + e_1a_{21}(\check{y})}{a_{11}(\check{y})}(\varphi'(y)e_1 + e_2)\\
\beta_4(y) &= \beta_2\left(y, \frac{\Lambda^4}{\lambda^3}e_2, -e_1\right)\\
&= -\frac{\Lambda^4}{\lambda^3}\varphi'(y)(e_2)^2 + \frac{\Lambda^4}{\lambda^3}e_1e_2 + \frac{\frac{\Lambda^4}{\lambda^3}e_2a_{12}(\check{y}) - e_1 \det A(\check{y})}{a_{11}(\check{y})}(\varphi'(y)e_1 + e_2)\\
\beta_5(y) &= \beta_1(y, e_1, e_2) = \varphi'(y)e_2^2 - e_2e_1 + \frac{e_1 - e_2a_{21}(\check{y})}{a_{11}(\check{y})}(\varphi'(y)e_1 + e_2)\\
\beta_6(y) &= \beta_2(y, e_1, e_2) = e_1^2 - \varphi'(y)e_1e_2 + \frac{e_1a_{12}(\check{y}) + e_2 \det A(\check{y})}{a_{11}(\check{y})}(\varphi'(y)e_1 + e_2)
\end{aligned}$$

where $\check{y} = \psi_1(y)$.

Then if $B_i = I\beta_i$,

$$\|\tilde{T}'B_3\|_{BMO} \leq C + C\|(T^t)'B_4\|_{BMO}, \quad \|(T^t)'B_6\|_{BMO} \leq C + C\|\tilde{T}'B_5\|_{BMO}$$

as desired; we need only show that B_3, B_6 satisfy (6.11), B_5 is small and all the B_i are bounded.

Note the following:

- $\mathbf{Re}\,\frac{1}{a_{11}} = \mathbf{Re}\,\frac{\overline{a_{11}}}{|a_{11}|^2} \geq \frac{\lambda}{\Lambda^2}$.
- $\beta_3, \dots, \beta_6$ are all bounded by a constant depending only on λ, Λ.
- If $\|\varphi'\|_{L^\infty}$ is small enough, then

$$\beta_3(y) \approx (e_1)^2 + \frac{\frac{\Lambda^4}{\lambda^3}(e_2)^2 + e_1e_2a_{21}(\check{y})}{a_{11}(\check{y})}$$

and so

$$\mathbf{Re}\,\beta_3(y) \gtrsim (e_1)^2 + \frac{\Lambda^2}{\lambda^2}(e_2)^2 - |e_1 e_2|\frac{\Lambda}{\lambda} \geq \frac{1}{2}$$

since $e_1^2 + e_2^2 = 1$.

- Similarly, if $\|\varphi'\|_{L^\infty}$ is small enough, then

$$\beta_6(y) \approx \frac{1}{a_{11}(\check{y})}\left(a_{11}(\check{y})(e_1)^2 + a_{12}(\check{y})(e_1 e_2) + (e_2)^2 \det A(\check{y})\right)$$

and if $\|A - A_0\|$ is small enough, so that $a_{11} \approx 1$, $a_{21} \approx 0$, then

$$\beta_6(y) \approx \frac{1}{a_{11}(\check{y})}\begin{pmatrix} e_1 & e_2 \end{pmatrix} A(\check{y}) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

so $\mathbf{Re}\,\beta_6(y) \geq \frac{\lambda^2}{\Lambda^2}$.

- Finally, if $\|\varphi'\|_{L^\infty}$ is small, and if $\|A - A_0\|$ is small, so that $a_{11} \approx 1$, $a_{21} \approx 0$, then

$$\beta_5(y) = \frac{1}{a_{11}(\check{y})}\left(e_1 e_2\,(1 - a_{11}(\check{y})) + \left(e_2^2 a_{11}(\check{y}) + e_1^2 - e_1 e_2 a_{21}(\check{y})\right)\varphi'(y) - e_2^2 a_{21}(\check{y})\right)$$

is also small.

6.7 $\|T\|_{L^2 \mapsto L^2}$ is finite

In this section, we establish that $\|T\|_{L^2 \mapsto L^2}$ is finite; this allows us to use results of the form $\|T\|_{L^2 \mapsto L^2} \leq C(\lambda, \Lambda) + \epsilon \|T\|_{L^2 \mapsto L^2}$ to show that $\|T\|_{L^2 \mapsto L^2} \leq C(\lambda, \Lambda)$.

We can only prove this under our a priori assumptions that A and φ are smooth, and that for some (large) R_0, $A(y) = I$ and $\varphi(y) = 0$ for $|y| > R_0$. While the following analysis will yield a bound on $\|T\|_{L^2 \mapsto L^2}$, the bound will depend on A', φ'', R_0, and $1/|e_2|$ (if $e_2 \neq 0$); thus, the previous analysis was necessary (it yields a bound independent of these ugly quantities).

We assume $\delta_0 < 1/2$, so that $\|\varphi\|_{L^\infty} \leq R_0/2$.

Again, we need only find some bounded function $\beta : \mathbf{R} \mapsto \mathbf{C}$ with $\mathbf{Re}\,\beta(x) > \mu$ such that, for every $R > 0$, there is some $\eta \in C_0^\infty(\mathbf{R})$ with $\eta \equiv 1$ on $(-R, R)$ such that

$$\int \nabla_X \Gamma_X(\psi(y))\beta(y)\eta(y)\,dy$$

is bounded uniformly in X.

First, we consider the case where $e_2 = 0$ (that is, we are working in a domain that looks like the left or right half-planes). Then either $e_1 = 1$ or $e_1 = -1$; for simplicity we consider only the case where $e_1 = 1$. Let

$$g(x,t) = \int_{\rho(x)\varphi(-t)}^{x} \frac{1}{a_{11}(w)}dw$$

where $\rho \equiv 1$ on $(-R_0, R_0)$, $\rho \equiv 0$ outside of $(-2R_0, 2R_0)$, and $|\rho'| < C/R_0$, $|\rho''| \le C/R_0^2$.

Then $\partial\Omega = \{(\varphi(-t), t) : t \in \mathbf{R}\}$, so $g \equiv 0$ on $\partial\Omega$. Furthermore,

$$\nabla g(x,t) = \begin{pmatrix} \frac{1}{a_{11}(x)} - \frac{\rho'(x)\varphi(-t)}{a_{11}(\rho(x)\varphi(-t))} \\ \frac{\rho(x)\varphi'(-t)}{a_{11}(\rho(x)\varphi(-t))} \end{pmatrix}$$

and so outside of $(-2R_0, R_0) \times (-2R_0, R_0)$, we have that $\nabla g(x,t) = \begin{pmatrix} \frac{1}{a_{11}(x)} & 0 \end{pmatrix}$.

Therefore, $|\operatorname{div} A^T \nabla g| \le C(\|\varphi''\|_{L^\infty} + \|A'\|_{L^\infty})$, and $\operatorname{div} A^T(y)\nabla g(y,s) = 0$ outside of $B(0, \sqrt{2}R_0)$.

By (6.21), for any $R > 0$ there is some $\eta \in C_0^\infty(\mathbf{R})$ with $\eta \equiv 1$ on $(-R, R)$ such that

$$\begin{aligned} &O(\lambda,\Lambda)(1 + R_0\|\varphi''\|_{L^\infty} + R_0\|A'\|_{L^\infty}) \\ &\qquad = \int_{\mathbf{R}} \nabla_X \Gamma_X(\psi(y))\,\eta(y)\,\nu(y)\cdot A^T(\psi(y))\nabla g(\psi(y))\sqrt{1+\varphi'(y)^2}\,dy \\ &\qquad\quad + \int_{\mathbf{R}} \nabla_X \tilde{\Gamma}_X(\psi(y))\,\eta(y)\,\frac{d}{dy}g(\psi(y))\,dy. \end{aligned}$$

But $\frac{d}{dy}g(\psi(y)) = 0$, and since $\psi(y) = \begin{pmatrix} \varphi(y) & -y \end{pmatrix}$, we have that

$$\nu(y)\cdot A^T(\psi(y))\nabla g(\psi(y))\sqrt{1+\varphi'(y)^2} = -\frac{1}{a_{11}(\varphi(y))}\begin{pmatrix} 1 \\ \varphi'(y) \end{pmatrix} \cdot A^T(\varphi(y))\begin{pmatrix} 1 \\ \varphi'(y) \end{pmatrix}.$$

So

$$\|T^t(B_8)\|_{BMO} \leq C(\lambda, \Lambda, R_0, \|A'\|_{L^\infty}, \|\varphi''\|_{L^\infty})$$

where

$$B_8(y) = \frac{1}{a_{11}(\varphi(y))} \begin{pmatrix} 1 \\ \varphi'(y) \end{pmatrix} \cdot A^T(\varphi(y)) \begin{pmatrix} 1 \\ \varphi'(y) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $B_8(y) = \beta_8(y)I$ where β_8 is bounded and, if $\|\varphi'\|_{L^\infty}$ or $|\mathbf{Im}\, a_{11}|$ is small enough, then $\mathbf{Re}\, \beta_8(y)$ is positive and bounded away from zero. Thus, under our a priori smoothness assumptions, $\|T\|_{L^2 \mapsto L^2}$ is finite.

Now, consider the case where $e_2 \neq 0$. If $|x| > R_0/|e_2|$, then $\varphi(x) = 0$ and $\psi(x) = (xe_2, -xe_1)$ and $|\psi_1(x)| = |xe_2| > R_0$, so $A(\psi(x)) = I$.

Assume that $R_1 \geq R_0/|e_2|$ is large enough (to be determined later). Choose $\rho \in C^\infty$ such that $\rho \equiv 1$ on $(-R_1, R_1)$, $\rho \equiv 0$ outside of $(-2R_1, 2R_1)$, and $|\rho'| < C/R_1$, $|\rho''| < C/R_1^2$. Let

$$g(x,t) = e_2 t + (1-\rho(t)) \left(\int_{-R_0}^x \frac{e_1 - e_2 a_{21}(w)}{a_{11}(w)} dw \right) + \rho(t) \left(e_1 x - \varphi(e_2 x - e_1 t) + e_1 R_0 \right).$$

If $|x| \geq R_0$, our a priori assumption $\int_{-R_0}^{R_0} \frac{e_1 - e_2 a_{21}}{a_{11}} = 2R_0 e_1$ means that

$$g(x,t) = e_2 t + e_1 x + e_1 R_0 - \rho(t)\varphi(e_2 x - e_1 t)$$

So if $|y| \geq R_0/|e_2| \geq R_0$, then $\varphi(y) = 0$, and

$$g(\psi(y)) = g(ye_2, -ye_1) = e_1 R_0 - \rho(-ye_1)\varphi(y) = e_1 R_0.$$

Conversely, we may take R_1 large enough that if $|y| < R_0/|e_2|$, then $|\psi(y)| < R_1$. So $\rho(\psi_2(y)) = 1$, and so

$$\begin{aligned} g(\psi(y)) &= g(ye_2 + \varphi(y)e_1, -ye_1 + \varphi(y)e_2) \\ &= e_2(-ye_1 + \varphi(y)e_2) + e_1(ye_2 + \varphi(y)e_1) - \varphi(y) + e_1 R_0 = e_1 R_0. \end{aligned}$$

So g is constant on $\partial\Omega$.

Now, consider

$$\nabla g(x,t) = \begin{pmatrix} (1-\rho(t))\frac{e_1 - e_2 a_{21}(x)}{a_{11}(x)} + e_1\rho(t) - e_2\rho(t)\varphi'(e_2x - e_1t) \\ e_2 + e_1\rho(t)\varphi'(e_2x - e_1t) \end{pmatrix}$$
$$+ \rho'(t)\begin{pmatrix} 0 \\ -\int_{-R_0}^{x} \frac{e_1 - e_2 a_{21}(w)}{a_{11}(w)}dw + e_1x - \varphi(e_2x - e_1t) + e_1R_0 \end{pmatrix}.$$

If $|t| > 2R_1$, or if $|x|$, $|e_1t - e_2x| > R_0$, then

$$\nabla g(x,t) = \begin{pmatrix} \frac{e_1 - e_2 a_{21}(x)}{a_{11}(x)} \\ e_2 \end{pmatrix}$$

and so $\operatorname{div} A^T(x)\nabla g(x,t) = 0$. So $\operatorname{div} A^T\nabla g$ is zero outside a bounded set. Clearly, it is bounded in this set by a constant which depends on A', φ'', R_0, λ, Λ.

But $\frac{d}{dy}g(\psi(y)) = 0$. So as before,

$$\|T^t(B_8)\|_{BMO} \le C(\lambda, \Lambda, R_0, 1/|e_2|, \|A'\|_{L^\infty}, \|\varphi''\|_{L^\infty})$$

where

$$B_8(y) = \nu(y)\cdot A^T(\psi(y))\nabla g(\psi(y))\sqrt{1+\varphi'(y)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But on $\partial\Omega$,

$$\nabla g(x,t) = \begin{pmatrix} e_1 - e_2\varphi'(e_2x - e_1t) \\ e_2 + e_1\varphi'(e_2x - e_1t) \end{pmatrix}.$$

So

$$-\nu(y)\cdot A^T(\psi(y))\nabla g(\psi(y))\sqrt{1+\varphi'(y)^2} = \begin{pmatrix} e_1 - e_2\varphi'(y) \\ e_2 + e_1\varphi'(y) \end{pmatrix}\cdot A^T(\varphi(y))\begin{pmatrix} e_1 - e_2\varphi'(y) \\ e_2 + e_1\varphi'(y) \end{pmatrix}$$

which is bounded and whose real part is positive and bounded away from zero. Thus, $\|T\|_{L^2\mapsto L^2}$ is finite.

CHAPTER 7

INVERTIBILITY OF LAYER POTENTIALS

Recall that in the proof of Theorem 1.5, we used boundedness and invertibility of layer potentials. In Chapter 6 we proved boundedness; in this chapter we will prove invertibility.

Theorem 7.1. *Let $1 < p < \infty$, and let $\epsilon_0 > 0$. Let V be a Lipschitz domain with Lipschitz constants k_i. Let A_0 be real and satisfy (1.2), and let $A_s = sI + (1-s)A_0$. Assume that*

(7.2) There exists an ϵ_0 such that, if A satisfies (1.2) and $\|A - A_0\|_{L^\infty} \leq \epsilon_0$, or if $A = A_s$ for some $0 \leq s \leq 1$, then $(\mathcal{K}_\pm^A)^t$ and $(\mathcal{L}^A)^t$ are bounded linear operators on $H^1(\partial V)$ and $L^p(\partial V)$.

(7.3) The layer potentials $(\mathcal{K}_\pm^I)^t : L^p(\partial V) \cap H^1(\partial V) \mapsto L^p(\partial V) \cap H^1(\partial V)$ are onto.

(7.4) $(N)_p^{A_s}$, $(R)_p^{A_s}$ hold in $V = V_+$ and $\bar{V}^C = V_-$ with constants at most C_p, for all $0 \leq s \leq 1$.

Then there is some $\epsilon > 0$ such that, if A satisfies (1.2) and $\|A - A_0\|_{L^\infty} < \epsilon$, then $(\mathcal{K}_\pm^A)^t$, $(\mathcal{L}^A)^t$ are one-to-one and onto on the space $L^p(\partial V) \cap H^1(\partial V)$. Furthermore, $\epsilon \geq 1/C(\lambda, \Lambda, k_i, C_p, p, \epsilon_0)$, and

$$\|f\|_{L^p} \leq C(\lambda, \Lambda, k_i, C_p, p, \epsilon_0) \|(\mathcal{K}_\pm^A)^t f\|_{L^p},$$
$$\|f\|_{L^p} \leq C(\lambda, \Lambda, k_i, C_p, p, \epsilon_0) \|(\mathcal{L}^A)^t f\|_{L^p}$$

for all $f \in L^p(\partial V) \cap H^1(\partial V)$.

If in addition

(7.5) There exists a number $C_1 > 0$ such that, if a is a $H^1(\partial V)$ atom with support in some $B(X_0, R) \cap \partial V$ and H^1 norm 1, and if $\operatorname{div} A_0 \nabla u = 0$ in V, $\nu \cdot A_0 \nabla u = a$ or $\partial_\tau u = a$ on ∂V, then

$$\int_{\partial V} N(\nabla u)(X)(1 + |X - X_0|/R)^\alpha \, d\sigma(X) \leq C_1$$

then $\mathcal{K}_{\pm}^t$, $\mathcal{L}^t$ are invertible with bounded inverse on $H^1(\partial V)$ as well.

The ϵ produced by this theorem depends on p. Therefore, we intend to use this theorem only for some fixed $p_0 > 1$ and H^1, and interpolate to get an ϵ that will work for all p with $1 < p \leq p_0$.

We use the notational shorthand that $\mathcal{K} = \mathcal{K}^A$, $\mathcal{L} = \mathcal{L}^A$, $\mathcal{K}^s = \mathcal{K}^{A_s}$, $\mathcal{L}^s = \mathcal{L}^{A_s}$, for any $0 \leq s \leq 1$.

By (5.5), if $V_{\pm}$ is bounded then $\mathcal{K}_{\pm}1 = \pm 1$, so $(\mathcal{K}_{\pm})^t$ is invertible on all of $L^p(\partial V)$. Since $L^p(\partial V) \cap H^1(\partial V)$ is dense in $L^p(\partial V)$ if ∂V is unbounded, we have that $(\mathcal{K}_{\pm})^t$, $(\mathcal{L}_{\pm})^t$ are invertible on all of $L^p(\partial V)$.

Proof. In Lemma 7.10, we will show that $\mathcal{K}_+^A - \mathcal{K}_-^A = I$ on $L^p(\partial V)$ and on $H^1(\partial V)$ for any elliptic matrix A; so, taking the norm in either space,

$$\|f\| = \|(\mathcal{K}_+^0)^t f - (\mathcal{K}_-^0)^t f\| \leq \|(\mathcal{K}_+^0)^t f\| + \|(\mathcal{K}_-^0)^t f\|.$$

The idea of using this jump relation to prove invertibility comes from [30], where it was used in the case where $A \equiv I$, that is, for harmonic functions. We generalize to other A.

In Lemma 7.13 we will show that (7.4) implies $\|(\mathcal{K}_V^0)^t f\|_{L^p(\partial V)} \approx \|(\mathcal{L}_V^0)^t f\|_{L^p(\partial V)}$. In Lemma 7.14, we will show that (7.5) implies $\|(\mathcal{K}_V^0)^t f\|_{H^1(\partial V)} \approx \|(\mathcal{L}_V^0)^t f\|_{H^1(\partial V)}$. In either case, the comparability constants that may depend on p. In Lemma 7.11, we will show that $\mathcal{L}_{V_+}^t = \mathcal{L}_{V_-}^t$ on $L^p(\partial V)$ and $H^1(\partial V)$.

In either case,

$$\|f\| \leq \|(\mathcal{K}_+^0)^t f\| + \|(\mathcal{K}_-^0)^t f\| \leq C_p \|\mathcal{L}_0^t f\| \leq C_p \|(\mathcal{K}_{\pm}^0)^t f\|. \tag{7.6}$$

By Theorem 5.17 and Theorem 6.1, we have that

$$\begin{aligned} \|\mathcal{K}_{\pm}^A - \mathcal{K}_{\pm}^{A_0}\|_{L^p \mapsto L^p} &\leq C_p \|A - A_0\|_{L^\infty}, \quad \|(\mathcal{K}_{\pm}^A)^t - (\mathcal{K}_{\pm}^{A_0})^t\|_{H^1 \mapsto H^1} \leq C \|A - A_0\|_{L^\infty} \\ \|\mathcal{L}^A - \mathcal{L}^{A_0}\|_{L^p \mapsto L^p} &\leq C_p \|A - A_0\|_{L^\infty}, \quad \|(\mathcal{L}^A)^t - (\mathcal{L}^{A_0})^t\|_{H^1 \mapsto H^1} \leq C \|A - A_0\|_{L^\infty} \end{aligned}$$

for all A sufficiently near A_0.

So

$$\|f\| \leq C_p \|\mathcal{L}^t f\| + C_p \|\mathcal{L}_0^t f - \mathcal{L}^t f\| \leq C_p \|\mathcal{L}^t f\| + C_p \|A - A_0\|_{L^\infty} \|f\|$$

and similarly for $\mathcal{K}^t_\pm$; so if $\|A - A_0\|_{L^\infty}$ is small enough, then

$$\|f\| \leq C_p \|\mathcal{L}^t f\|, \qquad \|f\| \leq C_p \|\mathcal{K}^t_\pm f\|.$$

This implies the following:

- $\mathcal{L}^t$ and $\mathcal{K}^t_\pm$ are one-to-one on $H^1(\partial V) \cap L^p(\partial V)$.
- If $\mathcal{L}^t$ and $\mathcal{K}^t_\pm$ are onto, then their inverses have norms at most $C(p)$.

So we need only show that $\mathcal{K}^t_\pm$, $\mathcal{L}^t$ are onto $L^p(\partial V) \cap H^1(\partial V) \mapsto L^p(\partial V) \cap H^1(\partial V)$. In Theorem 7.15, we will show that if G is onto and satisfies $\|f\| \leq C\|Gf\|$ and $\|G - G'\|$ is small, then G' is also onto and satisfies $\|f\| \leq C\|G'f\|$. So we need only show that $(\mathcal{K}^0_\pm)^t$ and $(\mathcal{L}^0)^t$ are onto.

Consider $\mathcal{K}^0_\pm$ first. Let $A_s = (1-s)I + sA_0$. Then $\|f\| \leq C\|(\mathcal{K}^s_\pm)^t f\|$ uniformly in s; applying Theorem 7.15 to A_s and $A_{s-\eta}$ for sufficiently small η and repeating, we see that since $(\mathcal{K}^1_\pm)^t = (\mathcal{K}^I_\pm)^t$ is onto, $(\mathcal{K}^0_\pm)^t$ is as well.

Now we wish to show that $\mathcal{L}^0$ is onto. We may assume that V is either bounded or special. Let $f \in L^p \cap H^1$. Since $(R)^{A_0}_p$ holds, there is some u with $\operatorname{div} A_0^T \nabla u = 0$ in V, $\partial_\tau u = f$ on ∂V and $\|N(\nabla u)\|_{L^p(\partial V)} \leq C\|a\|_{L^p(\partial V)}$.

Then there is some $g \in L^p(\partial V)$ with $g = \nu \cdot A_0^T \nabla u$. If V is bounded, then $\fint_{\partial V} g \, d\sigma = \fint_V \nabla 1 \cdot A_0^T \nabla u = 0$, so $g \in H^1(\partial V)$. If V is special, then $H^1(\partial V)$ is dense in $L^p(\partial V)$ for $p > 1$ and so $(\mathcal{K}^0_\pm)^t$ is invertible on all of $L^p(\partial V)$.

Let $h = ((\mathcal{K}^0_+)^t)^{-1} g \in L^p(\partial V)$. Then by uniqueness $\mathcal{S}_0^T h = u$, and so $(\mathcal{L}^0)^t h = f$, as desired. Thus $(\mathcal{L}^0)^t$ as well as $(\mathcal{K}^0)^t$ is invertible on L^p and on H^1. □

7.1 Domains to which Theorem 7.1 applies

Theorem 7.1 has four conditions.

By Theorem 6.1, if V is a good Lipschitz domain, then (7.2) holds.

In Theorem 8.1, we will show that if V is a good Lipschitz domain and there exists a $p > 1$ such that (7.4) holds in all domains with Lipschitz constants at most $C = C(V)$, then (7.5) holds.

I claim that, for any k, we can find a (possibly small) $p = p(k) > 1$ such that, if V is a good Lipschitz domain with constants at most k, then (7.4) holds in V.

By [25] and [26], if V is a special or bounded Lipschitz domain, then there exists some $p = p_0 > 1$, depending only on λ, Λ and the Lipschitz constants of V, such that $(N)_p^{A_0}$, $(R)_p^{A_0}$ hold in V with constants depending on the same quantities. Since the complement of a special Lipschitz domain is also a special Lipschitz domain, we need only show that $(N)_p^{A_0}$ or $(R)_p^{A_0}$ holds in the complement of a bounded Lipschitz domain to have (7.4) hold for all good Lipschitz domains. The proof of this result is complicated; we will first discuss Condition (7.3).

From [30, Theorem 4.2 and Corollary 4.4], Condition (7.3) holds if V is a bounded, simply connected Lipschitz domain. It also holds if $V = \Omega$ is a special Lipschitz domain:

Lemma 7.7. *If Ω is a special Lipschitz domain, then $(\mathcal{K}^I)^t$ and $(\mathcal{L}^I)^t$ are surjective on $H^1(\partial\Omega) \cap L^p(\partial\Omega)$ for p small enough.*

Proof. By symmetry of the Laplacian it suffices to prove this for domains of the form $\{(x,t) : t > \varphi(x)\}$ for some Lipschitz function φ.

In fact, since (7.2) and (7.4) hold for special Lipschitz domains, we may proceed as in the proof of Theorem 7.1 to see that if $(\mathcal{K}^B)^t$ and $(\mathcal{L}^B)^t$ are invertible on $L^p(\partial\Omega) \cap H^1(\partial\Omega)$, for any real matrix B satisfying (1.2), then $(\mathcal{K}^I)^t$ and $(\mathcal{L}^I)^t$ are as well.

Let $\mathbf{H} = \{(x,t) : t > 0\} = \mathbf{R}^2_+$ be the upper half-plane. Then $\mathcal{D}^I_{\mathbf{H}}$ is half of the Poisson integral for the upper half-plane, so $\mathcal{K}^I_{\mathbf{H}} f = \frac{1}{2} f$. Thus $\mathcal{K}^I_{\mathbf{H}}$, $(\mathcal{K}^I_{\mathbf{H}})^t$ are clearly onto.

It is easy to check that $\Gamma^B_{(x,t)}(y,s) = \Gamma^I_{(x,t-\varphi(x))}(y, s - \varphi(y))$, where

$$B = \begin{pmatrix} 1 & -\varphi'(x) \\ -\varphi'(x) & 1 + \varphi'(x)^2 \end{pmatrix}.$$

But for any good function f,

$$\begin{aligned}
\mathcal{K}^B_\Omega f(x,\varphi(x)) &= \lim_{t\to 0^+}\int_{\partial\Omega} \nu(Y)\cdot B\nabla\Gamma^B_{(x,t+\varphi(x))}(Y)\,f(Y)\,d\sigma(Y)\\
&= \lim_{t\to 0^+}\int_{\mathbf{R}} \begin{pmatrix}\varphi'\\ -1\end{pmatrix}\cdot\begin{pmatrix}1 & \varphi'(y)\\ \varphi'(y) & 1+\varphi'(y)^2\end{pmatrix}\nabla\Gamma^B_{(x,t+\varphi(x))}(y,\varphi(y))\,f(y)\,dy\\
&= \lim_{t\to 0^+}\int_{\mathbf{R}} \begin{pmatrix}\varphi'\\ -1\end{pmatrix}\cdot\begin{pmatrix}1 & \varphi'(y)\\ \varphi'(y) & 1+\varphi'(y)^2\end{pmatrix}\begin{pmatrix}1 & -\varphi'(y)\\ 0 & 1\end{pmatrix}\nabla\Gamma^I_{(x,t)}(y,0)\,f(y)\,dy\\
&= \lim_{t\to 0^+}\int_{\mathbf{R}} \begin{pmatrix}0\\ -1\end{pmatrix}\cdot\nabla\Gamma^I_{(x,t)}(y,0)\,f(y)\,dy = \mathcal{K}^I_{\mathbf{H}} f(x) = \frac{1}{2}f(x).
\end{aligned}$$

Thus, $\mathcal{K}^B_\Omega = \frac{1}{2}$, and so $(\mathcal{K}^B_\Omega)^t = \frac{1}{2}$ is also invertible. □

We now complete Condition (7.4) for bounded simply connected domains.

Theorem 7.8. *Let V be a domain with compact boundary, which satisfies all the conditions of a good Lipschitz domain, except that ∂V need not be connected. There is some k depending only on λ, Λ and the Lipschitz constants of V such that, if $(N)^A_p$ holds in all bounded simply connected Lipschitz domains U with Lipschitz constants at most k, then $(N)^A_p$, $(R)^{\tilde{A}}_p$ hold in V with constants depending on the same quantities.*

If ∂V is not connected, then when we say u solves $(R)^A_p$ with boundary data f, then we mean that $\tau\cdot\nabla u = \partial_\tau f$ on ∂V, not that $u = f$ on ∂V. This weaker formulation of $(R)^A_p$ is necessary; consider boundary data f which is constant on each connected component of ∂V. This data satisfies $\|\partial_\tau f\|_{L^p} = 0$, but if f is not constant on all of ∂V then the solution u cannot satisfy $\|N(\nabla u)\|_{L^p} = 0$.

Similarly, when we say that $(N)^A_p$ holds in V, we mean only that we can solve the Neumann problem for boundary data which integrates to zero on each connected component of ∂V. This requirement may be dispensed with (by adding multiples of Γ^A_X with poles in various components of V^C), but this introduces unnecessary complications, and so we do not explore this topic.

Proof. Uniqueness (with conditions) will be shown in Theorem 12.1. By Lemma 3.10, we know that a function u exists which satisfies $\operatorname{div} A\nabla u = 0$ in V, $\nu\cdot A\nabla u = g$ on ∂V, such that $\|\nabla u\|_{L^2(V)} \leq C\|g\|_{H^1(\partial V)}$. We need only show that $\|N(\nabla u)\|_{L^p} \leq C_p\|g\|_{L^p}$.

By (3.8), we have that $|\nabla u(Y)| \leq \frac{C}{\operatorname{dist}(Y,\partial V)} \|\nabla u\|_{L^2(V)}$. But by (2.10),

$$\frac{C}{\operatorname{dist}(Y,\partial V)} \|\nabla u\|_{L^2(V)} \leq \frac{C}{\operatorname{dist}(Y,\partial V)} \|g\|_{H^1(\partial V)} \leq \frac{C_p \|g\|_{L^p(\partial V)} \sigma(\partial V)^{1/q}}{\operatorname{dist}(Y,\partial V)}.$$

Define $N_1 F(X) = \sup\{|F(Z)| : Z \in \gamma(X),\ \operatorname{dist}(Z,\partial V) < \sigma(\partial V)/\beta\}$, $N_2 F(X) = \sup\{|F(Z)| : Z \in \gamma(X),\ \operatorname{dist}(Z,\partial V) \geq \sigma(\partial V)/\beta\}$, for some constant β to be chosen later. Then $NF(X) = \max(N_1 F(X), N_2 F(X)) \leq N_1 F(X) + N_2 F(X)$, so to bound $\|N(\nabla u)\|_{L^p}$ we need only bound $\|N_1(\nabla u)\|_{L^p}$ and $\|N_2(\nabla u)\|_{L^p}$.

Now, let $Z \in V$, $\operatorname{dist}(Z,\partial V) \geq \sigma(\partial V)/\beta$. Then

$$|\nabla u(Z)| \leq C_p \|g\|_{L^p(\partial V)} \sigma(\partial V)^{-1/p}$$

and so $\|N_2(\nabla u)\|_{L^p} \leq C_p \|g\|_{L^p(\partial V)}$.

We now consider N_1. Define X_j, r_j as in Definition 2.4. and Let r be any number with $\frac{3}{2} r_j < r < 2r_j$, and let $Q(X_j, r)$ be as in (2.6). We have that $Q(X_j, r) \subset V$ and $B(X_j, r_j/2) \cap V \subset B(X_j, r) \cap V \subset Q(X_j, r)$. Furthermore, we have that $\sigma(\partial V) \leq C r_j$.

I claim that if $X \in B(X_j, r_j) \cap \partial V$ then $N_1(\nabla u)(X) \leq N_{Q(X_j,r)}(\nabla u)(X)$. If $Y \in \gamma_V(X)$ and $\operatorname{dist}(Y,\partial V) < \sigma(\partial V)/\beta$, then $|X - Y| \leq (1+a)\operatorname{dist}(Y,\partial V) \leq C\beta\sigma(\partial V) \leq C\beta r_j$. So taking $\beta < 1/4C$ we have that $Y \in B(X_j, \frac{5}{4} r_j)$. This implies that $\operatorname{dist}(Y, \partial Q(X_j, r) \setminus \partial V) > r_j/4 > |X - Y|$, so $Y \in \gamma_{Q(X_j,r)}(X)$, which establishes the claim.

Thus

$$\int_{B_j \cap \partial V} N_1(\nabla u)^p \, d\sigma \leq \int_{\partial Q(X_j,r)} N_{Q(X_j,r)}(\nabla u)^p \, d\sigma$$

for $\frac{3}{2} r_j < r < 2r_j$. But $Q(X_j, r)$ is a simply connected bounded Lipschitz domain; let p be small enough that $(N)^A_p$ holds in all the $Q(X_j, r)$s, and assume $p \leq 2$. Then for each r,

$$\begin{aligned}\int_{B_j \cap \partial V} N_1(\nabla u)^p \, d\sigma &\leq \int_{\partial Q(X_j,r)} N_{Q(X_j,r)}(\nabla u)^p \, d\sigma \leq C \int_{\partial Q(X_j,r)} |\nu \cdot A\nabla u|^p \, d\sigma \\ &\leq C \int_{\Delta(X_j,r)} |g|^p \, d\sigma \, dr + C \int_{\partial Q(X_j,r)\setminus \partial V} |\nabla u|^p \, d\sigma\end{aligned}$$

Taking the average over r in $(\frac{3}{2}r_j, 2r_j)$, we have that

$$\begin{aligned}\int_{B_j\cap\partial V} N_1(\nabla u)^p\,d\sigma &\leq C\|g\|^p_{L^p(\Delta(X_j,2r_j))} + \frac{C}{r_j}\int_{3r_j/2}^{2r_j}\int_{\partial Q(X_j,r)\setminus\partial V}|\nabla u|^p\,d\sigma\,dr\\ &\leq C\|g\|^p_{L^p(\Delta(X_j,2r_j))} + \frac{C}{r_j}\int_{Q(X_j,2r_j)}|\nabla u|^p\\ &\leq C\|g\|^p_{L^p(\Delta(X_j,2r_j))} + \frac{Cr_j}{r_j^p}\left(\int_{Q(X_j,2r_j)}|\nabla u|^2\right)^{p/2} \qquad (7.9)\end{aligned}$$

But by our bound on $\|\nabla u\|_{L^2}$, this is at most $C\|g\|^p_{L^p(\partial V)}$. Since there are at most k_2 such balls, we have that $\int_{\partial V} N_1(\nabla u)^p\,d\sigma \leq C\|g\|^p_{L^p}$, as desired. So $(N)^A_p$ holds in V for V any good Lipschitz domain, for p small enough. (Note that p depends only on k_1, although the constants in the definition of $(N)^A_p$ may depend on k_2, k_3.)

We now pass to the regularity problem $(R)^{\tilde{A}}_p$. Pick some $g \in L^p(\partial V)$ with $\int_\omega g = 0$ for every connected component ω of ∂V; if $p > 1$ and $\sigma(\partial\Omega) < \infty$ this condition is equivalent to requiring $g \in H^1$.

If $p > 1$ is small enough, then there is some u such that $\operatorname{div} A\nabla u = 0$ in V and $\nu\cdot A\nabla u = g$ on ∂V.

I claim that $\tilde{u}$ is defined on V. (Section 4.4 will only guarantee this if V is simply connected.) We need only show that $\int_\omega \nu\cdot A\nabla u = 0$ for all Jordan curves $\omega \subset V$; we may assume $\omega = \partial U$ for some simply connected bounded domain U.

But

$$\int_{\partial U\cup(U\cap\partial V)} \nu\cdot A\nabla u = \int_{\partial(U\cap V)} \nu\cdot A\nabla u = \int_{U\cap V}\nabla 1\cdot A\nabla u = 0$$

by the weak definition of $\nu\cdot A\nabla u$. But $U\cap\partial V$ is the union of one or more entire components of ∂V; therefore,

$$\int_{\partial U}\nu\cdot A\nabla u = -\int_{U\cap\partial V}\nu\cdot A\nabla u = -\int_{U\cap\partial V} g = 0.$$

So $\tilde{u}$ is well-defined on V. By Section 4.4, $\operatorname{div}\tilde{A}\nabla\tilde{u} = 0$ in V.

By (4.5), if $\nu\cdot A\nabla u = g$ on ∂V, then $\partial_\tau\tilde{u} = g$ on ∂V. Clearly $N(\nabla u)(X) \approx N(\nabla\tilde{u})(X)$. So if $(N)^A_p$ holds in V, then there exist solutions to $(R)^{\tilde{A}}_p$.

So if $(N)^A_p$ holds in V, so does $(R)^{\tilde{A}}_p$. $\square$

7.2 Comparing layer potentials on the two sides of a boundary

We remind ourselves of a few classic results from layer potential theory, and confirm that they still hold for complex coefficients on Lipschitz domains.

Lemma 7.10. *If f is a L^p Lipschitz function, then $\mathcal{K}_+ f(X) - \mathcal{K}_- f(X) = f(X)$.*

Proof. Let $V_\pm^\rho = V_\pm \backslash B(X,\rho)$, $\Psi_\pm^\rho = V_\pm \cap B(X,\rho)$. Recall (5.4): if U is a bounded domain, and $X \in U$, then

$$\int_{\partial U} \nu \cdot A^T \nabla \Gamma_X^T \, d\sigma = 1.$$

So, fixing some $\rho > 0$ small, letting $\mathbf{e}$ be a vector such that $X \pm t\mathbf{e}$ is in a nontangential cone in $V_\pm$ for all sufficiently small positive t, and extending f in some reasonable fashion to $\mathbf{R}^2$, we have

$$\begin{aligned}\mathcal{K}_+ f(X) - \mathcal{K}_- f(X) &= \lim_{t\to 0^+} \int_{\partial V} \nu \cdot A^T \nabla \Gamma_{X+t\mathbf{e}}^T f \, d\sigma - \lim_{t\to 0^+} \int_{\partial V} \nu \cdot A^T \nabla \Gamma_{X-t\mathbf{e}}^T f \, d\sigma \\ &= \lim_{t\to 0^+} \int_{\partial V_+^\rho} \nu \cdot A^T \nabla \Gamma_{X+t\mathbf{e}}^T f \, d\sigma + \lim_{t\to 0^+} \int_{\partial V_-^\rho} \nu \cdot A^T \nabla \Gamma_{X-t\mathbf{e}}^T f \, d\sigma \\ &\quad + \lim_{t\to 0^+} \int_{\partial \Psi_+^\rho} \nu \cdot A^T \nabla \Gamma_{X+t\mathbf{e}}^T f \, d\sigma + \lim_{t\to 0^+} \int_{\partial \Psi_-^\rho} \nu \cdot A^T \nabla \Gamma_{X-t\mathbf{e}}^T f \, d\sigma\end{aligned}$$

and so

$$\begin{aligned}\mathcal{K}_+ f(X) - \mathcal{K}_- f(X) &= -\int_{\partial B(X,\rho)} \nu \cdot A^T \nabla \Gamma_X^T f \, d\sigma \\ &\quad + \lim_{t\to 0^+} \int_{\partial \Psi_+^\rho} \nu \cdot A^T \nabla \Gamma_{X+t\mathbf{e}}^T f \, d\sigma + \int_{\partial \Psi_-^\rho} \nu \cdot A^T \nabla \Gamma_{X-t\mathbf{e}}^T f \, d\sigma \\ &= f(X) - \int_{\partial B(X,\rho)} \nu(Y) \cdot A^T(Y) \nabla \Gamma_X^T(Y)(f(Y) - f(X)) \, d\sigma \\ &\quad + \lim_{t\to 0^+} \int_{\partial \Psi_+^\rho} \nu(Y) \cdot A^T(Y) \nabla \Gamma_{X+t\mathbf{e}}^T(Y)(f(Y) - f(X)) \, d\sigma \\ &\quad + \lim_{t\to 0^+} \int_{\partial \Psi_-^\rho} \nu(Y) \cdot A^T(Y) \nabla \Gamma_{X-t\mathbf{e}}^T(Y)(f(Y) - f(X)) \, d\sigma.\end{aligned}$$

The integrands are at most $C\|f'\|_{L^\infty}$, and so the integrals are each at most $C\rho\|f'\|_{L^\infty}$. Taking the limit as $\rho \to 0$ yields the desired result. ☐

Since $\mathcal{K}_\pm$ are bounded as operators on L^p and H^1, and Lipschitz functions are dense in those spaces, we have that $\mathcal{K}_+ - \mathcal{K}_-$ is the identity on those spaces as well.

Lemma 7.11. *Suppose that u is defined on V, $N(\nabla u) \in L^p(\partial V)$ for $1 < p \leq \infty$. Then there exists an extension of u to $\bar{V}$ which is Hölder continuous on compact subsets of $\bar{V}$. If in particular $u = \mathcal{S}f$ for some $f \in L^p$, then u is Hölder continuous on all compact subsets of $\mathbf{R}^2$.*

This implies that $\mathcal{L}_+^t f = \tau \cdot \nabla \mathcal{S}^T f = \mathcal{L}_-^t f$ for all $f \in L^p$; since L^p functions are dense in H^1, this must hold in H^1 as well.

Proof. If Ω is a special Lipschitz domain, $N(\nabla u) \in L^p(\partial\Omega)$, and $0 \leq \tau < t$ or $0 \geq \tau > t$, then by (3.1)

$$|u(\psi(x,t)) - u(\psi(x,\tau))| \leq \int_{|\tau|}^{|t|} |\nabla u(\psi(x, \pm s))|\, ds \leq \int_{|\tau|}^{|t|} C s^{-1/p} \|N(\nabla u)\|_{L^p}\, ds$$

But since t, τ have the same sign,

$$\begin{aligned}\int_{|\tau|}^{|t|} s^{-1/p}\, ds &\leq \min\left(\int_0^{|t|} s^{-1/p}\, ds, |t-\tau|\, |\tau|^{-1/p}\right)\\ &= \min\left(q|t|^{1/q}, |t-\tau|^{1/q}\left(\frac{|t-\tau|}{|\tau|}\right)^{1/p}\right) \leq C|t-\tau|^{1/q}\end{aligned}$$

by considering the cases $|t-\tau| < |t|/2$, $|t-\tau| \geq |t|/2$ separately.

If $t \neq 0$, then by Hölder's inequality

$$\begin{aligned}|u(\psi(x,t)) - u(\psi(y,t))| &\leq \left|\int_x^y N(\nabla u)(\psi(z))\sqrt{1+\varphi'(z)^2}\, dz\right|\\ &\leq C(p)|x-y|^{1/q}\|N(\nabla u)\|_{L^p}.\end{aligned}$$

So if X, $Y \in \bar{\Omega}$, then

$$|u(X) - u(Y)| \leq C(p)|X-Y|^{1/q}\|N(\nabla u)\|_{L^p}. \tag{7.12}$$

Thus, u is Hölder continuous on $\bar{\Omega}$.

If V is a good Lipschitz domain, then this result holds near its boundary; by (3.1) ∇u is bounded away from ∂V, and so u is Hölder continuous on all compact subsets of $\bar{V}$.

By Theorem 5.12, if $f \in L^p$, then $N(\nabla \mathcal{S} f) \in L^p$ and so $\mathcal{S} f$ extends continuously to each of $\overline{V_+} = \bar{V}$ and $\overline{V_-} = V^C$; we need only show that the two extensions agree.

Pick some $X \in \partial V$, $t > 0$ small, $\mathbf{e}$ as in the proof of Lemma 7.10. Then

$$|\mathcal{S} f(X + t\mathbf{e}) - \mathcal{S} f(X - t\mathbf{e})| = \left| \int_{\partial V} (\Gamma_Y(X + t\mathbf{e}) - \Gamma_Y(X - t\mathbf{e}))\, f(Y)\, d\sigma(Y) \right|$$

But

$$|\Gamma_Y(X + t\mathbf{e}) - \Gamma_Y(X - t\mathbf{e})| \leq \left| \int_{-t}^{t} \nabla \Gamma_Y(X + r\mathbf{e})\, dr \right| \leq \left| \int_0^{Ct} \frac{C}{\sqrt{|X - Y|^2 + r^2}}\, dr \right|$$

But that integral is at most $\frac{Ct}{|X-Y|}$, and if $|X - Y| < t$, then

$$\int_0^{Ct} \frac{C}{\sqrt{|X - Y|^2 + r^2}}\, dr = \int_0^{Ct/|X-Y|} \frac{C}{\sqrt{1 + r^2}}\, dr \leq C \ln \frac{Ct}{|X - Y|}.$$

So

$$\begin{aligned} &|\mathcal{S} f(X + t\mathbf{e}) - \mathcal{S} f(X - t\mathbf{e})| \\ &\qquad = \left| \int_{\partial V} (\Gamma_Y(X + t\mathbf{e}) - \Gamma_Y(X - t\mathbf{e}))\, f(Y)\, d\sigma(Y) \right| \\ &\qquad \leq \int_{|X-Y|>|t|} |f(Y)| \frac{C|t|}{|X - Y|}\, d\sigma(Y) + C \int_{|X-Y|<|t|} |f(Y)| \ln \frac{Ct}{|X - Y|} d\sigma(Y). \end{aligned}$$

Applying Hölder's inequality to each integral, we see that $|\mathcal{S} f(X + t\mathbf{e}) - \mathcal{S} f(X - t\mathbf{e})| \leq C(p) t^{1/q} \|f\|_{L^p}$ provided $1 < p < \infty$.

So u is continuous across the boundary, and so is continuous on $\mathbf{R}^2$. □

7.3 Comparing norms of layer potentials

Lemma 7.13. *If $(N)_p^{A_0}$ and $(R)_p^{A_0}$ hold in V and $\bar{V}^C$, then for all $f \in L^p(\partial V)$,*

$$\|(\mathcal{K}_+^0)^t f\|_{L^p(\partial V)} \approx \|(\mathcal{L}^0)^t f\|_{L^p(\partial V)} \approx \|(\mathcal{K}_-^0)^t f\|_{L^p(\partial V)}.$$

Proof. By definition of $(N)_p^{A_0}$, $(R)_p^{A_0}$, there is some constant C depending on k_i, λ, Λ, such that if $\operatorname{div} A_0\nabla u = 0$ in V then

$$\|N(\nabla u)\|_{L^p(\partial V)} \leq C(p)\|\nu \cdot A_0^T \nabla u\|_{L^p(\partial V)}, \text{ and}$$
$$\|N(\nabla u)\|_{L^p(\partial V)} \leq C(p)\|\tau \cdot \nabla u\|_{L^p(\partial V)}.$$

Since $|\nu \cdot A_0^T \nabla u| \leq \Lambda N(\nabla u)$ and $|\tau \cdot \nabla u| \leq N(\nabla u)$, we may reverse either inequality (up to a multiplicative constant) and so

$$\|\nu \cdot A_0^T \nabla u\|_{L^p(\partial V_\pm)} \approx \|\tau \cdot \nabla u\|_{L^p(\partial V_\pm)}.$$

But by Lemma 7.11, $\mathcal{S}_0^T f$ is continuous on $\mathbf{R}^2$, so $\tau \cdot \nabla \mathcal{S}_0^T f$ must be the same on ∂V_+ and ∂V_-, and so

$$\|(\mathcal{K}_+^0)^t f\|_{L^p(\partial V)} \approx \|(\mathcal{L}^0)^t f\|_{L^p(\partial V)} \approx \|(\mathcal{K}_-^0)^t f\|_{L^p(\partial V)}. \qquad \square$$

Lemma 7.14. *If (7.2)–(7.5) hold, then for all* $f \in H^1$,

$$\|(\mathcal{K}_+^0)^t f\|_{H^1} \approx \|(\mathcal{L}^0)^t f\|_{H^1} \approx \|(\mathcal{K}_-^0)^t f\|_{H^1}.$$

Proof. Let $f \in H^1$. By Theorem 5.16, $(\mathcal{K}_\pm^0)^t f \in H^1$, $(\mathcal{L}^0)^t f \in H^1$. We first show that $\|(\mathcal{L}^0)^t f\|_{H^1} \leq C(p)\|(\mathcal{K}_+^0)^t f\|_{H^1}$. By the atomic decomposition, $(\mathcal{K}_+^0)^t f = \sum_i \lambda_i a_i$ for some H^1 atoms a_i and constants λ_i with $\sum |\lambda_i| = \|(\mathcal{K}_+^0)^t f\|_{H^1}$.

Let $u = \mathcal{S}_0^T f$, so that $(\mathcal{K}_+^0)^t f = \nu \cdot A^T \nabla u$. Then $(\mathcal{L}^0)^t f = \tau \cdot \nabla u$. We want to use Lemma 5.15 to bound $\|\tau \cdot \nabla u\|_{H^1}$.

Since $(N)_p^{A_0}$ holds in V, we can write $u = \sum_i \lambda_i u_i$ where $\nu \cdot A_0^T \nabla u_i = a_i$, $\|N(\nabla u)\|_{L^p} \leq C\|a_i\|_{L^p}$. Then by (7.4) and (7.5),

$$\|N(\nabla u_i)\|_{L^p} \leq C r_i^{1/p-1}, \qquad \int_{\partial V} N(\nabla u_i)(X)(1 + |X - X_i|/r_i)^\alpha \, d\sigma(X) \leq C$$

where a_i is supported in $B(X_i, r_i) \cap \partial V$. Since $|\tau \cdot \nabla u_i| \leq N(\nabla u_i)$, these inequalities hold for $\tau \cdot \nabla u_i$ as well.

If ∂V is bounded, then $\int_{\partial V} \tau \cdot \nabla u_i = 0$. Otherwise, we are working in a special Lipschitz domain Ω. Recall the $\chi_\pm$ of (2.6). Since $N(\nabla u_i)(Y)(1 + |Y|/r_i)^\alpha \in L^1$, for any $\epsilon > 0$ and any $R_0 > 0$ there must be some $R > R_0$ with $N(\nabla u)(\chi_\pm(X_i, R)) < \epsilon/R^{1+\alpha}$.

So

$$\left|\int_{\partial Q(X_i,R)\cap\partial\Omega} \tau\cdot\nabla u_i\,d\sigma\right| = \left|\int_{\partial Q(X_i,R)\setminus\partial\Omega} \tau\cdot\nabla u_i\,d\sigma\right| < CR\epsilon/R^{1+\alpha}.$$

This may be made arbitrarily small by making ϵ small or R large; so $\int_{\partial\Omega}\tau\cdot\nabla u_i\,d\sigma = 0$.

So by Lemma 5.15, $\|\tau\cdot\nabla u_i\|_{H^1}\leq C$; therefore,

$$\|\mathcal{L}^t f\|_{H^1} = \left\|\tau\cdot\sum_i \lambda_i u_i\right\|_{H^1} \leq C\sum_i\lambda_i = C\|\mathcal{K}^t f\|_{H^1}.$$

Since $(N)_p^{A_0}$ holds in $\bar{V}^C$, $\|(\mathcal{L}^0)^t f\|_{H^1(\partial V)}\leq C(p)\|(\mathcal{K}^0_-)^t f\|_{H^1(\partial V)}$. Similarly, we may say that $(\mathcal{L}^0)^t f = \sum_i \lambda_i a_i$ and let $\tau\cdot\nabla u_i = a_i$. We may repeat the above argument; we need only show that $\int_{\partial V}\nu\cdot A_0^T\nabla u_i = 0$ to establish $\|\mathcal{K}^t f\|_{H^1}\leq C\|\mathcal{L}^t f\|_{H^1}$. If V or V^C is bounded then this follows as in the proof of Theorem 5.16.

If $V=\Omega$ is a special Lipschitz domain, let $\eta\in C_0^\infty(\mathbf{R}^2)$ with $\eta\equiv 1$ on $Q(X_i,R)$, $\eta\equiv 0$ on $\Omega\setminus Q(X_i,2R)$, and $|\nabla\eta|\leq C/R$. Then by the weak definition of $\nu\cdot A_0^T\nabla u_i$,

$$\begin{aligned}\left|\int_{\partial\Omega}\eta\,\nu\cdot A_0^T\nabla u_i\,d\sigma\right| &= \left|\int_\Omega \nabla\eta\cdot A_0^T\nabla u_i\,d\sigma\right| \leq \frac{C}{R}\int_{Q(X_i,2R)\setminus Q(X_i,R)}|\nabla u_i|\\ &\leq \frac{C}{R}\int_R^{2R}\int_{\partial Q(X,r)}|\nabla u_i(Y)|\,d\sigma(Y)\,dr\\ &\leq C\int_R^{2R} N(\nabla u_i)(\chi_+(X_i,r)) + N(\nabla u_i)(\chi_-(X_i,r))\,dr.\end{aligned}$$

Since $N(\nabla u_i)\in L^1(\partial\Omega)$, this integral goes to zero as $R\to\infty$. This completes the proof. □

7.4 Some elementary analysis

Theorem 7.15. *Let G, G' be two bounded linear operators from a Banach space B to itself, and suppose that $\|f\|_B\leq\alpha\|Gf\|_B$, G is a bijection, and $\|G-G'\|_{B\mapsto B}\leq\epsilon$. If $\epsilon<1/\alpha$, then G' is also a bijection, and its inverse has norm at most $\alpha/(1-\alpha\epsilon)$.*

Proof. First, if $f\neq 0$ and $G'f=0$, then

$$\|f\|\leq\alpha\|Gf\| = \alpha\|(G-G')f\|\leq\alpha\epsilon\|f\|$$

and so $\epsilon \geq 1/\alpha$; conversely, if $\epsilon < 1/\alpha$, then G' is one-to-one.

Next, if $\epsilon < 1/\alpha$, then

$$\|f\| = \frac{\alpha}{1-\alpha\epsilon}\left(\frac{1}{\alpha} - \epsilon\right)\|f\| \leq \frac{\alpha}{1-\alpha\epsilon}\left(\|Gf\| - \|(G-G')f\|\right) \leq \frac{\alpha}{1-\alpha\epsilon}\|G'f\|$$

and so G' satisfies the same sort of useful inequality as G; in particular, if $(G')^{-1}$ exists it has norm at most $\alpha/(1-\alpha\epsilon)$. Furthermore, if $G'g_n \to f$ then $\{g_n\}$ is a Cauchy sequence; since B is a Banach space $g_n \to g$ for some g, and so $G'g = f$. Thus, G' has closed range.

Suppose G' is not onto. Let $f_0 \in B \setminus G'B$, and let $\eta = \inf\left\{\|f_0 - G'b\| : b \in B\right\}$; since $G'B$ is closed, η is positive. Pick some $\rho > 1$, and let b_0 be such that $\|f_0 - G'b_0\| \leq \rho\eta$. Let $f_1 = f_0 - G'b_0$, $f_2 = f_1/\|f_1\|$. Then $\|f_1\| \leq \rho\eta$, and if $b \in B$, then

$$\|f_1 - G'b\| = \|f_0 - G'(b + b_0)\| \geq \eta \geq \frac{1}{\rho}\|f_1\|,$$

so for all $b \in B$, $\|f_2 - G'b\| \geq \frac{1}{\rho}$.

Let $Gh = f_2$. Then $\|Gh - G'h\| = \|f_2 - G'h\| \geq \frac{1}{\rho}$ and so

$$\|G - G'\| \geq \|(G-G')h\|/\|h\| \geq \frac{1}{\rho\|h\|} \geq \frac{1}{\rho\alpha\|Gh\|} = \frac{1}{\rho\alpha\|f_2\|} = \frac{1}{\alpha\rho}.$$

Thus, if $\|G' - G\| < \frac{1}{\alpha}$, then G' is also one-to-one and onto. $\square$

CHAPTER 8

BOUNDARY DATA IN H^1

We have shown that $\mathcal{L}^t$, $\mathcal{K}^t_\pm$ are invertible on $H^1(\partial V)$ provided that the following theorem holds:

Theorem 8.1. *Let V be a good Lipschitz domain. Suppose that a is an atom of $H^1(\partial\Omega)$, that is, $\|a\|_{L^\infty} \leq 1/r$, $\int a = 0$, and $\operatorname{supp} a \subset \Delta$ for some connected set $\Delta \subset \partial V$ with $\sigma(\Delta) = r$.*

Suppose A satisfies the conditions of Theorem 1.5. Assume that there is some $1 < p < \infty$ such that $(N)^A_p$, $(R)^A_p$, $(D)^A_q$, $(N)^{\tilde{A}}_p$, $(R)^{\tilde{A}}_p$ and $(D)^{\tilde{A}}_q$ hold in all good Lipschitz domains whose constants are no bigger than some $C = C(\lambda, \Lambda, V)$ (to be chosen later).

If $\operatorname{div} A\nabla u = 0$ in V, u satisfies the conditions of Theorem 12.4, and either

(8.2) $\nu \cdot A\nabla u = a$ on ∂V, or

(8.3) $\partial_\tau u = a$ on ∂V

then there exist some numbers C, $\alpha > 0$ depending only on the ellipticity constants of A such that for any $X_0 \in \operatorname{supp} a$,

$$\int_{\partial V} N(\nabla u)(X)(1 + |X - X_0|/r)^\alpha \, d\sigma(X) \leq C.$$

The method of proof is essentially that of [7]. Theorem 12.4 is a uniqueness result; without this requirement, the theorem is false, as may be seen by examining the harmonic functions $u(x,t) = x$ or $u(x,t) = t$ in the upper half-plane.

Throughout this chapter, we assume without loss of generality that $X_0 = 0$.

8.1 A priori bounds on u

Suppose that u is a regularity solution in V, $u = f$ on ∂V, with $a = \partial_\tau f$ a H^1 atom with support in $B(0, r) \cap \partial V$.

If u satisfies the conditions of Theorem 12.4, then u is also a solution to $(D)_q^A$ with boundary data f, so $\|Nu\|_{L^q} \leq C\|f\|_{L^q} \leq Cr^{1/q}$. By (3.1),

$$|u(X)| \leq Cr^{1/q} \min(\operatorname{dist}(X, \partial V), \sigma(\partial V))^{-1/q}. \tag{8.4}$$

By Lemma 3.4 and (3.8),

$$|\nabla u(X)| \leq Cr^{1/q} \operatorname{dist}(X, \partial V)^{-1} \min(\operatorname{dist}(X, \partial V), \sigma(\partial V))^{-1/q}. \tag{8.5}$$

Furthermore, suppose that $X \in \partial V$ and $R > 0$ is small enough that $Q(X, 4R)$ exists, and that $a \equiv 0$ on $\Delta(X, 4R)$. Then by Lemma 3.4, Lemma 3.3 and Hölder's inequality,

$$\begin{aligned}\left(\int_{Q(X,2R)} |\nabla u|^2\right)^{1/2} &\leq C\left(\fint_{Q(X,4R)} |u|^2\right)^{1/2} \leq C\left(\fint_{Q(X,4R)} |u|^{2q}\right)^{1/2q} \\ &\leq \frac{C}{R^{1/q}}\left(\int_{Q(X,4R)} |u|^{2q}\right)^{1/2q} \leq \frac{C}{R^{1/q}}\|Nu\|_{L^q(\partial V)} \leq \frac{Cr^{1/q}}{R^{1/q}}.\end{aligned} \tag{8.6}$$

Now, suppose that $\operatorname{div} A\nabla u = 0$, $\nu \cdot A\nabla f = a$. Then as in the proof of Theorem 7.8, $\tilde{u}$ exists on V and satisfies $\operatorname{div} \tilde{A}\nabla\tilde{u} = 0$, $\partial_\tau \tilde{u} = a$, and $|\nabla u(X)| \approx |\nabla \tilde{u}(X)|$. It may be easily checked that $\tilde{u}$ satisfies the conditions of Theorem 12.4 if and only if u does. So u satisfies (8.5) and (8.6) as well.

If $V = \Omega$ is special, we may make some additional remarks. First, by the bound on ∇u, $\lim_{t\to\infty} u(\psi(x, t))$ exists for every $x \in \mathbf{R}$; furthermore, the limit u_∞ is independent of x. Then

$$|u(\psi(x, t)) - u_\infty| \leq \int_t^\infty Cr^{1/q} t^{-1-1/q}\, dt \leq C_p r^{1/q} t^{-1/q}.$$

If $|x| > 2r$ or $t > r$, let $R = \frac{1}{2}\operatorname{dist}(\psi(x, t), \operatorname{supp} a) \approx |\psi(x, t)|$; by applying Lemma 3.6, we see that

$$\begin{aligned}|u(\psi(x,t)) - u_\infty| &\leq C \left(\fint_{B(\psi(x,t),R)\cap\Omega} |u - u_\infty|^2 \right)^{1/2} \\ &\leq C \left(\fint_{B(\psi(x,t),R)\cap\Omega} C_p r^{2/q} s^{-2/q} dy\, ds \right)^{1/2} \\ &\leq C \left(\frac{C}{R} \fint_{x-R}^{x+R} \int_0^{Cr} C_p r^{2/q} |s|^{-2/q}\, ds\, dy \right)^{1/2} \leq C_p r^{1/q} R^{-1/q}.\end{aligned}$$

For the Neumann problem, we may assume $u_\infty = 0$; for the regularity problem, we know $u = 0$ on $\partial\Omega \setminus \operatorname{supp} a$, so we must have that $u_\infty = 0$.

Summarizing, we have that in a special Lipschitz domain,

$$|u(X)| \leq C_p r^{1/q} |X|^{-1/q}. \tag{8.7}$$

8.2 A bound on our integral in terms of itself

Lemma 8.8. *Suppose that V, p, A, a, u satisfy the conditions of Theorem 8.1. Assume that $\alpha < 1/q = 1 - 1/p$.*

Then there is some constant C depending only on p, λ, Λ and the Lipschitz constants of V such that if we define

$$I(R_0) = \int_{\Delta(0,R_0)} N(\nabla u)(X)(1 + |X|/r)^\alpha \, d\sigma(X)$$

then for any $h > 0$ and any R_0 small enough that $Q(0, 4R_0)$ exists,

$$I(R_0) \leq C h^{1/p} I(4R_0) + \frac{C}{h^{2-1/p}}.$$

In Corollary 8.9, we will show that this lemma implies that Theorem 8.1 holds in bounded Lipschitz domains. If $V = \Omega$ is a special Lipschitz domain, let $I = \lim_{R_0 \to \infty} I(R_0)$. Theorem 8.1 for special Lipschitz domains will follow from the lemma once we have shown that I is finite; we will do this in Section 8.3.

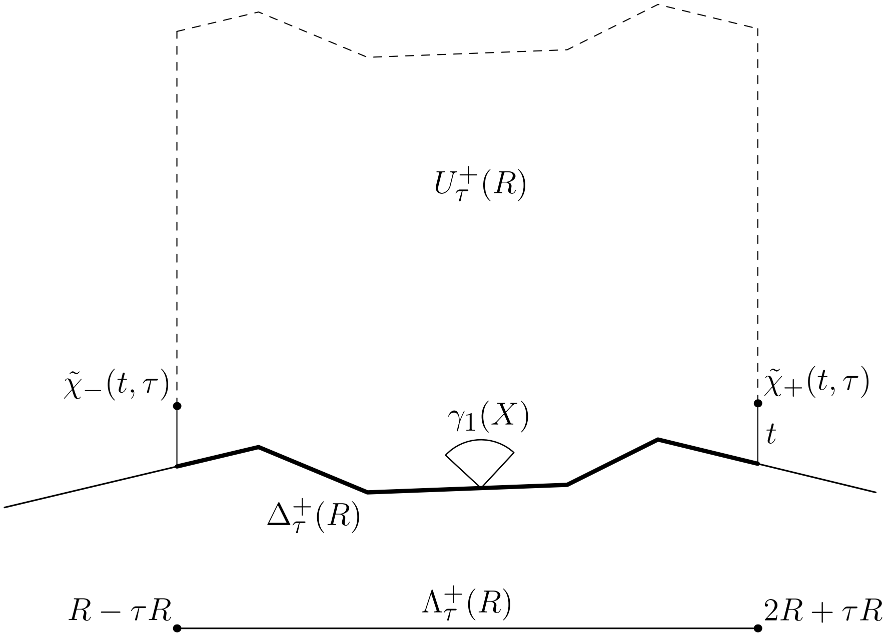

Figure 8.1: Domains, segments, and points useful in this chapter

Proof. It suffices to prove this for a supported in $\Delta(0,1)$; we may rescale to get the general result.

Suppose that $0 < R < 4R_0$. Recall that there is a $\psi : \mathbf{R}^2 \mapsto \mathbf{R}^2$ such that for all such R, $Q(0,R) = \{\psi(x,t) : -R < x < R, 0 < t < (1+k_1)R\}$. We will need the following definitions; see Figure 8.1.

$$\begin{aligned}
\gamma_1(X,R) &= \{Y \in \gamma(X) : |Y-X| < R/8\}\\
\gamma_2(X,R) &= \{Y \in \gamma(X) : |Y-X| \geq R/8\}\\
N_i(X,R) &= \sup\{|\nabla u(Y)| : Y \in \gamma_i(X)\}\\
Nf(x) &= N_\Omega f(\psi(x)) = \sup_{\gamma(\psi(x))} |f|\\
\Lambda^+(R) &= (R,2R), \quad \Lambda^-(R) = (-2R,-R)\\
\Lambda^\pm_\tau(R) &= \cup_{x\in\Lambda^\pm(R)}(x-\tau R, x+\tau R) = (-2R-\tau R, -R+\tau R) \text{ or } (R-\tau R, 2R+\tau R)\\
\Delta^\pm_\tau(R) &= \psi(\Lambda^\pm_\tau(R)) = \Delta(\psi(\pm\tfrac{3}{2}R), (\tfrac{1}{2}+\tau)R), \quad \Delta^\pm(R) = \Delta^\pm_0(R)\\
U^\pm_\tau(R) &= \left\{\psi(x,s) : x \in \Lambda^\pm_\tau(R), 0 < s < R\left(\tfrac{1}{2}+\tau\right)(1+k_1)\right\} = Q(\psi(\tfrac{3}{2}R), (\tfrac{1}{2}+\tau)R)
\end{aligned}$$

We will usually omit the R. Note the following:

- If $x \in \Lambda^\pm$ then $N_1(\psi(x)) \leq N_{U_\tau^\pm}(\nabla u)(x)$ for $\tau \geq \frac{1}{4}$.
- If $Y \in \gamma_2(X,R)$, then $\operatorname{dist}(Y,\partial\Omega) \geq \frac{1}{1+a}|Y-X| \geq \frac{R}{8+8a}$.

Let $j_0 = \lceil \log_2 R_0 \rceil - 1$, the smallest j_0 such that $\Delta(0,R_0) \subset \Delta(0,2) \cup \bigcup_{j=0}^{j_0} \Delta^\pm(2^j)$. Then

$$I(R_0) \leq C\int_{\Delta(0,2)} N(\nabla u)(X)\,d\sigma(X) + C\sum_{j=1}^{j_0} 2^{j\alpha} \int_{\Delta^+(2^j)\cup\Delta^-(2^j)} N(\nabla u)(X)\,d\sigma(X).$$

We can bound the first integral easily:

$$\int_{\Delta(0,2)} N(\nabla u)(X)\,d\sigma(X) \leq C\|N(\nabla u)\|_{L^p(\partial V)} \leq C\|a\|_{L^p(\partial V)} \leq C.$$

We just need to bound $\sum_j 2^{j\alpha}\int_{\Delta^\pm(2^j)} N(\nabla u)(X)\,d\sigma(X)$. Pick some $2^j = R$; we seek a bound on $\int_{\Delta^\pm(R)} N(\nabla u)$. We consider only $\Delta^+ = \Delta^+(R)$ and the Neumann problem; the case for $\Delta^-(R)$ and the regularity problem is similar.

By our a priori bounds on u, if $X \in \Delta^+ = \Delta^+(R)$, then $N_2(X) \leq \frac{C}{R^{1+1/q}}$. If $\tau \geq \frac{1}{4}$, then

$$\begin{aligned}\int_{\Delta^+(R)} N_1 \leq \int_{\partial U_\tau} N_{U_\tau}(\nabla u)\,d\sigma &\leq CR^{1/q}\left(\int_{\partial U_\tau} N_{U_\tau}(\nabla u)^p\,d\sigma\right)^{1/p} \\ &\leq CR^{1/q}\left(\int_{\partial U_\tau} |\nu\cdot A\nabla u|^p\,d\sigma\right)^{1/p}.\end{aligned}$$

If we let $\tilde\chi_-(t,\tau) = \psi(R-\tau R,t)$, $\tilde\chi_+(t,\tau) = \psi(2R+\tau R,t)$, and $\tilde\chi_\pm^\tau = \tilde\chi_\pm(0,\tau)$, then by (8.5)

$$\begin{aligned}\int_{\partial U_\tau} |\nu\cdot A\nabla u|^p\,d\sigma &= \int_{\{\tilde\chi_\pm(t,\tau):0<t<hR\}} |\nu\cdot A\nabla u|^p\,d\sigma + \int_{\{\tilde\chi_\pm(t,\tau):t\geq hR\}} |\nu\cdot A\nabla u|^p\,d\sigma \\ &\leq hRN(\nabla u)(\tilde\chi_-^\tau)^p + hRN(\nabla u)(\tilde\chi_+^\tau)^p + \frac{CR}{(hR)^{p+p/q}}.\end{aligned}$$

So, for any $h < 1/2$ and $1/4 \leq \tau \leq 1/2$, we have that

$$\left(\int_{\partial U_\tau} |\nu\cdot A\nabla u(y,s)|^p\right)^{1/p} \leq h^{1/p}R^{1/p}\big(N(\nabla u)(\tilde\chi_-^\tau) + N(\nabla u)(\tilde\chi_+^\tau)\big) + \frac{C}{h^{1+1/q}R^{2/q}}.$$

Taking the integral from $\tau = 1/4$ to $\tau = 1/2$, we get that

$$\begin{aligned}\int_{\Delta^+} N_1 &\le \int_{1/4}^{1/2} CR^{1/q}\left(\int_{\partial U_\tau} |\nu\cdot A\nabla u|^p d\sigma\right)^{1/p} d\tau\\ &\le \int_{1/4}^{1/2} CR\,h^{1/p}\big(N(\nabla u)(\tilde\chi_-^\tau) + N(\nabla u)(\tilde\chi_+^\tau)\big)\,d\tau + \frac{C}{h^{1+1/q}R^{2/q}}\\ &\le Ch^{1/p}\int_{1/4}^{1/2}\big(N(\nabla u)(\tilde\chi_-^\tau) + N(\nabla u)(\tilde\chi_+^\tau)\big)R\,d\tau + \frac{C}{h^{1+1/q}R^{2/q}}\\ &\le Ch^{1/p}\int_{\Delta_{1/2}^+} N(\nabla u)(\psi(x))\,dx + \frac{C}{h^{1+1/q}R^{2/q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Delta^+} N(\nabla u) &\le \int_{\Delta^+} N_1 + \int_{\Delta^+} N_2 \le \frac{C}{R^{1/q}} + Ch^{1/p}\int_{\Delta_{1/2}^+} N(\nabla u) + \frac{C}{h^{1+1/q}R^{2/q}}\\ &\le Ch^{1/p}\int_{\Delta_{1/2}^+} N(\nabla u) + \frac{C}{h^{1+1/q}R^{1/q}}.\end{aligned}$$

So

$$\begin{aligned}\int N(\nabla u)(1+|x|^\alpha) &\le C + \sum_{j=0}^{j_0} 2^{j\alpha}\int_{\Delta^+(2^j)\cup\Delta^-(2^j)} N(\nabla u)\\ &\le C + \sum_{j=0}^{j_0} Ch^{1/p}\int_{\Delta_{1/2}^+(2^j)\cup\Delta_{1/2}^-(2^j)} 2^{j\alpha}N(\nabla u) + \sum_{j=0}^{j_0} 2^{j\alpha}\frac{C}{h^{1+1/q}2^{j/q}}.\end{aligned}$$

But $\Delta_{1/2}(2^j)$ does not overlap $\Delta_{1/2}(2^{j+3})$, and so the first sum is at most

$$3Ch^{1/p}\int_{\Delta(0,2^{j_0+2})} N(\nabla u)(X)(1+|X|^\alpha)\,d\sigma(X) \le Ch^{1/p}I(4R_0).$$

If $\alpha < 1/q$, the second sum is a geometric series with ratio $2^{\alpha-1/q} < 1$, and so is at most $Ch^{-1-1/q}$. So $I(R_0) \le Ch^{1/p}I(4R_0) + Ch^{-1-1/q}$, as desired. □

Corollary 8.9. *Theorem 8.1 holds if ∂V is bounded.*

Proof. For any $\Delta \subset \partial V$, define $I(\Delta) = \int_\Delta N(\nabla u)(X)(1+|X|/r)^\alpha\,d\sigma(X)$. We wish to bound $I(\partial V)$.

First, note that

$$\begin{aligned} I(\partial V) &\leq \int_{\partial V} N(\nabla u)\left(\frac{\sigma(\partial V)}{r}\right)^{\alpha} d\sigma \leq \left(\frac{\sigma(\partial V)}{r}\right)^{\alpha} \sigma(\partial V)^{1/q}\left(\int_{\partial V} N(\nabla u)^p \, d\sigma\right)^{1/p} \\ &\leq \left(\frac{\sigma(\partial V)}{r}\right)^{\alpha} \sigma(\partial V)^{1/q} \|a\|_{L^p(\partial V)} = \left(\frac{\sigma(\partial V)}{r}\right)^{\alpha+1/q}. \end{aligned}$$

This is finite for all $r > 0$. Furthermore, we need only consider the case where $r \approx \sigma(\operatorname{supp} a)$ is small compared to $\sigma(\partial V)$.

We assume that r is small enough that $2r < R_0$ where $Q(0, 4R_0)$ exists and $R_0 \geq \sigma(\partial V)/C$. By Lemma 8.8,

$$I(\Delta(0, R_0)) \leq Ch^{1/p} I(\Delta(0, 4R_0)) + \frac{C}{h^{2-1/p}}$$

and so

$$\begin{aligned} I(\partial V) &\leq I(\partial V \setminus \Delta(0, R_0)) + \frac{C}{h^{2-1/p}} + Ch^{1/p} I(\Delta(0, 4R_0)) \\ &\leq I(\partial V \setminus \Delta(0, R_0)) + \frac{C}{h^{2-1/p}} + Ch^{1/p} I(\partial V) \end{aligned}$$

and so since $I(\partial V)$ is finite, I need only show that $I(\partial V \setminus \Delta(0, R_0)) \leq C$.

As in the proof of Theorem 7.8, let $N_1 F(X) = \sup\{|F(Z)| : Z \in \gamma(X), \operatorname{dist}(Z, \partial V) < \sigma(\partial V)/\beta\}$, $N_2 F(X) = \sup\{|F(Z)| : Z \in \gamma(X), \operatorname{dist}(Z, \partial V) \geq \sigma(\partial V)/\beta\}$, for some constant β.

If $\operatorname{dist}(X, \partial V) \geq \sigma(\partial V)/\beta$, then by (8.5) $|\nabla u(X)| \leq Cr^{1/q}\beta^{1+1/q}\sigma(\partial V)^{-1-1/q}$. So

$$\begin{aligned} \int_{\partial V} N_2(X)(1 + |X|/r)^{\alpha} \, d\sigma &\leq \int_{\partial V} Cr^{1/q}\beta^{1+1/q}\sigma(\partial V)^{-1-1/q}(1 + |X|/r)^{\alpha} \, d\sigma \\ &\leq \int_{\partial V} Cr^{1/q}\beta^{1+1/q}\sigma(\partial V)^{-1-1/q}(\sigma(\partial V)/r)^{\alpha} \, d\sigma \\ &\leq C\beta^{1+1/q}\left(\frac{r}{\sigma(\partial V)}\right)^{1/q-\alpha} \end{aligned}$$

which is bounded by a constant provided $\alpha \leq 1/q$.

Suppose that $X \in \partial V$ and R small enough that $Q(X, 4R)$ exists. Then as in (7.9),

$$\int_{B(X,R)\cap\partial V} N_1(\nabla u)^p \, d\sigma \le C\|a\|_{L^p(\Delta(X,2R))} + \frac{C}{R^{p-1}} \left(\int_{Q(X,2R)} |\nabla u|^2 \right)^{p/2}$$

for all $1 < p \le 2$ small enough that $(N)_p^A$ or $(R)_p^A$ holds in all the $Q(X, \rho)$s. If $a \equiv 0$ on $\Delta(X, 4R)$, then by (8.6),

$$\int_{B(X,R)\cap\partial V} N_1(\nabla u)^p \, d\sigma \le \frac{C}{R^{p-1}} \left(\frac{r^{2/q}}{R^{2/q}} \right)^{p/2} = C \frac{r^{p/q}}{R^{p/q+p-1}}$$

and so

$$\begin{aligned} &\int_{B(X,R)\cap\partial V} N_1(\nabla u)(X)(1 + |X|/r)^\alpha \, d\sigma(X) \\ &\qquad\qquad \le C R^{1-1/p} \frac{\sigma(\partial V)^\alpha}{r^\alpha} \left(\int_{B(X,R)\cap\partial V} N_1(\nabla u)^p \, d\sigma \right)^{1/p} \le C \frac{\sigma(\partial V)^\alpha}{r^\alpha} \frac{r^{1/q}}{R^{1/q}}. \end{aligned}$$

So if $\alpha < 1/q$ and $\sigma(\partial V)/R$ is not too large, this is bounded. We may cover $\partial V \setminus \Delta(0, R_0)$ by at most Ck_2 such balls; thus, $I(\partial V \setminus \Delta(0, R_0)) \le C$, as desired. $\square$

8.3 Finiteness

We want to show that I is finite if $V = \Omega$ is a special Lipschitz domain. Recall that $\Omega = \{X \in \mathbf{R}^2 : \varphi(X \cdot \mathbf{e}^\perp) < X \cdot \mathbf{e}\}$; we begin by assuming $A(x,t) - I$ and φ have compact support. (These are the same a priori assumptions we used in Chapter 6.) Let R be so large that $B(0, R)$ contains $\operatorname{supp} a$, $\{x\mathbf{e}^\perp + \varphi(x)\mathbf{e} : \varphi \ne 0\}$, and such that $A(x) \equiv I$ for $|x| > R$.

In $\Omega_- \setminus B(0, R)$, define

- $u(x\mathbf{e}^\perp - t\mathbf{e}) = u(x\mathbf{e}^\perp + t\mathbf{e})$, for the Neumann problem
- $u(x\mathbf{e}^\perp - t\mathbf{e}) = -u(x\mathbf{e}^\perp + t\mathbf{e})$, for the regularity problem.

Then $\nabla u(x,t) = \pm E \nabla u(x, -t)$, where E is a constant matrix that represents reflection about the line $\{x\mathbf{e}^\perp : x \in \mathbf{R}\}$. Note that $E^T = E^{-1} = E$.

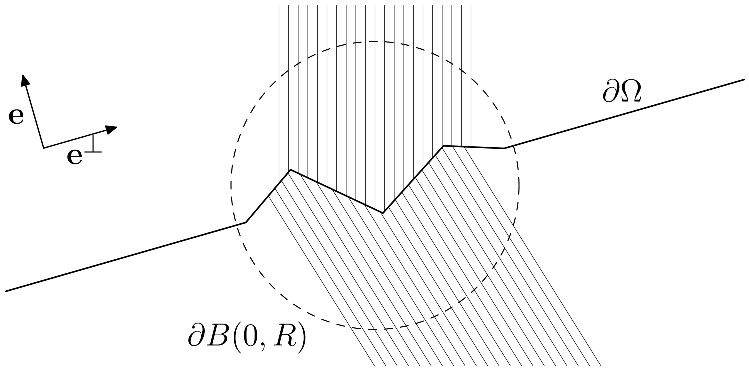

Figure 8.2: Redefining A in Ω^C

Redefine A in Ω^C such that $A(x\mathbf{e}^{\perp} - t\mathbf{e}) = EA(x\mathbf{e}^{\perp} + t\mathbf{e})E$ outside of $B(0,R)$. Figure 8.2 shows some lines on which A is constant after redefinition.

Then in $\Omega_- \setminus B(0,R)$, it is straightforward to check that $\operatorname{div} A\nabla u = 0$. But outside of $B(0,R)$, where $\partial\Omega$ coincides with $\{x\mathbf{e}^{\perp} : x \in \mathbf{R}\}$, we have that $\nu_+ \cdot A\nabla u|_{\partial\Omega_+} = -\nu_- \cdot A\nabla u|_{\partial\Omega_-}$, where $\nu_\pm = \mp\mathbf{e}$ are the outward normal vectors to $\Omega_\pm$. (In the Neumann case, this is because both conormal derivatives are zero.)

Then if $\eta \in C_0^\infty(\mathbf{R}^2 \setminus B(0,R)$, then

$$\int \nabla\eta \cdot A\nabla u = \int_{\Omega_+} \nabla\eta \cdot A\nabla u + \int_{\Omega_-} \nabla\eta \cdot A\nabla u = \int_{\partial\Omega_+} \eta\,\nu \cdot A\nabla u\,d\sigma + \int_{\partial\Omega_-} \eta\,\nu \cdot A\nabla u\,d\sigma = 0$$

and so $\operatorname{div} A\nabla u = 0$ in all of $\mathbf{R}^2 \setminus B(0,R)$.

Now, if $|X| > CR$, then A is independent of some direction in all of $B(X,|X|/4)$. So by (3.8), Lemma 3.4 and (8.7),

$$|\nabla u(X)| \leq C\left(\fint_{B(X,|X|/4)} |\nabla u|^2\right)^{1/2} \leq \frac{C}{|X|}\left(\fint_{B(X,|X|/2)} |u|^2\right)^{1/2} \leq \frac{C}{|X|^{1+1/q}}.$$

Since $N(\nabla u) \in L^p$, we have that

$$\int_{|X|<CR,X\in\partial\Omega} N(\nabla u)(X)(1+|X|/r)^\alpha\,d\sigma(X)$$

is finite. But for large $|X|$, $N(\nabla u)(X) \leq C/|X|^{1+1/q}$; so if $\alpha < 1/q$, then

$$\int_{|X|\geq CR, X\in\partial\Omega} N(\nabla u)(X)(1+|X|/r)^{\alpha}\,d\sigma(X)$$

is finite as well and we are done.

We now remove the assumption on φ and A. Assume $\varphi(0) = 0$, and let $\varphi_R = \varphi$ on $(-R, R)$ and let $\varphi_R = 0$ outside of $(-2R, 2R)$. Let $\Omega_R = \{X \in \mathbf{R}^2 : \varphi_R(X\cdot\mathbf{e}^{\perp}) < X\cdot\mathbf{e}\}$. Let $A_R(x) = A$ for $|x| < CR$ and $A_R(x) \equiv I$ for $|x| > 2CR$, C large enough. Let $\operatorname{div} A_R\nabla u_R = 0$ in Ω_R, $\nu\cdot A_R\nabla u_R = a$ or $\tau\cdot\nabla u_R = a$ on $\partial\Omega\cap\partial\Omega_R$, 0 on $\partial\Omega_R\backslash\partial\Omega$.

Suppose that $|Y|$ is small compared to R, S with $R < S$. Then $\operatorname{div} A\nabla(u_S - u_R) = 0$ in $B(0,R)\cap\Omega_R$, and $\nu\cdot A\nabla(u_S - u_R) = 0$ or $\tau\cdot\nabla(u_S - u_R) = 0$ on $\psi((-R,R))$.

By (8.4), $|u_R(X)| < Cr^{1/q}\operatorname{dist}(X,\partial V)^{-1/q}$. So by Lemma 3.6,

$$|u_S(Y) - u_R(Y)|^2 \leq \frac{C}{R^2}\int_{B(Y,R/2)} |u_S - u_R|^2 \leq \frac{C}{R}\int_0^{CR} Cr^{2/q}t^{-2/q}dt \leq C_p\frac{r^{2/q}}{R^{2/q}}.$$

Therefore, by (3.8), if $R \gg |Y|$ then

$$|\nabla u_S(Y) - \nabla u_R(Y)| \leq \frac{C_p r^{2/q}}{R^{2/q}\operatorname{dist}(Y,\partial\Omega)}.$$

Define $\breve{u} = \lim_{R\to\infty} u_R$. Then clearly $\operatorname{div} A\nabla\breve{u} = 0$ in Ω, $\nu\cdot A\nabla\breve{u} = a$ or $\tau\cdot\nabla\breve{u} = a$ on $\partial\Omega$, and $\int N(\nabla\breve{u})(1+|X|/r)^{\alpha} \leq C$. By Theorem 12.1, we have $u = \breve{u}$, and so we are done.

CHAPTER 9
INTERPOLATION

We have established that for V, A and A_0 as in Theorem 1.5, there exist some $\epsilon_0 > 0$, $p_0 > 1$ depending only on λ, Λ and k_i such that, if $\|A - A_0\|_{L^\infty} < \epsilon_0$, then

- $\mathcal{K}_\pm$, $\mathcal{L}$, $\mathcal{K}^t_\pm$, $\mathcal{L}^t$, are bounded $L^p(\partial V) \mapsto L^p(\partial V)$ for all $1 < p < \infty$.
- $\mathcal{K}^t_\pm$, $\mathcal{L}^t$ are invertible on $L^{p_0}_0(\partial V)$ with bounded inverse.
- $\mathcal{K}^t_\pm$, $\mathcal{L}^t$ are bounded and invertible on $H^1(\partial V)$ with bounded inverse.

We also know that, for any given p, if $\mathcal{K}^t_\pm$ is invertible with bounded inverse on $L^p_0(\partial V)$ then $(D)^A_q$, $(N)^A_p$ hold in V, and if $\mathcal{L}^t$ is invertible with bounded inverse on $L^p_0(\partial V)$ then $(R)^A_p$ holds in V. We now interpolate to complete the proof of Theorem 1.5.

Theorem 9.1. *Suppose that* $\|A - A_0\|_{L^\infty} < \epsilon_0$, *where* ϵ_0 *is as above. Then if* $1 < p < p_0$, *then*

$$\|f\|_{L^p} \leq C(p)\|\mathcal{K}^t_\pm f\|_{L^p}, \quad \|f\|_{L^p} \leq C(p)\|\mathcal{L}^t f\|_{L^p}$$

for all $f \in L^p_0(\partial V)$.

This ϵ_0 is independent of p. Since $L^{p_0}_0 \cap L^p_0$ is dense in L^p_0, we know that $\mathcal{K}^t$, $\mathcal{L}^t$ have dense range; thus, this suffices to establish that they are one-to-one, onto and their inverses have bounded norm.

Proof. Let $P = (\mathcal{L}^t)^{-1}$ or $(\mathcal{K}^t_\pm)^{-1}$. It suffices to prove that $\|Pg\|_{L^p} \leq C\|g\|_{L^p}$ for all $g \in \mathbf{C}^\infty_0$.

Let $E_\alpha = \{x : Mg(x) > \alpha\}$, where $Mg(x)$ is the maximal function defined in terms of connected subsets of ∂V (and not sets of the form $B(X, r) \cap \partial V$). Then E_α is an open subset of ∂V, which is one-dimensional; it is therefore a union of countably many disjoint open connected sets Δ_i.

If ∂V is bounded then assume that $\alpha > \fint_{\partial V} |g| \, d\sigma$. Then if $\fint_{\Delta_i} |g| > \alpha$, then $\sigma(\Delta_i) < \sigma(\partial V)$. But then we could enlarge Δ_i while maintaining $Mg \geq \alpha$ on Δ_i. This is a contradiction, so $\fint_{\Delta_i} |g| \leq \alpha$ for such α.

Define $a_i(x)$, $b(x)$ by

$$a_i(x) = \begin{cases} 0, & x \notin \Delta_i, \\ g(x) - \fint_{\Delta_i} g, & x \in \Delta_i, \end{cases} \qquad b(x) = \begin{cases} g(x), & x \notin E_\alpha, \\ \fint_{\Delta_i} g, & x \in \Delta_i. \end{cases}$$

Then $g = b + a$ where $a = \sum_i a_i$.

Now, $b \in L^{p_0}(\partial V)$ with $\|b\|_{L^{p_0}(\partial V)}^{p_0} \leq \|g\|_{L^p(\partial V \setminus E_\alpha)}^p \alpha^{p_0 - p} + \sigma(E_\alpha)\alpha^{p_0}$. Define $p_1 = (1+p)/2$, so $1 < p_1 < p$.

We know that $\int a_i = 0$, $\operatorname{supp} a_i \subset \Delta_i$, and

$$\int |a_i|^{p_1} = \int_{\Delta_i} |a_i|^{p_1} = \int_{\Delta_i} \left| g - \fint_{\Delta_i} g \right|^{p_1} \leq 2^{p_1 - 1} \int_{\Delta_i} |g|^{p_1} + \left| \fint_{\Delta_i} g \right|^{p_1} \leq 2^{p_1} \int_{\Delta_i} |g|^{p_1}.$$

So by [27, Section III.5.7],

$$\|a_i\|_{H^1} \leq C(p)\sigma(\Delta_i)^{1-1/p_1} \|a_i\|_{L^{p_1}} \leq C(p)\sigma(\Delta_i)^{1/q_1} \|g\|_{L^{p_1}(\Delta_i)}.$$

Therefore,

$$\begin{aligned} \|a\|_{H^1} &\leq C(p) \sum_i \sigma(\Delta_i)^{1/q_1} \|g\|_{L^{p_1}(\Delta_i)} \leq C(p) \left(\sum_i \sigma(\Delta_i) \right)^{1/q_1} \left(\sum_i \|g\|_{L^{p_1}(\Delta_i)}^{p_1} \right)^{1/p_1} \\ &\leq C(p)\sigma(E_\alpha)^{1/q_1} \|g\|_{L^{p_1}(E_\alpha)} \leq C(p) \left(\frac{\|Mg\|_{L^{p_1}(E_\alpha)}^{p_1}}{\alpha^{p_1}} \right)^{1/q_1} \|g\|_{L^{p_1}(E_\alpha)} \\ &\leq C(p) \frac{\|Mg\|_{L^{p_1}(E_\alpha)}^{p_1}}{\alpha^{p_1 - 1}}. \end{aligned}$$

Recall that

$$\int_{\partial V} |Pg|^p \, d\sigma = \int_0^\infty p\alpha^{p-1} \sigma\{X \in \partial V : |Pg(X)| > \alpha\} \, d\alpha.$$

But if $|Pg| > \alpha$, then either $\alpha < \alpha_0 = \fint_{\partial V} |g|\,d\sigma \leq \|g\|_{L^p}/\sigma(\partial V)^{1/p}$, or a, b are defined and $|Pa| > \alpha/2$ or $|Pb| > \alpha/2$. So, making the change of variables $\alpha \mapsto 2\alpha$,

$$\begin{aligned}\int_{\partial V} |Pg|^p\,d\sigma &\leq \|g\|_{L^p}^p + \int_{\alpha_0/2}^{\infty} p2^p\alpha^{p-1}\left(\sigma\{|Pa(X)| > \alpha\} + \sigma\{|Pb(X)| > \alpha\}\right)d\alpha \\ &\leq \|g\|_{L^p}^p + p2^p\int_{\alpha_0/2}^{\infty}\alpha^{p-1}\left(\frac{\|Pa\|_{L^1}}{\alpha} + \frac{\|Pb\|_{L^{p_0}}^{p_0}}{\alpha^{p_0}}\right)d\alpha.\end{aligned}$$

Since P is bounded $H^1 \mapsto L^1$ and $L^{p_0} \mapsto L^{p_0}$, this means that

$$\begin{aligned}\int_{\partial V} |Pg|^p\,d\sigma &\leq \|g\|_{L^p}^p + C(p)\int_{\alpha_0/2}^{\infty}\alpha^{p-1}\left(\frac{\|a\|_{H^1}}{\alpha} + \frac{\|b\|_{L^{p_0}}^{p_0}}{\alpha^{p_0}}\right)d\alpha \\ &\leq \|g\|_{L^p}^p + C(p)\int_{\alpha_0/2}^{\infty}\alpha^{p-1}\left(\frac{\|Mg\|_{L^{p_1}(E_\alpha)}^{p_1}}{\alpha^{p_1}} + \frac{\|g\|_{L^{p_0}(E_\alpha^C)}^{p_0}}{\alpha^{p_0}} + \sigma(E_\alpha)\right)d\alpha.\end{aligned}$$

Since $E_\alpha = \{X \in \partial V : Mg(X) > \alpha\}$, we have that $\sigma(E_\alpha) \leq \|Mg\|_{L^{p_1}(E_\alpha)}^{p_1}/\alpha^{p_1}$, and since $p_1 < p < p_0$, we can rewrite this as

$$\begin{aligned}\int_{\partial V} |Pg|^p\,d\sigma &\leq \|g\|_{L^p}^p + \int_0^\infty \frac{C(p)}{\alpha^{p_1-p+1}}\int_{Mg(X)>\alpha} Mg(X)^{p_1}\,d\sigma(X)\,d\alpha \\ &\quad + \int_0^\infty \frac{C(p)}{\alpha^{p_0-p+1}}\int_{Mg(X)\leq\alpha} |g(X)|^{p_0}\,d\sigma(X)\,d\alpha \\ &= \|g\|_{L^p}^p + C(p)\int_{\partial V}\int_0^{Mg(X)} \alpha^{p-p_1-1}\,d\alpha\,Mg(X)^{p_1}\,d\sigma(X) \\ &\quad + C(p)\int_{\partial V}\int_{Mg(X)}^\infty \alpha^{p-p_0-1}\,d\alpha\,|g(X)|^{p_0}\,d\sigma(X) \\ &= \|g\|_{L^p}^p + C(p)\int_{\partial V}\frac{1}{p_1-p}Mg(X)^{p-p_1}Mg(X)^{p_1}\,d\sigma(X) \\ &\quad + C(p)\int_{\partial V}\frac{1}{p_0-p}Mg(X)^{p-p_0}|g(X)|^{p_0}\,d\sigma(X) \\ &\leq C(p)\|Mg\|_{L^p}^p \leq C(p)\|g\|_{L^p}^p\end{aligned}$$

as desired. □

CHAPTER 10

BOUNDARY DATA IN *BMO* AND CARLESON-MEASURE ESTIMATES

We have solved the Neumann and regularity problems with boundary data in L^p for all $p > 1$ small. We also have an endpoint result for boundary data in the Hardy subspace H^1 of L^1.

Similarly, we can solve the Dirichlet problem for boundary data in L^q for all $q < \infty$ large; we would like an endpoint result for this problem as well.

In this dissertation, we will prove two endpoint results, one for boundary data in L^∞, and the other for data in its superspace BMO. We will delay the maximum principle (the L^∞ result) to Chapter 13; in this chapter, we will prove the following result for boundary data in BMO.

Theorem 10.1. *Suppose that A, V satisfy the conditions of Theorem 1.5. Then there is some $\epsilon_0 > 0$ such that, if $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, then for every $g \in BMO(\partial V)$ there exists a function u with $\operatorname{div} A\nabla u = 0$ in V, $u = g$ on ∂V, and such that*

$$\frac{1}{\sigma(B(X_0, R) \cap \partial V)} \int_{V \cap B(X_0,R)} |\nabla u(X)|^2 \operatorname{dist}(X, \partial V)\, dX \leq C\|g\|_{BMO}^2.$$

We will prove uniqueness (completing the proof of Theorem 1.7) in Corollary 12.12.

Recall from (2.11) that the $BMO(\partial V)$ norm of f is equivalent to

$$\|f\|_{BMO(\partial V)} \approx \sup_{X \in \partial V,\, R>0} \inf_{F \in \mathbf{C}} \frac{1}{R} \int_{B(X,R) \cap \partial V} |f - F|\, d\sigma.$$

This formulation of the BMO norm will be more convenient in this chapter than the formulation in terms of connected subsets of ∂V.

This u will turn out to be a layer potential. Recall that $\mathcal{D}f|_{\partial V} = \mathcal{K}f$, and that $\mathcal{K}^t$ is bounded and invertible with bounded inverse on $H^1(\partial V)$ (Theorem 7.1). Therefore by duality, $\mathcal{K}$ is bounded and invertible with bounded inverse on $BMO(\partial V)$.

So if $g \in BMO$, then $g = \mathcal{D}f$ for some f with $\|g\|_{BMO(\partial V)} \approx \|f\|_{BMO(\partial V)}$. Thus, to show that Theorem 10.1 holds, we need only show that

$$\frac{1}{\sigma(B(X_0,R)\cap\partial V)}\int_{V\cap B(X_0,R)} |\nabla\mathcal{D}f(X)|^2 \operatorname{dist}(X,\partial V)\,dX \le C\|f\|^2_{BMO(\partial V)} \tag{10.2}$$

for all $f \in BMO(\partial V)$.

10.1 L^2 estimates imply BMO estimates

In this section, we show that if

$$\int_V |\nabla\mathcal{D}f(X)|^2 \operatorname{dist}(X,\partial V)\,dX \le C\|f\|^2_{L^2(\partial V)} \tag{10.3}$$

holds for all $f \in L^2(\partial V)$, then (10.2) holds for all $f \in BMO(\partial V)$.

For most of this chapter, it will be convenient to work with more general operators. We have that

$$\nabla\mathcal{D}f(X) = \nabla_X \int_{\partial V} \left(\nu \cdot A^T \nabla\Gamma_X^T(Y)\right) f(Y)\,d\sigma(Y).$$

So if $TF(X) = \int_{\partial V} J(X,Y)F(Y)\,d\sigma(Y)$, where

$$J(X,Y) = \left(\nabla_X \left(\nu\cdot A^T\nabla\Gamma_X^T(Y)\right) \quad \nabla_X\left(\nu\cdot A^T\nabla\Gamma_X^T(Y)\right)\right),$$

then bounds on T will imply bounds on $\nabla\mathcal{D}$.

We have some useful conditions on this J and T. By (4.6), $|J(X,Y)| \le \frac{C}{|X-Y|^2}$. Also, by (5.5) $\mathcal{D}1(X)$ is constant on each connected component of $\mathbf{R}^2\setminus\partial V$, so $\nabla\mathcal{D}1(X)\equiv 0$ away from ∂V; thus, $TI(X)\equiv 0$.

We prove a lemma for T and J having these nice properties:

Lemma 10.4. *Suppose that $V \subset \mathbf{R}^2$ is a good Lipschitz domain, that*

$$TF(X) = \int_{\partial V} J(X,Y)F(Y)\,d\sigma(Y)$$

for some (matrix-valued function) J. Assume that there is some $\alpha > 0$ such that for any $X \in V$ and $Y \in \partial V$ we have that $|J(X,Y)| \le C\operatorname{dist}(X,\partial V)^{\alpha-1}/|X-Y|^{1+\alpha}$. Further *assume that* $TI(X) = 0$ *and that for any* $F \in L^2(\partial V \mapsto \mathbf{C}^{2\times 2})$,

$$\int_V |TF(X)|^2 \operatorname{dist}(X, \partial V)\, dX \leq C\|F\|_{L^2}^2. \tag{10.5}$$

Then if $B \in BMO(\partial V \mapsto \mathbf{C}^{2\times 2})$, *then* $TB(X)$ *converges for each* $X \in \partial V$ *and*

$$\frac{1}{\sigma(B(X_0, R) \cap \partial V)} \int_{B(X_0,R)\cap V} |TB(X)|^2 \operatorname{dist}(X, \partial V)\, dX \leq C\|B\|_{BMO}^2$$

for any $R > 0$, $X_0 \in \partial V$.

We remark that if $X \in V$ and $F \in L^p(\partial V)$ for any $1 \leq p \leq \infty$, then $TF(X)$ exists. This follows because, for any fixed $X \in V$, $J(X, \cdot) \in L^\infty(\partial V) \cap L^1(\partial V) \supset L^q(\partial V)$: $|J(X,Y)| \leq C/\operatorname{dist}(X, \partial V)^2$, and

$$\int_{\partial V} |J(X,Y)|\, d\sigma(Y) \leq \operatorname{dist}(X, \partial V)^{\alpha-1} \int_{\partial V} \frac{1}{|X-Y|^{1+\alpha}}\, d\sigma(Y) \leq C \operatorname{dist}(X, \partial V)^{\alpha-1}.$$

Proof. Let $D(X,R) = B(X,R) \cap \partial V$ for any $R > 0$ and any $X \in \partial V$. Recall from basic BMO theory that, if $B \in BMO(\partial V)$, then

$$⨍_{2^{k+1}D} |B - B_D|\, d\sigma \leq C(k+1)\|B\|_{BMO}$$

where $D = D(X_0, r)$, $B_D = ⨍_D B$, and $2^k D = 2^k D(X_0, r) = D(X_0, 2^k r)$.

Furthermore, by the John-Nirenberg inequality,

$$\int_D |B - B_D|^2 \leq C\sigma(D)\|B\|_{BMO}^2.$$

Fix some surface ball $D = B(X_0, R) \cap \partial V$. Now, $TB = T(B - B_D)$, and so without loss of generality $B_D = 0$. So

$$TB = T(B\mathbf{1}_D) + \sum_{k=0}^{\infty} T(B\mathbf{1}_{2^{k+1}D\setminus 2^k D}).$$

But

$$\int_V |T(B\mathbf{1}_D)(X)|^2 \operatorname{dist}(X, \partial V)\, dX \leq C\|B\mathbf{1}_D\|_{L^2(\partial V)}^2 \leq C\sigma(D)\|B\|_{BMO}^2.$$

If $X \in B(X_0, R)$, then

$$\begin{aligned}
|T(B\mathbf{1}_{2^{k+1}D\setminus 2^kD})(X)| &= \left|\int_{2^{k+1}D\setminus 2^kD} J(X,Y)B(Y)\,d\sigma(Y)\right| \\
&\leq C\int_{2^{k+1}D\setminus 2^kD} \frac{\operatorname{dist}(X,\partial V)^{\alpha-1}}{|X-Y|^{1+\alpha}}|B(Y)|\,dy \\
&\leq C\frac{\operatorname{dist}(X,\partial V)^{\alpha-1}}{2^kR^{1+\alpha}}\int_{2^{k+1}D} |B(Y)|\,d\sigma(Y) \\
&\leq C\frac{\operatorname{dist}(X,\partial V)^{\alpha-1}}{2^kR^{1+\alpha}}(k+1)\sigma(2^{k+1}D)\|B\|_{BMO}.
\end{aligned}$$

If $\sigma(D) < 2R$, then since ∂V is connected, $B(X_0, R)$ contains ∂V; so $\sigma(2^kD) \leq \sigma(\partial V) = \sigma(D)$ for all k. Otherwise, $\sigma(2^{k+1}D) \leq C2^kR \leq C2^k\sigma(D)$. In either case, $\sigma(2^{k+1}D) \leq C2^k\sigma(D)$ and so

$$\left|\sum_{k=0}^{\infty} T(B(\mathbf{1}_{2^{k+1}D\setminus 2^kD}))(X)\right| \leq C\|B\|_{BMO}\frac{\operatorname{dist}(X,\partial V)^{\alpha-1}\sigma(D)}{R^{1+\alpha}}\sum_{k=0}^{\infty}\frac{(k+1)}{2^{k\alpha}}.$$

Therefore, if $\beta = \sigma(D)\|B\|_{BMO}^2$, we have that

$$\begin{aligned}
\int_{B(X_0,R)\cap V} |TB(X)|^2 \operatorname{dist}(X,\partial V)\,dX &\leq C\beta + C\beta\sigma(D)\int_{B(X_0,R)\cap V}\frac{\operatorname{dist}(X,\partial V)^{2\alpha-1}}{R^{2+2\alpha}}\,dX \\
&\leq C\beta + C\beta\sigma(D)\int_{B(X_0,R)}\frac{|X-X_0|^{2\alpha-1}}{R^{2+2\alpha}}\,dX \\
&\leq C\beta + C\beta\frac{\sigma(D)}{R} \leq C\beta = C\sigma(D)\|B\|_{BMO}^2.
\end{aligned}$$

This concludes the proof. □

Therefore, to prove Theorem 10.1, we need only establish (10.3).

10.2 Operators on the half-plane

In this section, we establish a sufficient condition for (10.5) to hold if $V = \mathbf{R}^2_+$. In the next two sections, we will work with general operators T and kernels J; in Section 10.4, we will return to the Dirichlet problem and ∇D.

From [17, Theorem 1.1], we have

Theorem 10.6. *Let* $\theta_t f(x) = \int_{\mathbf{R}^n} \psi_t(x,y) f(y)\,dy$*, where*

$$|\psi_t(x,y)| \leq C \frac{t^\alpha}{(t+|x-y|)^{n+\alpha}} \tag{10.7}$$

$$|\psi_t(x,y) - \psi_t(x+h,y)| \leq C \frac{|h|^\alpha}{(t+|x-y|)^{n+\alpha}} \tag{10.8}$$

$$|\psi_t(x,y) - \psi_t(x,y+h)| \leq C \frac{|h|^\alpha}{(t+|x-y|)^{n+\alpha}} \tag{10.9}$$

whenever $|h| \leq t/2$*.*

Suppose that there exists a function $b : \mathbf{R}^n \mapsto \mathbf{C}$ *such that for some constants* λ*,* Λ*,* $C_0 > 0$*,*

$$\lambda \leq \mathbf{Re}\, b(x) \leq |b(x)| \leq \Lambda \quad \text{and} \quad \fint_Q \int_0^\infty |\theta_t b(x)|^2 \frac{dt}{t} dx \leq C_0 \tag{10.10}$$

for every $x \in \mathbf{R}^n$ *and every dyadic cube* $Q \subset \mathbf{R}^n$*.*

Then for all $f \in L^2(\mathbf{R}^n)$ *we have that*

$$\int_{\mathbf{R}^{n+1}_+} |\theta_t f(x)|^2 \frac{dx\,dt}{t} \leq C(\lambda, \Lambda, C_0) \|f\|^2_{L^2(\mathbf{R}^n)}. \tag{10.11}$$

(In [17], this is presented in more generality; the test function b is replaced by a system of test functions b_Q indexed by dyadic cubes Q. We do not need that generality here.)

We will need this theorem to hold if our kernel and test function take values in $\mathbf{C}^{2\times 2}$ rather than in $\mathbf{C}$. We begin with at trivial generalization:

Theorem 10.12. *Let* $T_t F(x) = \int_{\mathbf{R}^n} J_t(x,y) F(y)\,dy$*, where* $J_t(x,y)$*,* $F(y)$*, and* $T_t F(x)$ *are all square matrices. Assume that* J *satisfies*

$$|J_t(x,y)| \leq C \frac{t^\alpha}{t(t+|x-y|)^{n+\alpha}} \tag{10.13}$$

$$|J_t(x,y) - J_t(x+h,y)| \leq C \frac{|h|^\alpha}{t(t+|x-y|)^{n+\alpha}} \tag{10.14}$$

$$|J_t(x,y) - J_t(x,y+h)| \leq C \frac{|h|^\alpha}{t(t+|x-y|)^{n+\alpha}} \tag{10.15}$$

whenever $|h| \leq t/2$*.*

Suppose that there exists a function $b : \mathbf{R}^n \mapsto \mathbf{C}$ *such that for some constants* λ, Λ, $C_0 > 0$,

$$\lambda \leq \mathbf{Re}\, b(x) \leq |b(x)| \leq \Lambda \quad \textit{and} \quad \fint_Q \int_0^\infty |T_t(Ib)(x)|^2 t\, dt\, dx \leq C_0 \tag{10.16}$$

for every $x \in \mathbf{R}^n$ *and every dyadic cube* $Q \subset \mathbf{R}^n$.

Then

$$\iint_{\mathbf{R}^{n+1}_+} |T_t F(x)|^2 t\, dx\, dt \leq C(\lambda, \Lambda, C_0) \|F\|^2_{L^2(\mathbf{R}^n)} \tag{10.17}$$

for any L^2 *matrix-valued function* F.

Note that $t\, dx\, dt = \operatorname{dist}(X, \partial\mathbf{R}^2_+)\, dX$; this is why we use it instead of $\frac{dx\, dt}{t}$.

So if $T(Ib) \in BMO$ for some good scalar function b, and a few other conditions hold, then (10.17) holds. The problem is to generalize, so that we need only show $T(B) \in BMO$ for B in some larger class of test matrices.

Fortunately, [17] proves Theorem 10.6 from a similar theorem with the identity function replacing the b_Qs; we may prove a more general theorem from Theorem 10.12 using the same techniques.

The conditions on B analogous to (10.16) will turn out to be the same as the conditions on B_1 required by Theorem 6.10, that is, there exist constants C_0, $C_1 > 0$ and a smooth, compactly supported function v with $\int v = 1$ such that, if we define $v_t(x) = \frac{1}{t^n} v\left(\frac{x}{t}\right)$, then

$$\sup_{x \in \mathbf{R}^n} |B(x)| \leq C_0, \qquad \sup_{x \in \mathbf{R}^n,\, t \in \mathbf{R}^+} \left|(v_t * B(x))^{-1}\right| \leq C_1. \tag{10.18}$$

Theorem 10.19. *Suppose that* $B : \mathbf{R}^n \mapsto \mathbf{C}^{2\times 2}$ *is a bounded matrix-valued function which satisfies (10.18).*

Let $T_t F(x) = \int_{\mathbf{R}^n} J_t(x, y) F(y)\, dy$, *where* $J_t(x, y)$, $F(y)$, *and* $T_t F(x)$ *are all square matrices, and* J *satisfies (10.13–10.15).*

Suppose that for each dyadic cube Q,

$$\int_Q \int_0^{l(Q)} |T_t B(x)|^2 t\, dt\, dx \leq C_2 |Q|. \tag{10.20}$$

Then

$$\iint_{\mathbf{R}^{n+1}_+} |T_tF(x)|^2 t\,dx\,dt \le C(C_0, C_1, C_2)\|F\|^2_{L^2(\mathbf{R}^n)}.$$

Proof. Without loss of generality take $C_2 = 1$. It suffices to establish (10.16) for $b \equiv 1$.

As before, $T_tF(X)$ converges for $X \in V$ and $F \in L^p(\partial V)$, $1 \le p \le \infty$. In particular, $\left|\int J_t(x,z)\,dz\right| \le C/t$.

Let $A_tB(x) = v_t * B(x)$. Then

$$|T_tI(x)| = |T_tI(x)A_tB(x)(A_tB(x))^{-1}| \le C|T_tI(x)A_tB(x)|.$$

So we may write

$$\begin{aligned} T_tI(x)A_tB(x) &= \int J_t(x,z)\,dz \int \frac{1}{t^n} v\left(\frac{x-y}{t}\right) B(y)\,dy \\ &= \int J_t(x,z)\,dz \int \frac{1}{t^n} v\left(\frac{x-y}{t}\right) B(y)\,dy - T_tB(x) + T_tB(x) \\ &= \int \left(\frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - J_t(x,y)\right) B(y)\,dy + T_tB(x). \end{aligned}$$

Let $\check{T}_tF(x) = \int \check{J}_t(x,y)F(y)\,dy$, where

$$\check{J}_t(x,y) = \left(\frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - J_t(x,y)\right).$$

We have that

$$\begin{aligned} \int_0^{l(Q)} |T_tI(x)|^2 t\,dt \le C\int_0^{l(Q)} |T_tI(x)A_tB(x)|^2 t\,dt &\le C\int_0^{l(Q)} |\check{T}_tB(x) + T_tB(x)|^2 t\,dt \\ &\le C\int_0^{l(Q)} |\check{T}_tB(x)|^2 t\,dt + C\int_0^{l(Q)} |T_tB(x)|^2 t\,dt \end{aligned}$$

and so by (10.19),

$$\int_Q \int_0^{l(Q)} |T_tI(x)|^2 t\,dt\,dx \le C\int_Q\int_0^{l(Q)} |\check{T}_tB(x)|^2 t\,dt\,dx + C|Q|.$$

So we need only show that

$$\int_Q \int_0^{l(Q)} |\check{T}_t B(x)|^2 t\,dt\,dx \leq C|Q|.$$

We will do this by applying Theorem 10.12 and Lemma 10.4 to $\check{T}$.

By assumption, J_t satisfies (10.13–10.15). We need to show that $\check{J}_t$ does as well; it suffices to prove this for $J_t(x,y) + \check{J}_t(x,y) = \frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right)$.

Recall that $v_t(x-y) = 0$ if $|x-y| > Ct$. So (10.13–10.15) hold whenever $|x-y| > Ct$. If $|x-y| < Ct$, then $t \approx (t + |x-y|)$, and so

$$\begin{aligned}
|\check{J}_t(x,y) + J_t(x,y)| &\leq \left|\frac{C}{t^{n+1}} v\left(\frac{x-y}{t}\right)\right| \leq \frac{C}{t^{n+1}} \\
&\leq \frac{C}{t(t+|x-y|)^n} \leq \frac{C}{t(t+|x-y|)^n}\left(\frac{Ct}{t+|x-y|}\right)^\alpha .
\end{aligned}$$

If $|y-y'| < t/2$, then since $0 < \alpha \leq 1$,

$$\begin{aligned}
&\left|\frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - \frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y'}{t}\right)\right| \\
&\qquad \leq \frac{C}{t^{n+1}}\left|v\left(\frac{x-y}{t}\right) - v\left(\frac{x-y'}{t}\right)\right| \leq \frac{1}{t^{n+2}}\|\nabla v\|_{L^\infty}|y-y'| \\
&\qquad \leq C\frac{|y-y'|/(t+|x-y|)}{t(t+|x-y|)^n} \leq C\frac{\big(|y-y'|/(t+|x-y|)\big)^\alpha}{t(t+|x-y|)^n} = C\frac{|y-y'|^\alpha}{t(t+|x-y|)^{n+\alpha}}.
\end{aligned}$$

If $|x-x'| < t/2$, then

$$\begin{aligned}
&\left|\frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - \frac{\int J_t(x',z)\,dz}{t^n} v\left(\frac{x'-y}{t}\right)\right| \\
&\qquad\qquad \leq \left|\frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - \frac{\int J_t(x',z)\,dz}{t^n} v\left(\frac{x-y}{t}\right)\right| \\
&\qquad\qquad\quad + \left|\frac{\int J_t(x',z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - \frac{\int J_t(x',z)\,dz}{t^n} v\left(\frac{x'-y}{t}\right)\right|
\end{aligned}$$

The second term is at most $\frac{C|x-x'|^\alpha}{t(t+|x-y|)^{n+\alpha}}$ as before. Since $|x-y| < Ct$, the first term is at most

$$\frac{\|v\|_{L^\infty}}{t^n}\int |J_t(x,z)-J_t(x',z)|\,dz \le \frac{C}{t^n}\int \frac{|x-x'|^\alpha}{t(t+|x-z|)^{n+\alpha}}\,dz$$
$$\le \frac{C|x-x'|^\alpha}{t^{n+1+\alpha}} \le \frac{C|x-x'|^\alpha}{t(t+|x-y|)^{n+\alpha}}.$$

Thus $\check{J}_t$ satisfies (10.13–10.15).

Also, since $\int v = 1$, we have that

$$\begin{aligned}\check{T}_t I(x) &= \int \left(\frac{\int J_t(x,z)\,dz}{t^n} v\left(\frac{x-y}{t}\right) - J_t(x,y)\right) dy\\ &= \int J_t(x,z)\,dz \int \frac{1}{t^n} v\left(\frac{x-y}{t}\right) dy - \int J_t(x,y)\,dy = 0.\end{aligned}$$

So by Theorem 10.12,

$$\int_{\mathbf{R}^{n+1}_+} |\check{T}_t F(x)|^2 t\,dx\,dt \le C\|F\|^2_{L^2(\mathbf{R}^n)}$$

for any L^2 matrix-valued function F, and so by Lemma 10.4, we have that

$$\int_Q \int_0^{l(Q)} |\check{T}_t B(x)|^2 t\,dt\,dx \le C\|B\|^2_{BMO}|Q|.$$

This completes the proof. □

10.3 Operators on Lipschitz domains

We now wish to generalize the results of the previous section to arbitrary good Lipschitz domains V.

We begin with special Lipschitz domains.

Theorem 10.21. *Suppose that $\Omega \subset \mathbf{R}^2$ is a special Lipschitz domain with Lipschitz constant*

k_1, and that $J : \mathbf{R}^2 \times \mathbf{R}^2 \mapsto \mathbf{C}^{2\times 2}$ satisfies

$$\begin{aligned}
|J(X,Y)| &\leq C\frac{\operatorname{dist}(X,\partial\Omega)^\alpha}{\operatorname{dist}(X,\partial\Omega)|X-Y|^{1+\alpha}},\\
|J(X,Y) - J(X',Y)| &\leq C\frac{|X-X'|^\alpha}{\operatorname{dist}(X,\partial\Omega)|X-Y|^{1+\alpha}},\\
|J(X,Y) - J(X,Y')| &\leq C\frac{|Y-Y'|^\alpha}{\operatorname{dist}(X,\partial\Omega)|X-Y|^{1+\alpha}}.
\end{aligned}$$

Define $TF(X) = \int_{\partial\Omega} J(X,Y)F(Y)\,d\sigma(Y)$.

Let $B : \partial\Omega \mapsto \mathbf{C}^{2\times 2}$ satisfy

$$\sup_{X\in\partial\Omega} |B(X)| \leq C_0, \qquad \sup_{\Delta\subset\partial\Omega \textit{ connected}} \left| \left(\fint_\Delta B \right)^{-1} \right| \leq C_1. \tag{10.22}$$

Suppose that for any $X_0 \in \partial\Omega$ and any $R > 0$,

$$\int_{B(X_0,R)\cap\Omega} |TB(X)|^2 \operatorname{dist}(X,\partial\Omega)\,dX \leq C_2\sigma(B(X_0,R)\cap\partial\Omega). \tag{10.23}$$

Then for every $F \in L^2(\partial\Omega \mapsto \mathbf{C}^{2\times 2})$,

$$\int_\Omega |TF(X)|^2 \operatorname{dist}(X,\partial\Omega)\,dX \leq C(C_0,C_1,C_2,k_1)\|F\|^2_{L^2(\partial\Omega)}.$$

Proof. This follows from Theorem 10.19 by a change of variables. Define the matrix-valued function $\check{B}(x) = B(\psi(x))\sqrt{1+\varphi'(x)^2}$; then $\sup_{x\in\partial V} |\check{B}(x)| \leq C_0(1 + \|\varphi'\|_{L^\infty})$, and $\sup_{a,b\in\mathbf{R}} \left| \left(\fint_a^b \check{B} \right)^{-1} \right| \leq C_1(1 + \|\varphi'\|_{L^\infty})$. By Lemma 6.13, this means that $\check{B}$ satisfies (10.18).

Let $J_t(x,y) = J(\psi(x,t),\psi(y))$. Then J_t satisfies (10.13–10.15). If we let $T_tF(x) = \int_{\mathbf{R}} J_t(x,y)F(y)\,dy$, then

$$\begin{aligned}
T_t\check{B}(x) &= \int_{\mathbf{R}} J_t(x,y)\check{B}(y)\,dy = \int_{\mathbf{R}} J(\psi(x,t),\psi(y))B(\psi(y))\sqrt{1+\varphi'(y)^2}\,dy\\
&= \int_{\partial\Omega} J(\psi(x,t),Y)B(Y)\,d\sigma(Y) = TB(\psi(x,t))
\end{aligned}$$

and so if $Q \subset \mathbf{R}$ is an interval and $x_0 \in Q$, $X_0 = \psi(x_0)$, then

$$\begin{aligned}\int_Q \int_0^{l(Q)} |T_t B(x)|^2 t \, dt \, dx &= \int_Q \int_0^{l(Q)} |TB(\psi(x,t))|^2 t \, dt \, dx \\ &\leq \int_{\psi(Q \times (0, l(Q)))} |TB(X)|^2 C \operatorname{dist}(X, \partial\Omega) \, dX \\ &\leq \int_{B(X_0, Cl(Q)) \cap \Omega} |TB(X)|^2 C \operatorname{dist}(X, \partial\Omega) \, dX \\ &\leq C C_2 l(Q).\end{aligned}$$

Thus, by Theorem 10.19, we must have that

$$\iint_{\mathbf{R}^2_+} |T_t F(x)|^2 t \, dt \, dx \leq C \|F\|_{L^2}$$

for all $F \in L^2(\mathbf{R} \mapsto \mathbf{C}^{2\times 2})$. But since $T_t(\sqrt{1 + (\varphi')^2} F)(x) = TF(\psi(x,t))$, we must have that

$$\int_\Omega |TF(X)|^2 \operatorname{dist}(X, \partial\Omega) \, dX \leq C \|F\|_{L^2(\partial\Omega)}$$

for all $F \in L^2(\partial\Omega \mapsto \mathbf{C}^{2\times 2})$, as desired. $\square$

We now wish to move to good Lipschitz domains with compact boundary.

Theorem 10.24. *Suppose that* $J : \mathbf{R}^2 \times \mathbf{R}^2 \mapsto \mathbf{C}^{2\times 2}$ *satisfies*

$$|J(X,Y)| \leq C \frac{1}{|X-Y|^2}, \tag{10.25}$$

$$|J(X,Y) - J(X',Y)| \leq C \frac{|X-X'|^\alpha}{|X-Y|^{2+\alpha}}, \tag{10.26}$$

$$|J(X,Y) - J(X,Y')| \leq C \frac{|Y-Y'|^\alpha}{|X-Y|^{2+\alpha}}. \tag{10.27}$$

Define $T_U F(X) = \int_{\partial U} J(X,Y) F(Y) \, d\sigma(Y)$.

Assume that for each special Lipschitz domain $\Omega \subset \mathbf{R}^2$ *with Lipschitz constant* k_Ω*, there exists some* $B_\Omega : \partial\Omega \mapsto \mathbf{C}^{2\times 2}$ *which satisfies (10.22) and (10.23) with constants depending only on* k_Ω.

Then if $V \subset \mathbf{R}^2$ is a good Lipschitz domain with Lipschitz constants k_i, and if $F \in L^2(\partial V \mapsto \mathbf{C}^{2\times 2})$, we have that

$$\int_V |T_V F(X)|^2 \operatorname{dist}(X, \partial V)\, dX \leq C(k_i) \|F\|^2_{L^2(\partial V)}.$$

Proof. This follows trivially from the previous theorem if V is a special Lipschitz domain; thus we may assume that ∂V is bounded. By Definition 2.4, there are k_2 special Lipschitz domains Ω_i, with Lipschitz constant at most k_1, such that

$$\partial V \subset \bigcup_{i=1}^{k_2} \partial \Omega_i \cap B(X_i, r_i)$$

where $r_i > \sigma(\partial V)/C$ and $X_i \in \partial V$, with $\Omega_i \cap B(X_i, 2r_i) = V \cap B(X_i, 2r_i)$.

So we may write $F = \sum_{i=1}^{k_2} F_i$, where $F_i(X) = 0$ outside of $B(X_i, r_i)$, and $\sum_i |F_i(X)| = |F(X)|$ for all $X \in \partial V$.

Pick some i and note that $T_V F_i \equiv T_i F_i$, where $T_i = T_{\Omega_i}$. By Theorem 10.21,

$$\int_{\Omega_i} |T_i F_i(X)|^2 \operatorname{dist}(X, \partial \Omega_i)\, dX \leq C(k_1) \|F_i\|^2_{L^2(\partial \Omega_i)}.$$

Recall that $\partial \Omega_i \cap B(X_i, 2r_i) = \partial V \cap B(X_i, 2r_i)$. If $X \in B(X_i, \frac{3}{2} r_i) \cap V$, then either $\operatorname{dist}(X, \partial V) = \operatorname{dist}(X, \partial \Omega_i)$ or X is closer to some point in $B(X_i, 2r_i)^C$ than it is to $\partial V \cap B(X_i, 2r_i)$. In this case, $\operatorname{dist}(X, \partial \Omega_i) > r_i/2 > \frac{1}{3}|X - X_i| \geq \frac{1}{3} \operatorname{dist}(X, \partial V)$. So in either case,

$$\begin{aligned}
&\int_{V \cap B(X_i, 3r_i/2)} |T_i F_i(X)|^2 \operatorname{dist}(X, \partial V)\, dX \\
&\qquad \leq \int_{\Omega_i \cap B(X_i, 3r_i/2)} |T_i F_i(X)|^2\, 3 \operatorname{dist}(X, \partial \Omega_i)\, dX \\
&\qquad \leq 3 \int_{\Omega_i} |T_i F_i(X)|^2 \operatorname{dist}(X, \partial \Omega_i)\, dX \leq C(k_1) \|F_i\|^2_{L^2(\partial \Omega_i)}.
\end{aligned}$$

Conversely, suppose that $X \notin B(X_i, \frac{3}{2}r_i)$. Recall that $r_i \approx \sigma(\partial V)$. Then

$$\begin{aligned}|T_V F_i(X)| = \left|\int_{B(X_i,r_i)\cap\partial V} J(X,Y)F_i(Y)\,d\sigma(Y)\right| &\leq \int_{B(X_i,r_i)\cap\partial V} \frac{C}{|X-Y|^2}\,|F_i(Y)|\;d\sigma(Y)\\ &\leq \frac{C}{|X-X_i|^2}\int_{\partial V} |F_i(Y)|\;d\sigma(Y) \leq \frac{C}{|X-X_i|^2}\sqrt{\sigma(\partial V)}\|F_i\|_{L^2(\partial V)}.\end{aligned}$$

So

$$\begin{aligned}&\int_{V\setminus B(X_i,3r_i/2)} |T_V F_i(X)|^2 \operatorname{dist}(X,\partial V)\,dX\\ &\qquad\leq \int_{V\setminus B(X_i,3r_i/2)} \frac{C}{|X-X_i|^3}\sigma(\partial V)\|F_i\|^2_{L^2(\partial V)}\,dX \leq C\|F_i\|^2_{L^2(\partial V)}\sigma(\partial V)\frac{C}{r_i}\\ &\qquad= C\|F_i\|^2_{L^2(\partial V)}.\end{aligned}$$

Putting these together, we see that

$$\int_V |T_V F_i(X)|^2 \operatorname{dist}(X,\partial V)\,dX \leq C\|F_i\|^2_{L^2}.$$

So

$$\begin{aligned}\int_V |T_V F(X)|^2 \operatorname{dist}(X,\partial V)\,dX &= \int_V \Big|\sum_{i=1}^{k_2} T_V F_i(X)\Big|^2 \operatorname{dist}(X,\partial V)\,dX\\ &\leq k_2 \sum_{i=1}^{k_2}\int_V |T_V F_i(X)|^2 \operatorname{dist}(X,\partial V)\,dX\\ &\leq C\sum_{i=1}^{k_2}\|F_i\|^2_{L^2} \leq C\|F\|^2_{L^2}.\end{aligned}$$

This completes the proof. □

10.4 The Dirichlet problem on a Lipschitz domain

We now return to the Dirichlet problem. We want to show that

$$\int_V |\nabla Df(X)|^2 \operatorname{dist}(X,\partial\Omega)\,dX \leq C\|f\|^2_{L^2(\partial\Omega)} \tag{10.28}$$

for V a good Lipschitz domain.

Recall that $Df(X) = \int_{\partial\Omega} \nu \cdot A^T \nabla \Gamma_X^T f \, d\sigma$. So

$$\begin{aligned}\nabla Df(X) &= \nabla_X \int_{\partial V} \nu \cdot A^T \nabla \Gamma_X^T f \, d\sigma \\ &= \int_{\partial V} \begin{pmatrix} \partial_{x_1}\partial_{y_1}\Gamma_X^T(Y) & \partial_{x_1}\partial_{y_2}\Gamma_X^T(Y) \\ \partial_{x_2}\partial_{y_1}\Gamma_X^T(Y) & \partial_{x_2}\partial_{y_2}\Gamma_X^T(Y) \end{pmatrix} (A(Y)\nu(Y)) f(Y) \, d\sigma(Y)\end{aligned}$$

We would like to apply Theorem 10.24 to this expression. We must define J and the B_Ωs. The obvious candidate for $J(X,Y)$ is

$$\begin{pmatrix} \partial_{x_1}\partial_{y_1}\Gamma_X^T(Y) & \partial_{x_1}\partial_{y_2}\Gamma_X^T(Y) \\ \partial_{x_2}\partial_{y_1}\Gamma_X^T(Y) & \partial_{x_2}\partial_{y_2}\Gamma_X^T(Y) \end{pmatrix}.$$

Unfortunately, this matrix is not Hölder continuous; however, a nearby matrix is. Recall that

$$B_6(Y) = B_6^A(Y) = \begin{pmatrix} a_{11}(Y) & a_{21}(Y) \\ 0 & 1 \end{pmatrix}, \quad B_6^T(X) = \begin{pmatrix} a_{11}(X) & a_{12}(X) \\ 0 & 1 \end{pmatrix}.$$

Define

$$J(X,Y) = B_6^T(X) \begin{pmatrix} \partial_{x_1}\partial_{y_1}\Gamma_X^T(Y) & \partial_{x_1}\partial_{y_2}\Gamma_X^T(Y) \\ \partial_{x_2}\partial_{y_1}\Gamma_X^T(Y) & \partial_{x_2}\partial_{y_2}\Gamma_X^T(Y) \end{pmatrix} B_6(Y)^t$$

and let $TF(X) = \int_{\partial\Omega} J(X,Y)F(Y)\, d\sigma(Y)$ as usual. Then

$$\nabla Df(X) = B_6^T(X)^{-1} \int_{\partial\Omega} J(X,Y) B_6(Y)^{-1} (A(Y)\nu(Y)) f(Y) \, d\sigma(Y)$$

so if we let $F(Y) = B_6(Y)^{-1} A(Y) \begin{pmatrix} \nu(Y) & \nu(Y) \end{pmatrix} f(Y)$, then $|\nabla Df(X)| \leq C|TF(X)|$ and $|F(Y)| \leq C|f(Y)|$; thus, we need only show that J satisfies (10.25–10.27) and define the B_Ω appropriately. This will establish (10.3), which by Lemma 10.4 will prove Theorem 10.1.

We begin with the conditions on J. As in Section 10.1, $|J(X,Y)| \leq C/|X-Y|^2$. We may write

$$J(X,Y) = \begin{pmatrix} \partial_{y_1}\partial_{x_2}\tilde{\Gamma}_X^T(Y) & \partial_{y_2}\partial_{x_2}\tilde{\Gamma}_X^T(Y) \\ \partial_{y_1}\partial_{x_2}\Gamma_X^T(Y) & \partial_{y_2}\partial_{x_2}\Gamma_X^T(Y) \end{pmatrix} B_6(Y)^t.$$

Since all of the X-derivatives are now in terms of x_2, and $A(X)$ is independent of x_2, we have that each component is a solution to an elliptic equation in X. Thus, $J(X,Y)$ must be Hölder continuous in X, and in fact by Lemma 3.7 must satisfy

$$|J(X',Y) - J(X,Y)| \leq C\frac{|X-X'|^\alpha}{|X-Y|^{2+\alpha}}$$

for all $|X - X'| < \frac{1}{2}|X - Y|$.

Similarly,

$$|J(X,Y') - J(X,Y)| \leq C\frac{|Y-Y'|^\alpha}{|X-Y|^{2+\alpha}}$$

for all $|Y - Y'| < \frac{1}{2}|X - Y|$.

So (10.25–10.27) hold. Fix some special Lipschitz domain Ω; we need to find a $B = B_\Omega$ which satisfies (10.22) and (10.23).

Let

$$B(Y) = (B_6(Y)^t)^{-1}\begin{pmatrix} A(Y)\nu(Y) & \tau(Y)\end{pmatrix}.$$

Then $\int_{\psi((a,b))} B(Y)\,d\sigma(Y) = \int_a^b B_1(y)\,dy$ where $B_1(y)$ is as in (2.34). From Section 6.4, we know that B satisfies (10.22). Furthermore,

$$\begin{aligned}
\left(B_6^T(X)^t\right)^{-1} TB(X) &= \left(B_6^T(X)^t\right)^{-1}\int J(X,Y)B(Y)\,d\sigma(Y)\\
&= \int_{\partial\Omega}\begin{pmatrix}\partial_{x_1}\partial_{y_1}\Gamma_X^T & \partial_{x_1}\partial_{y_2}\Gamma_X^T\\ \partial_{x_2}\partial_{y_1}\Gamma_X^T & \partial_{x_2}\partial_{y_2}\Gamma_X^T\end{pmatrix}\begin{pmatrix}A\nu & \tau\end{pmatrix}\,d\sigma(Y)\\
&= \int \begin{pmatrix}\nabla_X(\nu\cdot A^T\nabla\Gamma_X^T(Y)) & \nabla_X(\partial_\tau\Gamma_X^T(Y))\end{pmatrix}\,d\sigma(Y)\\
&= \int \begin{pmatrix}\nabla_X(\partial_\tau\tilde\Gamma_X^T(Y)) & \nabla_X(\partial_\tau\Gamma_X^T(Y))\end{pmatrix}\,d\sigma(Y)\\
&= \int \partial_\tau\begin{pmatrix}\nabla_X\tilde\Gamma_X^T(Y) & \nabla_X\Gamma_X^T(Y)\end{pmatrix}\,d\sigma(Y)
\end{aligned}$$

which equals 0, since $\nabla_X\Gamma_X^T(Y), \nabla_X\tilde\Gamma_X^T(Y)$ go to zero as $|X - Y| \to \infty$. Thus, B satisfies (10.23) as well and (10.28) is proven.

CHAPTER 11

REMOVING THE SMOOTHNESS ASSUMPTIONS

We have proven (the existence halves of) Theorems 1.5 and 1.7 only under the assumption that A, A_0 are smooth.

Theorem 11.1. *Fix some* Λ, λ *and* ϵ_0. *Let* V *be a good Lipschitz domain, and suppose that solutions to* $(N)_p^A$, $(R)_p^A$, $(N)_1^A$, *and* $(R)_1^A$ *exist for all matrix-valued functions* A *satisfying (1.2),* $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, *and which in addition are smooth and satisfy* $A(x) \equiv I$ *for large* $|x|$.

Let A *be a matrix-valued function that satisfies (1.2) and* $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, *but not smoothness or* $A(x) \equiv I$ *for large* x. *Then there exist solutions to* $(N)_p^A$, $(R)_p^A$, $(N)_1^A$, *and* $(R)_1^A$ *in* V.

Proof. By Lemmas 3.9 and 3.10, there exist regularity or Neumann solutions for boundary data in (dense subspaces of) H^1 or L^p. We need only show that they have the desired bounds on their non-tangential maximal functions.

Let u be the Neumann or regularity solution to $\nu \cdot A\nabla u = g$ or $\partial_\tau u = g$, where $g \in H^1_{at}$. We have that $\nabla u \in L^2$.

Assume $\eta > 0$ is small, and let $A_\eta(x)$ be a smooth matrix-valued function satisfying (1.2) such that $|A(x) - A_\eta(x)| < \eta$, for all x outside a set E_η with $|E_\eta \cap (-1/\eta, 1/\eta)| < \eta$. (Here $|\cdot|$ denotes Lebesgue measure in $\mathbf{R}^1$.) Let $V_\eta = \{(x,t) : x \in E_\eta,\ (x,t) \in V\}$.

Assume that A_η satisfies all of the conditions of the theorem, so $(N)_p^{A_\eta}$, $(R)_p^{A_\eta}$, $(N)_1^{A_\eta}$, and $(R)_1^{A_\eta}$ hold in V.

Take u^η to be the solution to $(N)_p^{A_\eta}$ and $(N)_1^{A_\eta}$, or $(R)_p^{A_\eta}$ and $(R)_1^{A_\eta}$, developed in the rest of this paper with the same boundary data as u.

I claim that, if $Y \in V$ and $r < \operatorname{dist}(Y, \partial V)$, then

$$\fint_{B(Y,r)} |\nabla u - \nabla u^\eta|^2 \leq \frac{o(\eta)}{r^2}$$

where $o(\eta) \to 0$ as $\eta \to 0$.

Suppose my claim is true. By (3.8), if $Y \in \gamma(X)$ then

$$\begin{aligned}
|\nabla u(Y)|^2 &\leq C \fint_{B(Y, \operatorname{dist}(Y,\partial V)/2)} |\nabla u|^2 \\
&\leq C \fint_{B(Y, \operatorname{dist}(Y,\partial V)/2)} |\nabla u - \nabla u^\eta|^2 + C \fint_{B(Y, \operatorname{dist}(Y,\partial V)/2)} |\nabla u^\eta|^2 \\
&\leq C \frac{o(\eta)}{\operatorname{dist}(Y,\partial V)^2} + C N(\nabla u^\eta)(X)^2.
\end{aligned}$$

Therefore, if we let

$$N_\delta f(X) = \sup\{|f(Y)| : (1+|X|^2)\delta < |X-Y| < (1+a)\operatorname{dist}(Y,\partial V)\},$$

then $N_\delta(\nabla u)(X) \leq \frac{o(\eta)}{\delta(1+|X|)^2} + CN(\nabla u^\eta)(X)$. So $\|N_\delta(\nabla u)\|_{L^p} \leq C\frac{o(\eta)}{\delta} + C\|N(\nabla u^\eta)\|_{L^p}$ for all $1 \leq p \leq \infty$. (The decay in X is necessary to ensure that this is true for ∂V not compact, that is, $V = \Omega$ a special Lipschitz domain.)

But $\|N(\nabla u^\eta)\|_{L^1} \leq C\|g\|_{H^1}$, $\|N(\nabla u^\eta)\|_{L^p} \leq C\|g\|_{L^p}$ for $1 < p < p_0$. By taking the limit as $\eta \to 0$, we see that $\|N_\delta(\nabla u)\|_{L^p} \leq C_p\|g\|_{L^p}$ and $\|N_\delta(\nabla u)\|_{L^1} \leq C\|g\|_{H^1}$ uniformly in δ; thus these inequalities must hold for $N(\nabla u)$ as well.

So I need only show that

$$\fint_{B(Y,r)} |\nabla u - \nabla u^\eta|^2 \leq \frac{o(\eta)}{r^2}.$$

Fix some choice of η. We have that u^η is a Neumann or regularity solution with boundary data in H^1 or $H^1 \cap L^p$. Since our solutions are of the form $\mathcal{S}((\mathcal{K}_-^t)^{-1}g)$, the $(N)_1^{A_\eta}$ and the $(N)_p^{A_\eta}$ solutions are the same. So $N(\nabla u^\eta) \in L^1(\partial V)$.

If V is bounded or special, then by Lemma 3.3, $\nabla u^\eta \in L^2(V)$, uniformly in η.

If V^C is bounded, recall that $u^\eta(X) = \mathcal{S}^\eta(P_\eta^{-1}g)$ where $P_\eta = (\mathcal{L}^{A_\eta})^t$ or $(\mathcal{K}_-^{A_\eta})^t$. Since both these operators are invertible on H^1 we have that $u^\eta = \mathcal{S}^\eta f$ for some $f \in H^1$. But then $f = \partial_\tau F$ for some F with $\|F\|_{L^\infty} \leq \|f\|_{L^1} \leq C\|g\|_{H^1}$; then

$$|u^\eta(X)| = |\mathcal{S}^\eta(\partial_\tau F)(X)| = \left| \int_{\partial V} \tau \cdot \Gamma_X^T \, F \, d\sigma \right| \leq C\|g\|_{H^1}\sigma(\partial V)/\operatorname{dist}(X,\partial V)$$

and so by Lemma 3.4 and (3.8) we have that $|\nabla u(X)| \leq C\|g\|_{H^1}\sigma(\partial V)/\operatorname{dist}(X,\partial V)^2$. So in particular ∇u^η is still in L^2 uniformly in η.

Pick some $W \subset V$ compact and large enough that $B(Y,r) \subset W$ and $\operatorname{supp} g \subset \partial V \cap \partial W$. Then if u_W is any constant, we have that

$$\begin{aligned}
\int_W |\nabla u - \nabla u^\eta|^2 &\leq C\,\mathbf{Re}\int_W (\nabla\bar u - \nabla\overline{u^\eta})\cdot A_\eta(\nabla u - \nabla u^\eta)\\
&= C\,\mathbf{Re}\int_W \nabla(\bar u - \overline{u^\eta})\cdot(A\nabla u - A_\eta\nabla u^\eta) + (\nabla\bar u - \nabla\overline{u^\eta})(A_\eta - A)\nabla u\\
&= C\,\mathbf{Re}\int_{\partial W\cap\partial V} \mathbf{Tr}(\overline{u - u^\eta - u_W})(\nu\cdot A\nabla u - \nu\cdot A_\eta\nabla u^\eta)\,d\sigma\\
&\quad + C\,\mathbf{Re}\int_{\partial W\setminus\partial V} \mathbf{Tr}(\overline{u - u^\eta - u_W})(\nu\cdot A\nabla u - \nu\cdot A_\eta\nabla u^\eta)\,d\sigma\\
&\quad + C\,\mathbf{Re}\int_W (\nabla\bar u - \nabla\overline{u^\eta})(A_\eta - A)\nabla u.
\end{aligned}$$

But since $|A - A_\eta| < \eta$ on $V \setminus V_\eta$,

$$\begin{aligned}
\left|\mathbf{Re}\int_V (\nabla\bar u - \nabla\overline{u^\eta})(A_\eta - A)\nabla u\right| &\leq C\eta\|\nabla u\|_{L^2(V\setminus V_\eta)}(\|\nabla u\|_{L^2(V\setminus V_\eta)} + \|\nabla u^\eta\|_{L^2(V\setminus V_\eta)})\\
&\quad + C\|\nabla u\|_{L^2(V_\eta)}(\|\nabla u\|_{L^2(V_\eta)} + \|\nabla u^\eta\|_{L^2(V_\eta)})
\end{aligned}$$

So the third term is at most $o(\eta)$, independently of W. If V is compact let $W = V$ and $u_W = 0$; since u, u^η are Neumann or regularity (Dirichlet) solutions with the same boundary data, $\int_V |\nabla u - \nabla u^\eta|^2 \leq o(\eta)$.

If V^C is compact then let $W = W(R) = B(0,R)\cap V$ for large R; by Lemma 3.11 $\int_{\partial V}\nu\cdot A\nabla u = \int_{\partial V}\nu\cdot A_\eta\nabla u^\eta$, even for regularity solutions. If $V = \Omega$ is a special Lipschitz domain, let $W = W(R) = Q(0,R)$ for R large; if we are solving the regularity problem, we must assume that $u_W = 0$. In any case, the first integral is zero.

Averaging our inequality over a range of R, we get that

$$\begin{aligned}
\int_{W(R_0)} |\nabla u - \nabla u^\eta|^2 &\leq o(\eta) + \frac{C}{R_0}\int_{R_0}^{2R_0}\int_{\partial W(R)\setminus\partial V} |u - u^\eta - u_W|\,|A\nabla u - A_\eta\nabla u^\eta|\,d\sigma\,dR\\
&\leq o(\eta) + \frac{C}{R_0}\int_{W(2R_0)\setminus W(R_0)} |u - u^\eta - u_W|\,|A\nabla u - A_\eta\nabla u^\eta|.
\end{aligned}$$

But $A\nabla u - A_\eta \nabla u^\eta \in L^2(V)$, so

$$\lim_{R_0\to\infty} \|A\nabla u - A_\eta \nabla u^\eta\|_{L^2(W(2R_0)\setminus W(R_0))} = 0.$$

We use the Poincaré inequality. If V^C is bounded or we are solving the Neumann problem, we may choose u_W such that

$$\|u - u^\eta - u_W\|_{L^2(W(2R_0)\setminus W(R_0))} \leq CR_0 \|\nabla u - \nabla u^\eta\|_{L^2(W(2R_0)\setminus W(R_0))}.$$

If V is special and we are solving the regularity problem, then $u - u^\eta = 0$ on ∂V, a large subset of $\partial(W(2R_0) \setminus W(R_0))$, and so

$$\|u - u^\eta\|_{L^2(W(2R_0)\setminus W(R_0))} \leq CR_0 \|\nabla u - \nabla u^\eta\|_{L^2(W(2R_0)\setminus W(R_0))}.$$

By taking the limit as $R_0 \to \infty$ we see that

$$\int_V |\nabla u - \nabla u^\eta|^2 \leq o(\eta) \tag{11.2}$$

as desired. □

Theorem 11.3. *Fix some Λ, λ and ϵ_0. Let V be a Lipschitz domain, and suppose that solutions to $(D)^A_q$ and $(R)^A_p$, $(R)^A_1$ exist and are unique for all matrix-valued functions A satisfying (1.2), $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, and which in addition are smooth and satisfy $A(x) \equiv I$ for large $|x|$.*

Let A be a matrix-valued function that satisfies (1.2) and $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, but not smoothness or $A(x) \equiv I$ for large x. Then there exist solutions to $(D)^A_q$ in V.

Proof. Suppose that f is smooth and compactly supported on ∂V. Construct u, A_η, u^η as in the proof of Theorem 11.1, with $\mathbf{Tr}\, u = \mathbf{Tr}\, u^\eta = f$, $\operatorname{div} A\nabla u = \operatorname{div} A_\eta \nabla u^\eta = 0$.

As in the proof of Theorem 11.1, by using Lemma 3.6 instead of (3.8) it suffices to show that

$$\int_{B(Y,r)} |u - u^\eta|^2 \leq o(\eta).$$

In fact, it suffices to show that $\fint_{B(Y,r)} |u - u^\eta|^2 \leq C(R)o(\eta)$ for any $B(Y,r) \subset B(0,R)$; we then get a uniform bound on the L^q norm of $N_{\delta,R}u(X) = \sup\{|u(Y)| : \delta < |X-Y| < (1+a)\operatorname{dist}(Y,\partial V),\ |Y| < R\}$ which becomes a bound on $\|Nu\|_{L^q(\partial V)}$ as before.

Fix $W \subset V$ compact with $B(0,2R) \cap V \subset W$ for R large. Then by the Poincaré inequality,

$$\|v - \fint_{\partial V \cap \partial W} \mathbf{Tr}\, v\|_{L^2(W)} \leq C(V,W)\|\nabla v\|_{L^2(W)}$$

for all functions $v \in W^{1,2}(W)$.

Recall u^η is the Dirichlet solution constructed in Theorem 1.5, with $\mathbf{Tr}\, u^\eta = f$ and $\|Nu\|_{L^q} \leq C\|f\|_{L^q}$. We will show (Theorem 12.4) that Dirichlet solutions and regularity solutions with the same boundary data are equal; so u^η is also the regularity solution used in the previous theorem, and so (11.2) still holds. So since $\mathbf{Tr}(u - u^\eta) \equiv 0$ on ∂V,

$$\|u - u^\eta\|_{L^2(W)} \leq C(W)\|\nabla u - \nabla u^\eta\|_{L^2(W)} \leq C(W)\|\nabla u - \nabla u^\eta\|_{L^2(V)} \leq o(\eta).$$

This completes the proof. □

Theorem 11.4. *Fix some Λ, λ and ϵ_0. Let V be a good Lipschitz domain, and suppose that solutions to $(R)^A_p$ and Theorem 1.7 exist for all matrix-valued functions A satisfying (1.2), $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, and which in addition are smooth and satisfy $A(x) \equiv I$ for large $|x|$.*

Let A be a matrix-valued function that satisfies (1.2) and $\|\mathbf{Im}\, A\|_{L^\infty} < \epsilon_0$, but not smoothness or $A(x) \equiv I$ for large x. Then solutions to Theorem 1.7 exist for A in V.

Proof. First take f to be a smooth, compactly supported function on ∂V. We may construct a u with $\operatorname{div} A\nabla u = 0$ in V, $\mathbf{Tr}\, u = f$ and $\|N(\nabla u)\|_{L^p} \leq C\|\partial_\tau f\|_{L^p}$. We need only show that (1.8) holds.

Define A_η, u^η as before. Then as in Theorem 11.1, $|\nabla u(Y) - \nabla u^\eta(Y)| \leq \frac{o(\eta)}{\operatorname{dist}(Y,\partial V)}$. So if $X \in \partial V$ and $R > 0$, and if $V_\delta = \{X \in V : \operatorname{dist}(X,\partial V) > \delta\}$, then

$$\begin{aligned}
\frac{1}{R}\int_{B(X,R)\cap V_\delta} &|\nabla u(Y)|^2 \operatorname{dist}(Y,\partial V)\,dY \\
&\leq \frac{2}{R}\int_{B(X,R)\cap V_\delta} \left(|\nabla u(Y) - \nabla u^\eta(Y)|^2 + |\nabla u^\eta(Y)|^2\right)\operatorname{dist}(Y,\partial V)\,dY \\
&\leq \frac{2}{R}\int_{B(X,R)\cap V_\delta} \frac{o(\eta)}{\operatorname{dist}(Y,\partial V)}\,dY + C\|f\|^2_{BMO}
\end{aligned}$$

and so by letting $\eta \to 0$,

$$\frac{1}{R}\int_{B(X,R)\cap V_\delta} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial V)\,dY \leq C\|f\|_{BMO}^2$$

uniformly in δ; by letting $\delta \to 0$ we recover Theorem 1.7 for smooth, compactly supported boundary data.

We now consider moving to nonsmooth, compactly supported boundary data. (I did not do this explicitly when proving Theorems 11.1 and 11.3 because smooth functions are dense in L^p; they are not dense in BMO.)

Since f is compactly supported, $f \in L^p$ for $1 \leq p < \infty$. Let p be large enough that $(D)_p^A$ holds in V. Let $f_n \to f$ in L^p, f_n smooth and compactly supported, with $\|f_n\|_{BMO} \leq C\|f\|_{BMO}$. Let u_n be the solution with boundary data f_n.

Then $\|N(u_n - u_m)\|_{L^p} \leq C\|f_n - f_m\|_{L^p}$, so

$$|u_n(X) - u_m(X)| \leq \frac{C\|f_n - f_m\|_{L^p}}{\operatorname{dist}(X,\partial V)^{1/p}}, \quad |\nabla u_n(X) - \nabla u_m(X)| \leq \frac{C\|f_n - f_m\|_{L^p}}{\operatorname{dist}(X,\partial V)^{1+1/p}}.$$

Thus, $\{u_n\}$ converges almost uniformly to some u with the right boundary data; we see that for any $X \in \partial V$ and any $R > 0$,

$$\begin{aligned}&\frac{1}{R}\int_{B(X,R)\cap V_\delta} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial V)\,dY \\ &\qquad \leq \frac{2}{R}\int_{B(X,R)\cap V_\delta} \left(|\nabla u(Y) - \nabla u_n(Y)|^2 + |\nabla u_n(Y)|^2\right) \operatorname{dist}(Y,\partial V)\,dY \leq C\|f\|_{BMO}\end{aligned}$$

for n large enough; by letting $\delta \to 0$ we complete the proof.

We now want to pass to non-compactly supported f. If ∂V is bounded this is trivial, so we may assume that $V = \Omega$ is a special Lipschitz domain. Without loss of generality, $0 \in \partial\Omega$ and $\psi(0) = 0$.

Let $f \in BMO(\partial\Omega)$ with $\|f\|_{BMO} = 1$. Let $\Delta_n = \Delta(0, 2^n)$, and let $f_n = f$ on Δ_n, $f_n = \fint_{\Delta_n} f\,d\sigma$ on $\partial\Omega \setminus \Delta_{n+1}$; we may define f_n on $\Delta_{n+1} \setminus \Delta_n$ in such a way that $\|f_n\|_{BMO} \leq C$.

Then f_n is a BMO function. Furthermore, $f_n = g_n + \alpha_n$, where $\alpha_n = \fint_{\Delta_n} f\,d\sigma$ is a constant and g_n is supported in Δ_{n+1}, $\fint_{\Delta_{n+1}} g_n \leq C$. By the John-Nirenberg inequality, this means that $\|g_n\|_{L^p} \leq C_p(C + \|g_n\|_{BMO})|\Delta_{n+1}|^{1/p} \leq C_p 2^{n/p}$.

Let u_n be the Dirichlet solution to $\operatorname{div} A\nabla u = 0$ in Ω with boundary data g_n. Then $u_{n+1} - u_n$ is also a solution, which is equal to the constant $\alpha_n - \alpha_{n+1}$ on Δ_n.

Let $v_n = u_{n+1} - u_n + \alpha_{n+1} - \alpha_n$. Then v_n is a solution to $\operatorname{div} A\nabla u = 0$ in Ω which is equal to zero on Δ_n. By Lemma 3.6, if $|X|$ is small enough compared to 2^n, then

$$\begin{aligned}
|v_n(X)| &\leq \left(\fint_{B(X,2^n/C)} |v_n|^p \right)^{1/p} \\
&\leq \left(\fint_{B(X,2^n/C)} |\alpha_{n+1} - \alpha_n|^p \right)^{1/p} + \left(\fint_{B(X,2^n/C)} |u_{n+1} - u_n|^p \right)^{1/p} \\
&\leq C + \left(\frac{C}{2^{2n}} \int_0^{C2^n} \|u_{n+1} - u_n\|^p_{L^p(\psi(\mathbf{R},t))} \right)^{1/p} \\
&\leq C + \frac{C}{2^{n/p}} \|N(u_{n+1} - u_n)\|_{L^p(\partial\Omega)} \\
&\leq C + \frac{C}{2^{n/p}} \|g_n\|_{L^p(\partial\Omega)} + \frac{C}{2^{n/p}} \|g_{n+1}\|_{L^p(\partial\Omega)} \leq C.
\end{aligned}$$

By Lemma 3.7,

$$|v_n(X) - v_n(0)| \leq C \frac{\|v_n\|_{L^2(B(0,2^n/C)\cap\Omega)}}{2^{n+n\alpha}} |X|^\alpha \leq C \frac{C}{2^{n\alpha}} |X|^\alpha.$$

Thus, $|v_n(X)| \leq C|X|^\alpha / 2^{n\alpha}$ for sufficiently large n. Therefore, $u_0(X) + \alpha_0 + \sum_n v_n(X)$ converges for all X, uniformly on compact sets. The sum $u(X)$ has the proper boundary values, and since

$$u_0(X) + \alpha_0 + \sum_{k=1}^{n} v_k(X) = u_{n+1}(X) + \alpha_{n+1},$$

u satisfies the Carleson-measure condition; thus, solutions to Theorem 1.7 with arbitrary BMO boundary data exist. $\square$

CHAPTER 12

CONVERSES AND UNIQUENESS

We have show that, if V is a bounded or special Lipschitz domain and f is a function defined on ∂V, then there is some u with $\operatorname{div} A\nabla u = 0$ in V and such that

- If $f \in H^1(\partial V)$, then $\nu \cdot A\nabla u = f$ (or $\tau \cdot \nabla u = f$) on ∂V and $N(\nabla u) \in L^1(\partial V)$.
- If $f \in L^p(\partial V) \cap H^1(\partial V)$ for $p > 1$ small enough, then $\nu \cdot A\nabla u = f$ (or $\tau \cdot \nabla u = f$) on ∂V and $N(\nabla u) \in L^p(\partial V)$.
- If $f \in L^p(\partial V)$ for $p < \infty$ large enough, then $u = f$ on ∂V and $N(u) \in L^p(\partial V)$.
- If $f \in BMO(\partial V)$, then $u = f$ on ∂V and (1.8) holds.

We wish to prove the converses, and to show that such u are unique.

12.1 Uniqueness for the Neumann and regularity problems

Theorem 12.1. *Suppose that* $\operatorname{div} A\nabla u = 0$ *in* V *for some Lipschitz domain* V*. Assume that either* $\nu \cdot A\nabla u = 0$ *on* ∂V *or* $u \equiv C$ *on* ∂V *for some constant* C*.*

If $\nabla u \in L^2_{loc}(V)$*, with* $\lim_{R\to\infty} \fint_{B(0,2R)\setminus B(0,R)} |\nabla u|^2 = 0$*, then* u *is a constant.*

In addition, if $V = \Omega$ *is a special Lipschitz domain, then by previous results there is some* $p_0 > 1$*, depending only on ellipticity and the Lipschitz constant of* Ω*, such that if* $1 < p \leq p_0$*, then solutions to* $(N)^A_p$ *and* $(R)^A_p$ *exist in* $Q(0,R)$ *for all* $R > 0$*. If* $N(\nabla u) \in L^p(\partial\Omega)$ *for some* $1 \leq p \leq p_0$*, then* u *is a constant.*

If either $\nabla u \in L^2(V)$ or if $N(\nabla u) \in L^p(\partial V)$ and $\lim_{|X|\to\infty} u(X)$ exists, then by Lemmas 3.3 and 3.4 u satisfies the conditions of the theorem. We phrase our conditions in this way to ensure that solutions obtained via layer potentials and the solutions of Lemmas 3.9 and 3.10 are equal.

In the regularity case we may assume without loss of generality that $u \equiv 0$ on ∂V.

Proof. Pick some R large, and let $W(R) = V \cap B(0,R)$ (if ∂V is bounded) or $W(R) = Q(0,2R)$ (if $V = \Omega$ is special). For any $\epsilon > 0$ there is some $\eta \in C_0^\infty(B(0,2R))$ such that $\|\nabla\eta - \nabla u\|_{L^2(V\cap B(0,R))} < \epsilon$. (See [11, p. 252].) We may further require that

$$\begin{aligned}\|\nabla\eta\|_{L^2(W(2R)\setminus W(R))} &\leq C\|\nabla u\|_{L^2(W(2R)\setminus W(R))} + \frac{C}{R}\left\|u - ⨍_{W(R)} u\right\|_{L^2(W(2R)\setminus W(R))}\\ &\leq C\|\nabla u\|_{L^2(W(2R)\setminus W(R))}.\end{aligned}$$

If $u \equiv 0$ on ∂V, then we may require that $\eta \equiv 0$ on ∂V as well.

But by the weak definition of $\nu \cdot A\nabla u = 0$ or $\operatorname{div} A\nabla u = 0$, we have that

$$\int_{V\cap B(0,2R)} \nabla\bar\eta \cdot A\nabla u = 0$$

and so

$$\begin{aligned}\left|\int_{W(2R)} \nabla\bar u \cdot A\nabla u\right| &= \left|\int_{W(2R)} (\nabla\bar u - \nabla\bar\eta)\cdot A\nabla u\right|\\ &\leq \int_{W(R)} |\nabla\bar u - \nabla\bar\eta|\,|A\nabla u| + \int_{W(2R)\setminus W(R)} (|\nabla\eta| + |\nabla u|)|A\nabla u|\\ &\leq C\epsilon\|\nabla u\|_{L^2(V\cap B(0,R))} + C\int_{W(2R)\setminus W(R)} |\nabla u|^2.\end{aligned}$$

We first take the limit as $\epsilon \to 0$ to eliminate the first term; we then take the limit as $R \to \infty$, which eliminates the second term. Thus by ellipticity of A $\nabla u = 0$ almost everywhere, so u must be a constant.

We now consider $N(\nabla u) \in L^p(\partial\Omega)$ for p small. Since $N(\nabla u) \in L^p(\partial\Omega)$, for any fixed ϵ, $R_0 > 0$, there must be some $R > R_0$ such that $N(\nabla u)(\psi(\pm R)) \leq \epsilon R^{-1/p}$. Recall that we assumed a was large enough that $\partial Q(0,R) \subset \gamma_a(\chi_+) \cup \gamma_a(\chi_-)$.

Pick some R_0, ϵ. Then

$$\|\nu \cdot A\nabla u\|_{L^p(\partial Q(0,R))} \leq C\epsilon \quad \text{or} \quad \|\tau \cdot \nabla u\|_{L^p(\partial Q(0,R))} \leq C\epsilon$$

depending on whether $\nu \cdot A\nabla u = 0$ or $\tau \cdot \nabla u = 0$ on $\partial\Omega$. (If $p = 1$, then the $H^1(\partial Q(0,R))$ norm is at most $C\epsilon$ as well.)

If p is small enough, then there exists a v with $\operatorname{div} A\nabla v = 0$ in $Q(0,R)$, $v = u$ or $\nu \cdot A\nabla v = \nu \cdot A\nabla u$ on $\partial Q(0,R)$, and $\|N(\nabla v)\|_{L^p(\partial\Omega_R)} \leq C\epsilon$. Since uniqueness holds in bounded Lipschitz domains, $u = v$ and so by (3.1) $|\nabla u(X)| \leq C\epsilon \operatorname{dist}(X, \partial\Omega)^{-1/p}$ for all $|X| \leq R_0/C$. By taking the limits as $R_0 \to \infty$ and $\epsilon \to 0$, we see that $\nabla u \equiv 0$, as desired. $\square$

12.2 L^p uniqueness for the Dirichlet problem

Theorem 12.2. *Let V be a bounded good Lipschitz domain. Assume that $p > 1$ is small enough that $(R)_p^{A^T}$ holds in all bounded Lipschitz domains with constants at most $C(k_i)$, where the k_i are the Lipschitz constants of V, and $C(k_i)$ is a constant depending on the k_i (to be chosen later).*

Then if $\operatorname{div} A\nabla u = 0$ in V, $Nu \in L^q(\partial V)$, and $u \equiv 0$ on ∂V, then $u \equiv 0$ in V.

Remark 12.3. As in the proof of Theorem 12.1, if this theorem holds, and in addition $q < \infty$ is large enough that $(D)_q^A$ holds in all of the subdomains $Q(0,R)$, then we have uniqueness in special Lipschitz domains.

Proof. Define $V_\delta = \{X \in V : \operatorname{dist}(X, \partial V) > \delta\}$ as before. By the dominated convergence theorem, for any $\epsilon > 0$ we can find some δ_0 such that, if $\delta < \delta_0$, then $\|u\|_{L^q(\partial V_\delta)} < \epsilon$. The proof will work (roughly) by constructing the Green's function in V_δ and bounding its normal derivative in $L^p(\partial V_\delta)$ uniformly in δ, which will force $u \equiv 0$.

Let $f_\delta = u|_{\partial V_\delta}$. By continuity of u, we know that f_δ is bounded (if large).

Let $v_\delta(X) = u(X)\eta_\delta(X)$, where $\eta_\delta \equiv 1$ on V_δ and $\eta_\delta \in C_0^\infty(V_{\delta/2})$. We may extend v_δ by 0; we then have that v_δ is continuous on $\mathbf{R}^2$, compactly supported, and has a bounded gradient.

Therefore, if $X \in V_\delta$, then

$$u(X) = v_\delta(X) = -\int \nabla v_\delta \cdot A^T \nabla \Gamma_X^T = -\int_{V_\delta} A\nabla v_\delta \cdot \nabla \Gamma_X^T - \int_{V_\delta^C} \nabla v_\delta \cdot A^T \nabla \Gamma_X^T$$

The second integral is simply

$$\int_{V_\delta^C} \nabla v_\delta \cdot A^T \nabla \Gamma_X^T = -\int_{\partial V_\delta} v_\delta \, \nu \cdot A^T \nabla \Gamma_X^T \, d\sigma$$

and $\|\nu \cdot A^T \nabla \Gamma_X^T\|_{L^p(\partial V_\delta)} \leq C\|\nabla \Gamma_X^T\|_{L^p(\partial V_\delta)} \leq C\sigma(\partial V^\delta)^{1/p}/\operatorname{dist}(X, \partial V_\delta)$ by (4.1). Thus this integral is at most $\epsilon C\sigma(\partial V^\delta)^{1/p}/\operatorname{dist}(X, \partial V_\delta)$, which goes to zero pointwise as $\epsilon \to 0$.

Again by (4.1), we have that on ∂V_δ, $\tau \cdot \nabla \Gamma_X$ is bounded and integrates to zero (hence is in H^1) and satisfies $\|\partial_\tau \Gamma_X\|_{L^p(\partial V)} \leq C\sigma(\partial V^\delta)^{1/p}/\operatorname{dist}(X, \partial V_\delta)$. Let Φ_X be the solution to $(R)_p^{A^T}$ in V_δ with boundary data $\Phi_X = \Gamma_X^T$. This means that $\|N(\nabla \Phi_X)\|_{L^p} \leq C\sigma(\partial V^\delta)^{1/p}/\operatorname{dist}(X, \partial V_\delta)$, and so Φ_X is bounded in V_δ and $\nabla \Phi_X \in L^2_{loc}$.

Therefore,

$$\begin{aligned}\int_{V_\delta} A\nabla v_\delta \cdot \nabla \Gamma_X^T &= \int_{\partial V_\delta} \Gamma_X^T \, \nu \cdot A\nabla v_\delta = \int_{\partial V_\delta} \Phi_X \, \nu \cdot A\nabla v_\delta \, d\sigma = \int_{V_\delta} \nabla \Phi_X \cdot A\nabla v_\delta \\ &= \int_{\partial V_\delta} v_\delta \, \nu \cdot A^T \nabla \Phi_X \, d\sigma = \int_{\partial V_\delta} u \, \nu \cdot A^T \nabla \Phi_X \, d\sigma\end{aligned}$$

Again, this is at most $\epsilon C\sigma(\partial V^\delta)^{1/p}/\operatorname{dist}(X, \partial V_\delta)$ and so we see that $u(X) = 0$ for all $X \in V$. □

Theorem 12.4. *Suppose that V is a good Lipschitz domain. Let f be defined on ∂V such that $f \in L^q(\partial V)$, $\partial_\tau f \in L^{\check{p}}(\partial V)$, where $q \leq \infty$ is large enough, $\check{p} \geq 1$ is small enough that solutions to $(D)_q^A$, $(R)_{\check{p}}^A$ exist and are unique in V (and, if $V = \Omega$ is special, in all the $Q(0, R)$s).*

If u is the solution to $(D)_p^A$ and v is the solution to $(R)_p^A$ with boundary data f, then $u \equiv v$ in V.

In this theorem, we do not require that $\check{p}$, q be conjugate.

Proof. If ∂V is compact then v is bounded in compact sets by Lemma 3.3. If V^C is bounded then $\lim_{|X| \to \infty} v(X)$ exists by definition of regularity solution, so in any case v is bounded in V. But then Nv is bounded. Since ∂V is bounded, v is a Dirichlet solution and we need only apply Theorem 12.2.

If $V = \Omega$ is a special Lipschitz domain, then for every $R_0 > 0$, $\epsilon > 0$, there is some $R > R_0$ such that $\lim_{Z \to \psi(\pm R) \text{ n.t.}} u(Z) = u(\psi(\pm R))$, $Nu(\psi(\pm R)) < \epsilon R^{-1/q}$, $N(\nabla v)(\pm R) < \epsilon R^{-1/\check{p}}$.

Define u_R in $Q(0, R)$ as follows: $\operatorname{div} A\nabla u_R = 0$ in $Q(0, R)$, $u_R = u = v$ on $\partial Q(0, R) \cap \partial \Omega$, and on $\partial Q(0, R) \setminus \partial \Omega$, $u_R = 0$ except for the two segments of length $R^{1-1/q}$ near $\partial \Omega$, where

u_R is to decrease linearly from $u(\psi(\pm R))$ to zero. Then $|\partial_\tau u_R| < \epsilon/R$ on $\partial Q(0,R) \setminus \partial\Omega$, so if R is large enough then

$$\|u_R\|_{L^q(\partial Q(0,R)\setminus\partial\Omega)} \leq 2\epsilon R^{-1/q^2} < \epsilon, \quad \|\partial_\tau u_R\|_{L^{\check{p}}(\partial Q(0,R)\setminus\partial\Omega)} \leq 2\epsilon R^{1/\check{p}-1/q\check{p}-1} < \epsilon.$$

Since $Q(0,R)$ is bounded, u_R is both a Dirichlet and regularity solution, so

$$\|Nu_R\|_{L^q(\partial\Omega_R)} \leq \|u_R\|_{L^q(\partial\Omega_R)}, \quad \|N(\nabla u_R)\|_{L^{\check{p}}(\partial\Omega_R)} \leq \|\partial_\tau u_R\|_{L^{\check{p}}(\partial\Omega_R)}.$$

I claim that as $R_0 \to \infty$ and $\epsilon \to 0$, $\nabla u_R(X)$ approaches both $\nabla u(X)$ and $\nabla v(X)$ pointwise (not uniformly); this suffices to show that $\nabla u \equiv \nabla v$, and so $u \equiv v$ up to an additive constant (which must be 0).

First,

$$\|u - u_R\|_{L^q(\partial\Omega_R)} = \|u - u_R\|_{L^q(\partial\Omega_R\setminus\partial\Omega)} \leq \|u_R\|_{L^q(\partial\Omega_R\setminus\partial\Omega)} + \|u\|_{L^q(\partial\Omega_R\setminus\partial\Omega)} \leq C\epsilon,$$

and so $\|N(u-u_R)\|_{L^q} \leq C\epsilon$; therefore, if $X \in Q(0,R/C)$ for C large enough,

$$|u(X) - u_R(X)| \leq C\epsilon \operatorname{dist}(X,\partial\Omega_R)^{-1/q} = C\epsilon \operatorname{dist}(X,\partial\Omega)^{-1/q}.$$

Therefore, by Lemma 3.4,

$$|\nabla u(X) - \nabla u_R(X)| \leq C\epsilon \operatorname{dist}(X,\partial\Omega)^{-1-1/q}.$$

Next, note that if $\check{p} > 1$,

$$\begin{aligned}\|\partial_\tau v - \partial_\tau u_R\|_{L^{\check{p}}(\partial\Omega_R)} &= \|\partial_\tau v - \partial_\tau u_R\|_{L^{\check{p}}(\partial\Omega_R\setminus\partial\Omega)} \\ &\leq \|\partial_\tau u_R\|_{L^{\check{p}}(\partial\Omega_R\setminus\partial\Omega)} + \|\partial_\tau v\|_{L^{\check{p}}(\partial\Omega_R\setminus\partial\Omega)} \leq C\epsilon.\end{aligned}$$

So by Theorem 12.1, $\|N(\nabla v - \nabla u_R)\|_{L^q} \leq C\epsilon$.

If $\check{p} = 1$ then

$$\|\partial_\tau u_R - \partial_\tau v\|_{H^1(\partial\Omega_R)} = \|\partial_\tau u_R - \partial_\tau v\|_{H^1(\partial\Omega_R\setminus\partial\Omega)}$$

This is a set of size at most CR, and $|\partial_\tau u_R|$, $|\partial_\tau v|$ are each of size at most ϵ/R. So $\|N(\nabla u_R - \nabla v)\|_{L^1(\partial\Omega_R)} \leq C\|\partial_\tau u_R - \partial_\tau v\|_{H^1} \leq C\epsilon$.

In either case, if $X \in Q(0, R/C)$, then

$$|\nabla u(X) - \nabla u_R(X)| \leq C\epsilon \operatorname{dist}(X, \partial Q(0,R))^{-1/\check{p}} = C\epsilon \operatorname{dist}(X, \partial\Omega)^{-1/\check{p}}.$$

Thus, by letting $\epsilon \to 0$, we see that $\nabla u(X) \equiv \nabla v(X)$, as desired. □

In Chapter 13, we will use the preceding theorems in bounded Lipschitz domains to prove a maximum principle. We may use it to establish results for bounded solutions.

Lemma 12.5. *If V is a good Lipschitz domain and u is a bounded solution to* $\operatorname{div} A\nabla u = 0$ *in V, and if $u \equiv 0$ on ∂V, then $u \equiv 0$ in V.*

Proof. If ∂V is bounded then the theorem follows trivially from Theorem 12.2. Without loss of generality assume that $\|u\|_{L^\infty} \leq 1$.

If we knew that $\nabla u \in L^2_{loc}$, then we could use Lemma 3.4 to show that $\|\nabla u\|_{L^2(\Omega)} \leq C$. Unfortunately, we do not.

Pick some $X \in \partial V$, and let R be small enough that $Q(X,R)$ exists. For every ϵ, $R > 0$, let $u_{R,\epsilon} = u$ on $\partial Q(X,R)$, except on $\{X \in \partial Q(X,R) : 0 < \operatorname{dist}(X, \partial\Omega) < \epsilon\}$; on this set $u_{R,\epsilon}$ is to increase from 0 to u smoothly. Take $\operatorname{div} A\nabla u_{R,\epsilon} = 0$. By Chapter 13, $\|u_{R,\epsilon}\|_{L^\infty(Q(X,R))} \leq C$.

Since u is bounded, we know that for q large enough, $\|N_{Q(X,R)}(u - u_{R,\epsilon})\|_{L^q} \leq C\epsilon^{1/q}$. By Lemma 3.4, (3.1) and (3.8), if $Y \in Q(X,R)$, then

$$|\nabla u(Y) - \nabla u_{R,\epsilon}(Y)| \leq \frac{C\epsilon^{1/q}}{\operatorname{dist}(Y, \partial Q(X,R))^{1+1/q}}.$$

From Lemma 3.4 and (3.8), we know that $|\nabla u(X)| \leq C/\operatorname{dist}(X, \partial V)$. So $|\partial_\tau u_{R,\epsilon}| \leq C/\epsilon$ on $\partial Q(X,R)$. We know from Theorem 12.4 that $u_{R,\epsilon}$ must equal the regularity solution, and so by Lemma 3.3 $\nabla u_{R,\epsilon} \in L^2(Q(X,R))$ (with norm at most CR/ϵ).

So we may apply Lemma 3.4 to get that

$$\int_{Q(X,R/2)} |\nabla u_{R,\epsilon}|^2 \leq C \fint_{Q(X,R)} |u_{R,\epsilon}|^2 \leq C.$$

So if $E \subset Q(X, R/2)$ with $\operatorname{dist}(E, \partial Q(X, R/2)) > \delta$, then

$$\int_E |\nabla u|^2 \leq 2 \int_E |\nabla u - \nabla u_{R,\epsilon}|^2 + 2 \int_E |\nabla u_{R,\epsilon}|^2 \leq \frac{C|E|\epsilon^{1/q}}{\delta^{1+1/q}} + C.$$

By letting $\epsilon \to 0$ and then letting $\delta \to 0$, we see that $\nabla u \in L^2(Q(X, R/2))$ for all $X \in \partial V$ and all R small enough that $Q(X, R)$ exists. So $\nabla u \in L^2_{loc}(V)$; so we may use Lemma 3.4 to see that $\nabla u \equiv 0$, so u is a constant (and in fact zero). ☐

Corollary 12.6. *Suppose that $(D)^A_q$ holds in a good Lipschitz domain V with bounded complement. Suppose that $Nu \in L^q$, $\operatorname{div} A\nabla u = 0$ in V, and $u = 0$ on ∂V. Then $u \equiv 0$ in V.*

Proof. We need only show that u is bounded in V. But by (3.1),

$$|u(X)| \leq \|Nu\|_{L^q} / \min(\operatorname{dist}(X, \partial V), \sigma(\partial V))^{1/q}$$

so we need only bound u near ∂V.

Let $X \in \partial V$, and let r be small enough that $Q(X, r)$ exists. Assume $Nu(\chi_\pm(X, r)) < \infty$. Then by Chapter 13, u is bounded in $Q(X, r)$. For any sufficiently small $\delta > 0$, we may cover $\{X \in V : \operatorname{dist}(X, \partial V) < \delta\}$ by finitely many such $Q(X, r)$, and so u is bounded on V, as desired. ☐

12.3 Square-function converse and uniqueness

In this section, we will prove the following theorem:

Theorem 12.7. *Suppose that V is a bounded Lipschitz domain,* $\operatorname{div} A\nabla u = 0$ *in V, and*

$$\sup_{Y_0 \in \partial V, R > 0} \frac{1}{\sigma(\partial V \cap B(Y_0, R))} \int_{V \cap B(Y_0, R)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY \leq \tilde{C}^2. \tag{12.8}$$

If $\|A - A_0\|_{L^\infty}$ is small enough, $q < \infty$ is large enough, and if $X_0 \in V$, then

$$\left(\fint_{\partial V} N(u - u(X_0))^q \, d\sigma \right)^{1/q} \leq C\tilde{C} \frac{\sigma(\partial V)}{\operatorname{dist}(X_0, \partial V)}. \tag{12.9}$$

To prove this theorem, we need the following lemma:

Lemma 12.10. *If (12.8) holds in V, and $U \subset V$ is a bounded Lipschitz domain, then (12.8) holds in U as well, with constants that may depend on the Lipschitz constants of V.*

The theorem and lemma together have several useful corollaries:

Corollary 12.11. *Suppose that V is a good Lipschitz domain, $\operatorname{div} A\nabla u = 0$ in V, and u satisfies (12.8).*

If $\|A - A_0\|_{L^\infty}$ is small enough, then $u|_{\partial V} \in BMO(\partial V)$ with BMO norm at most $C\tilde{C}$.

Corollary 12.12. *Suppose that u satisfies (12.8) in a good Lipschitz domain V and that $u|_{\partial V}$ is a constant. Then u is constant.*

Proof of Corollary 12.11. Let $\Delta \subset \partial V$ be connected. We need to show that $\fint_\Delta |u - u_\Delta|\,d\sigma \leq C\tilde{C}$ for some constant u_Δ.

If ∂V is compact, we may assume that $\sigma(\Delta) < \sigma(\partial V)/C$. So in particular, we may assume that Δ is small enough that $\Delta = \partial Q(X,r) \cap \partial V$ for some $X \in \partial V$ and some r small enough that $Q(X,r)$ is defined.

By Lemma 12.10, (12.8) holds in $Q(X,r)$. Let $X_0 \in Q(X,r)$ with $\operatorname{dist}(X_0, \partial Q(X,r)) \approx r \approx \sigma(\partial V)$. Then if q is as in Theorem 12.7,

$$\fint_\Delta |u - u(X_0)|\,d\sigma \leq \left(\fint_\Delta |u - u(X_0)|^q\,d\sigma\right)^{1/q} \leq \left(\fint_{\partial Q(X,r)} N(u - u(X_0))^q\,d\sigma\right)^{1/q} \leq C\tilde{C}.$$

Choosing $u_\Delta = u(X_0)$ completes the proof. □

Proof of Corollary 12.12. By Lemma 12.5, we need only show that u is bounded. If $V = \Omega$ is a special Lipschitz domain and $X \in \Omega$, then $X = \psi(x,t)$ for some $x \in \mathbf{R}$, $t > 0$. Let $X^* = \psi(x,0)$. By Lemma 12.10, (12.8) holds in $Q(X^*, 2t)$. But $\operatorname{dist}(X, \partial Q(X^*, 2t)) \approx t \approx \sigma(\partial Q(X^*, 2t))$, so

$$\begin{aligned} |u(X)| &\leq \left| u(X) - \fint_{\Delta(X^*,2t)} u \right| + \|u\|_{L^\infty(\partial\Omega)} \\ &\leq \|u\|_{L^\infty(\partial\Omega)} + C\left(\fint_{\partial Q(X^*,2t)} N(u - u(X))^q\,d\sigma\right)^{1/q} \leq \|u\|_{L^\infty(\partial\Omega)} + C\tilde{C} \end{aligned}$$

and so u is bounded in Ω.

If ∂V is bounded, examining the domains $Q(X,r)$ for small r as above shows that u is bounded near ∂V. Let R_0 be large enough that $\partial V \subset B(0,R_0)$. Then since u is continuous away from ∂V (Lemma 3.7), we can bound u inside of $B(0,3R_0)$ but away from ∂V as well; so we can bound u on all of $B(0,3R_0)\cap V$.

If V is bounded we are done. If V is not bounded, then if $|X|>3R_0$,

$$\frac{1}{\sigma(\partial V)}\int_{B(X,|X|/3)} |\nabla u|^2 \frac{|X|}{3} \leq C\tilde{C}^2.$$

So by (3.8), $|\nabla u(X)| \leq C\sqrt{\tilde{C}^2\sigma(\partial V)/|X|^3}$ for $|X|>3R_0$. Thus $\lim_{|Y|\to\infty} u(Y)$ exists, and

$$\big|u(X) - \lim_{|Y|\to\infty} u(Y)\big| \leq \frac{C\tilde{C}\sqrt{\sigma(\partial V)}}{\sqrt{|X|}}$$

for $|X|>3R_0$. So u is bounded on all of V. □

We now prove the theorem and the lemma.

Proof of Lemma 12.10. Let $Y_0\in\partial U$, and let $R>0$. We are trying to establish (12.8) with V replaced by U. We need only show that

$$\frac{1}{\sigma(\partial U\cap B(Y_0,R))}\int_{U\cap B(Y_0,R)} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial U)\,dY \\ \leq \frac{1}{\sigma(\partial V\cap B(Y_0^*,R^*))}\int_{V\cap B(Y_0^*,R^*)} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial V)\,dY$$

for some $R^*>0$ and some $Y_0^*\in\partial V$.

Either $\partial U\subset B(Y_0,R)$ or $\sigma(\partial U\cap B(Y_0,R))\geq 2R$; so if $R\leq\sigma(\partial U)$, then $R\leq\sigma(\partial U\cap B(Y_0,R))$. Since U is bounded, we have that $\bar{U}\subset B(Y_0,\sigma(\partial U))$ for any $Y_0\in\bar{U}$; so we may assume that $R\leq\sigma(\partial U)$. Since we allow our constants to depend on the Lipschitz constants of V (but not of U), we have that $\sigma(\partial V\cap B(Y_0^*,R^*))\leq CR^*$.

So we need only show that

$$\frac{1}{R}\int_{U\cap B(Y_0,R)} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial U)\,dY \leq \frac{1}{R^*}\int_{V\cap B(Y_0^*,R^*)} |\nabla u(Y)|^2 \operatorname{dist}(Y,\partial V)\,dY.$$

If $Y_0 \in \partial V$, then since $U \subset V$, we have that $\operatorname{dist}(Y, \partial U) \leq \operatorname{dist}(Y, \partial V)$ for all $Y \in U$, and that $U \cap B(Y_0, R) \subset V \cap B(Y_0, R)$. So

$$\frac{1}{R} \int_{U \cap B(Y_0,R)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial U)\, dY \leq \frac{1}{R} \int_{V \cap B(Y_0,R)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY.$$

Suppose $Y_0 \in \partial U \backslash \partial V$. Let $Y_0^* \in \partial V$ with $|Y_0 - Y_0^*| = \operatorname{dist}(Y_0, \partial V)$. If $R \geq \frac{1}{2} \operatorname{dist}(Y_0, \partial V)$, then

$$\frac{1}{R} \int_{U \cap B(Y_0,R)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial U)\, dY \leq 3 \frac{1}{3R} \int_{V \cap B(Y_0^*,3R)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY.$$

If $R < \frac{1}{2} \operatorname{dist}(Y_0, \partial V)$, then let $R^* = \operatorname{dist}(Y_0, \partial V) + R$. So $R^* > 3R$, and for all $Y \in B(Y_0, R)$ we have that

$$\frac{1}{R} \operatorname{dist}(Y, \partial U) \leq \frac{|Y - Y_0|}{R} < 1 \leq \frac{\operatorname{dist}(Y, \partial V)}{\operatorname{dist}(Y_0, \partial V) - |Y - Y_0|} \leq \frac{\operatorname{dist}(Y, \partial V)}{R^* - 2R} \leq 3 \frac{\operatorname{dist}(Y, \partial V)}{R^*}$$

and so

$$\frac{1}{R} \int_{U \cap B(Y_0,R)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial U)\, dY \leq \frac{3}{R^*} \int_{V \cap B(Y_0^*,R^*)} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY. \quad \square$$

Proof of Theorem 12.7. First, assume that A is smooth and that $u = Df$ for some $f \in BMO(\partial V)$. Recall that $\mathcal{K}^t$ is bounded and invertible on H^1, so $\|\mathcal{K} f\|_{BMO} \approx \|f\|_{BMO}$.

We know from [24, Section 3] that if A_0 is real, and $\operatorname{div} A_0 \nabla v = 0$ in a bounded Lipschitz domain W, then

$$\int_{\partial W} |N(v - v(X_0))|^2 \, d\sigma(X) \leq C \int_W |\nabla v(X)|^2 \operatorname{dist}(X, \partial W)\, dX$$

for any $X_0 \in W$ with $\operatorname{dist}(X_0, \partial W) \geq \frac{1}{C} \sigma(\partial W)$.

Let $\mathcal{K}_0 = \mathcal{K}_V^{A_0}$, $D_0 = D_V^{A_0}$. We can bound $\|\mathcal{K}_0 f\|_{BMO}$ as follows. Pick some $\Delta \subset \partial V$; as usual we may assume that $\Delta = \Delta(X, r)$ for some r small enough that $Q(X, r)$ exists. Let $X_\Delta = X + \frac{1}{2} r\mathbf{e}$, so $r \approx \operatorname{dist}(X_\Delta, \Delta) \approx \operatorname{dist}(X_\Delta, \partial V) \approx \operatorname{dist}(X_\Delta, \partial Q(X, r)) \approx \sigma(\Delta) \approx$

$\sigma(\partial Q(X,r))$. Then

$$\begin{aligned}\left(\fint_{\Delta} |\mathcal{K}_0 f - D_0 f(X_\Delta)|\, d\sigma\right)^2 &\le \fint_{\Delta} |\mathcal{K}_0 f - D_0 f(X_\Delta)|^2\, d\sigma \\ &\le \frac{C}{\sigma(\Delta)} \int_{\partial Q(X,r)} N(D_0 f - D_0 f(X_\Delta))^2 \\ &\le \frac{C}{\sigma(\Delta)} \int_{Q(X,r)} |\nabla D_0 f(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY \\ &\le \frac{C}{\sigma(\Delta)} \int_{B(X, C\sigma(\Delta))\cap V} |\nabla D_0 f(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY.\end{aligned}$$

This implies that

$$\|\mathcal{K}_0 f\|^2_{BMO(\partial V)} \le \sup_{X\in\partial V,\, R>0} \frac{C}{R} \int_{B(X,R)\cap V} |\nabla D_0 f(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY.$$

Recall from Chapter 10 that

$$\sup_{X\in\partial V,\, R>0} \frac{1}{R} \int_{B(X,R)\cap V} |\nabla D f(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY \le C\|f\|_{BMO}.$$

So by analyticity, if we let $\alpha = \|A - A_0\|_{L^\infty} \|f\|^2_{BMO} \approx \|A - A_0\|_{L^\infty} \|\mathcal{K} f\|^2_{BMO}$,

$$\begin{aligned}\|\mathcal{K} f\|^2_{BMO} &\le \|\mathcal{K}_0 f\|^2_{BMO} + C\alpha \\ &\le \sup_{X\in\partial V,\, R>0} \frac{C}{R} \int_{B(X,R)\cap V} |\nabla D_0 f(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY + C\alpha \\ &\le \sup_{X\in\partial V,\, R>0} \frac{C}{R} \int_{B(X,R)\cap V} |\nabla D f(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY + C\alpha \\ &= \sup_{X\in\partial V,\, R>0} \frac{C}{R} \int_{B(X,R)\cap V} |\nabla u(Y)|^2 \operatorname{dist}(Y, \partial V)\, dY + C\alpha \\ &\le C\tilde{C}^2 + C\alpha \le C\tilde{C}^2 + C\|A - A_0\|_{L^\infty} \|\mathcal{K} f\|^2_{BMO}.\end{aligned}$$

Thus if $\|A - A_0\|_{L^\infty}$ is small enough, we may hide the last term. Thus, $\mathcal{K} f = u|_{\partial V}$ is in BMO with norm $C\tilde{C}$.

But since ∂V is compact, by the John-Nirenberg inequality ([27, p. 144]), if $1 \le q < \infty$, then for $C_f = \fint_{\partial V} \mathcal{K} f$ we have that

$$\|\mathcal{K} f - C_f\|_{L^p(\partial V)} \leq C\|\mathcal{K} f\|_{BMO(\partial V)} \sigma(\partial V)^{1/p} \leq C\tilde{C}\sigma(\partial V)^{1/p}.$$

But since V is bounded, $\mathcal{K}1 = 1$, so since $\mathcal{K}$ is invertible on $L^q(\partial V)$,

$$\|f - C_f\|_{L^q(\partial V)} \leq C\|\mathcal{K}(f - C_f)\|_{L^q(\partial V)} \leq C\|\mathcal{K} f - C_f\|_{L^q(\partial V)} \leq C\tilde{C}\sigma(\partial V)^{1/q}.$$

So by Theorem 5.12, $\|N(Df - C_f)\|_{L^q(\partial V)} \leq C\tilde{C}\sigma(\partial V)^{1/q}$.

We may restate our goal (12.9) as

$$\|N(Df - Df(X_0))\|_{L^q(\partial V)} \leq C\frac{\sigma(\partial V)}{\operatorname{dist}(X_0, \partial V)}\tilde{C}\sigma(\partial V)^{1/q}.$$

But since $D1 \equiv 1$,

$$\begin{aligned}
|Df(X_0) - C_f| = |D(f - C_f)(X_0)| &= \left|\int_{\partial V} \nu \cdot A^T \nabla \Gamma_{X_0}^T (f - C_f)\, d\sigma\right| \\
&\leq \int_{\partial V} \frac{C}{\operatorname{dist}(X_0, \partial V)} |f - C_f|\, d\sigma \\
&\leq \frac{C\sigma(\partial V)}{\operatorname{dist}(X_0, \partial V)} \fint_{\partial V} |f - C_f|\, d\sigma \\
&\leq \frac{C\sigma(\partial V)}{\operatorname{dist}(X_0, \partial V)} \|f\|_{BMO} \leq \frac{C\sigma(\partial V)}{\operatorname{dist}(X_0, \partial V)}\tilde{C}.
\end{aligned}$$

So

$$\begin{aligned}
\|N(Df - Df(X_0))\|_{L^q(\partial V)} &\leq \|N(Df - C_f)\|_{L^q(\partial V)} + |C_f - Df(X_0)|\sigma(\partial V)^{1/q} \\
&\leq C\frac{\sigma(\partial V)}{\operatorname{dist}(X_0, \partial V)}\tilde{C}\sigma(\partial V)^{1/q}
\end{aligned}$$

and the theorem holds for $u = Df$, $f \in BMO$, and A smooth.

We now move on to more general u and A. Let $V_\delta = \{X \in V : \operatorname{dist}(X, \partial V) > \delta\}$. By Lemma 12.10, u satisfies (12.8) in V_δ. But $\overline{V_\delta} \subset V$, so ∇u is bounded on $\overline{V_\delta}$; thus $g_\delta = u|_{\partial V_\delta} \in L^\infty \subset BMO$. Furthermore, $\partial_\tau g_\delta$ is bounded.

Let A_η be as in Chapter 11, and let $D_\delta^\eta = D_{V_\delta}^{A_\eta}$, $\mathcal{K}_\delta^\eta = \mathcal{K}_{V_\delta}^{A_\eta}$.

Then $g_\delta = \mathcal{K}^\eta_\delta f_\delta$ for some $f_\delta \in BMO \cap L^q$, where q is large. Let $u^\eta_\delta = D^\eta_\delta f^\eta_\delta$, so (12.9) holds for u^η_δ.

Then $u^\eta_\delta = u$ on ∂V_δ. Furthermore, $\partial_\tau u^\eta_\delta \in L^p$ for some p small enough that $(R)^A_p$ and Theorem 12.4 hold for all the A^ηs in all the V_δs.

So as in Theorem 11.1, for any fixed δ,

$$\int_{V_\delta} |\nabla u - \nabla u^\eta_\delta|^2 \to 0$$

as $\eta \to 0$, so by (3.8), $|\nabla u(X) - \nabla u^\eta_\delta(X)| \leq o(\eta)/\operatorname{dist}(X, \partial V_\delta)$.

So for any $W \subset V_\delta$,

$$\begin{aligned}
\int_W & |\nabla u^\eta_\delta(X)|^2 \operatorname{dist}(X, \partial V_\delta)\, dX \\
&\leq \int_W |\nabla u^\eta_\delta(X) - \nabla u(X)|^2 \operatorname{dist}(X, \partial V_\delta)\, dX + \int_W |\nabla u(X)|^2 \operatorname{dist}(X, \partial V_\delta)\, dX \\
&\leq o(\eta) + \int_W |\nabla u(X)|^2 \operatorname{dist}(X, \partial V_\delta)\, dX.
\end{aligned}$$

In particular,

$$\sup_{X_0 \in \partial V_\delta,\, R>0} \frac{1}{R} \int_{B(X_0,R)\cap \partial V_\delta} |\nabla u^\eta_\delta(X)|^2 \operatorname{dist}(X, \partial V_\delta)\, dX \leq C\tilde{C}^2 + o(\eta)$$

so if $X_0 \in V_\delta$,

$$\left(\fint_{\partial V_\delta} N(u^\eta_\delta - u^\eta_\delta(X_0))^p \, d\sigma \right)^{1/p} \leq (C\tilde{C} + o(\eta)) \frac{\sigma(\partial V_\delta)}{\operatorname{dist}(X_0, \partial V_\delta)}.$$

But as in Theorem 11.3, $u^\eta_\delta(X) - u(X) \to 0$ as $\eta \to 0$, so

$$\left(\fint_{\partial V_\delta} N(u - u(X_0))^p \, d\sigma \right)^{1/p} \leq (C\tilde{C} + o(\eta)) \frac{\sigma(\partial V_\delta)}{\operatorname{dist}(X_0, \partial V_\delta)}.$$

Letting $\eta \to 0$ and then letting $\delta \to 0$ completes the proof. $\square$

12.4 The L^1 converse

We want to show that, if $N(\nabla u) \in L^1(\partial V)$, then $\nu \cdot A\nabla u$, $\partial_\tau u$ exist (in some weak sense) on ∂V and have H^1 norms at most $C\|N(\nabla u)\|_{L^1}$.

We will use the fact that H^1 is dual to BMO. We will need a lemma that lets us extend functions in $BMO(\partial V)$ to functions defined on V with certain nice properties:

Lemma 12.13. *Suppose that V is a good Lipschitz domain and that f is compactly supported on ∂V with $\partial_\tau f$ bounded. Then there exists some function F, with ∇F bounded in V, such that $F \to f$ nontangentially a.e. in ∂V, and such that*

$$\|\nabla F\|_{\mathcal{C}} = \sup_{X_0 \in \partial V, R>0} \frac{1}{\sigma(B(X_0,R)\cap \partial V)} \int_{B(X_0,R)\cap V} |\nabla F| \leq C\|f\|_{BMO}$$

where the constant C depends only on the Lipschitz constants of V.

If ∂V is unbounded then F is compactly supported; otherwise, $F - \fint_{\partial V} f$ is compactly supported.

Proof. We begin with the half-plane $\mathbf{R}^2_+$. By [28] and [29], if f is a compactly supported function on $\mathbf{R} = \partial\mathbf{R}^2_+$, with $\|f\|_{BMO(\mathbf{R})} = 1$, then there exists an $F \in C^\infty(\mathbf{R}^2_+)$ such that $\lim_{t\to 0} F(x,t) = f(x)$ for a.e. $x \in \mathbf{R}$ and $|\nabla F(x,t)|\,dx\,dt$ is a Carleson measure on $\mathbf{R}^2_+$.

We will need a few nice properties of this function F. By carefully examining the construction in those papers, we see that we may require $|\nabla F(x,t)| \leq C/t$.

Furthermore, we have that $|F(x,t) - \fint_{x-t}^{x+t} f(y)\,dy| \leq C\|f\|_{BMO}$. To see this fact, we must examine the construction more carefully; we delay this until the end of the proof.

If $\partial_\tau f$ is bounded, then f is Lipschitz. We want to modify F so that ∇F is bounded. Let $\mu = \|f\|_{BMO}/\|f'\|_{L^\infty}$. Let G be the function on $\mathbf{R}^2_+$ constructed above, and let $F(x,t) = f(x)\eta(t) + G(x,t)(1-\eta(t))$, where $\eta = 1$ on $[0,\mu]$, $\eta = 0$ on $[2\mu,\infty)$, and $|\eta'| \leq 2/\mu$.

I claim that F satisfies our desired conditions. Clearly, $F \to f$ nontangentially everywhere; we need only show that $|\nabla F|$ is bounded and $|\nabla F(x,t)|\,dx\,dt$ is a Carleson measure.

First,

$$\nabla F(x,t) = \begin{pmatrix} f'(x)\eta(t) \\ (f(x) - F(x,t))\eta'(t) \end{pmatrix} + (1-\eta(t))\nabla G(x,t).$$

Since $|\nabla G(x,t)|\,dx\,dt$ is a Carleson measure, so is $|(1-\eta(t))\nabla G(x,t)|\,dx\,dt$. Furthermore, $|(1-\eta(t))\nabla G(x,t)| \leq |\nabla G(x,t)| \leq C\|f\|_{BMO}/t$; since this term is nonzero only for $t > \mu = \|f\|_{BMO}/\|f'\|_{L^\infty}$, this implies $|(1-\eta(t))\nabla G(x,t)| \leq |\nabla G(x,t)| \leq C\|f'\|_{L^\infty}$ as well.

Now, $|f'(x)\eta(t)| \leq \|f'\|_{L^\infty}$, and

$$\frac{1}{T}\int_X^{X+T}\int_0^T |f'(x)|\eta(t)\,dt\,dx \leq \|f'\|_{L^\infty} 2\mu = 2\|f\|_{BMO}.$$

So $|f'(x)\eta(t)|$ is bounded and a Carleson measure.

We are left with the term $(f(x) - G(x,t))\eta'(t)$. But

$$|f(x) - G(x,t)| \leq \left| f(x) - \fint_{x-t}^{x+t} f(y)\,dy \right| + \left| \fint_{x-t}^{x+t} f(y)\,dy - G(x,t) \right| \leq t\|f'\|_{L^\infty} + C\|f\|_{BMO}$$

But since $\eta'(t)$ is nonzero only for $\mu < t < 2\mu$, we have that $|f(x) - G(x,t))\eta'(t)| < C\|f'\|_{L^\infty}$ and also $|f(x) - G(x,t))\eta'(t)| < C\|f\|_{BMO}/t$.

We need only show that this term is a Carleson measure. But if $T > 0$,

$$\frac{1}{T}\int_{x-T}^{x+T}\int_0^T |(f(y) - G(y,t))\eta'(t)|\,dt\,dy \leq \frac{1}{T}\int_{x-T}^{x+T}\int_\mu^{2\mu} \frac{C\|f\|_{BMO}}{t}\,dt\,dy \leq C\|f\|_{BMO}$$

as desired.

We now show that we may restrict the support of F by multiplying F by a cutoff function. Suppose f is supported in an interval $(a, a+b)$. Let η be smooth, supported in $(a-c, a+b+c) \times (0, b+c)$, with $\eta(y,s) = 1$ on $(a, a+b) \times (0,b)$, $|\eta| \leq 1$ and $|\nabla\eta| < 2/c$ everywhere. Let F be as above with $F \to f$ non-tangentially. Then $F\eta \to f$ nontangentially, and $|\nabla(F\eta)| \leq |\nabla F| + |F|\,|\nabla\eta|$.

If $(x,t) \in \operatorname{supp}\nabla\eta$, then either $f \equiv 0$ on $(x, x+t$ or $(x-t, x)$, or $\operatorname{supp} f \subset (x-t, x+t)$. In either case $\left|\fint_{x-t}^{x+t} f\right| \leq C\|f\|_{BMO}$, so $|F(x,t)| \leq C\|f\|_{BMO}$. We have that $\|\nabla F\|_{\mathcal{C}} \leq C\|f\|_{BMO}$ and $\|\nabla\eta\|_{\mathcal{C}} \leq C$, so $\|\nabla(F\eta)\|_{\mathcal{C}} \leq C\|f\|_{BMO}$.

So we may assume that F is supported in a small neighborhood of $\operatorname{supp} f \times (0, |\operatorname{supp} f|)$. This completes the proof for the upper half-plane.

We now pass to more general Lipschitz domains. If $f \in BMO(\partial\Omega)$ for Ω an arbitrary special Lipschitz domain, we may construct such an F by letting $g(x) = f(\psi(x))$, letting

$G \to g$ with $|\nabla G(x,t)|\,dx\,dt \in \mathcal{C}$ as before, and by letting

$$F(\psi(x,t)) = G(x,t) \Rightarrow F(x\mathbf{e}^\perp + t\mathbf{e}) = G(x, t-\varphi(x)).$$

Then $F \to f$ and $|\nabla F| \leq C|\nabla G|$, so

$$\|\,|\nabla F(x,t)|\,dx\,dt\|_{\mathcal{C}} \leq C\|f\|_{BMO}.$$

If V is a Lipschitz domain which is good but not special and $f \in BMO(\partial V)$, we may proceed as in the proof of [12, Lemma 2.3]. $\sigma(\partial V)$ is finite and so $f \in L^1(\partial V)$; we may assume without loss of generality that $\fint_{\partial V} f = 0$.

Let Ω_j, R_j, X_j be as in Definition 2.4. Let $\{\eta_j\}_{j=1}^{k_2}$ be a set of smooth, compactly supported functions such that $\sum_j \eta_j(X) = 1$ for all X with $\operatorname{dist}(X, \partial V) \leq \sigma(\partial V)/C$, $|\nabla\eta_j| \leq C/\sigma(\partial V)$ and $\operatorname{supp}\eta_j$ is contained in

$$\tilde{R}_j = \{X : |(X - X_j)\cdot \mathbf{e}_j^\perp| < \frac{3}{2}r_j, |(X-X_j)\cdot\mathbf{e}_j| \leq \frac{3}{2}(1+k_1)r_j\}.$$

Then $\|f\eta_j\|_{BMO(\partial V)} \leq C\|f\|_{BMO(\partial V)}$. Let $F_j : \Omega_j \mapsto \mathbf{C}$ be the function constructed above in Ω_j with boundary value f_j. Assume F_j is supported in R_j. Then by adding the F_js, we get our desired F.

We now show that in the upper half-plane, we have that $|F(x,t) - \fint_{x-t}^{x+t} f(y)\,dy| \leq C\|f\|_{BMO}$.

Without loss of generality take $\operatorname{supp} f \subset (0,1)$. We review the construction of [28] and [29] as follows: if $f \in BMO(\mathbf{R})$ and $I \subset \mathbf{R}$ is a dyadic interval, then there exists a family $W = W(I)$ of dyadic intervals $\omega \subset I$ and a function $\alpha : W \mapsto \mathbf{C}$ such that

- $|\alpha(\omega)| \leq C\|f\|_{BMO}$ for all intervals $\omega \in \Omega(I)$,
- $\sum_{\omega\subset\check{I},\ \omega\in\Omega(I)} |\omega| \leq C|\check{I}|$ for all intervals $\check{I} \subset I$, and
- $f = b + \fint_I f + \sum_{\omega\in W}\alpha(\omega)\chi_\omega$ for some function b such that $\|b\|_{L^\infty(I)} \leq C\|f\|_{BMO}$.

If $I \supset \operatorname{supp} f$, then $|\fint_I f| \leq C\|f\|_{BMO}$, so we may redefine b to include this term.

Let $W = W((0,1))$. Define

$$\check{F}(x,t) = \sum_{\omega \in W} \alpha_\omega \chi_\omega(x) \chi_{[0,\sigma(\omega)]}(t) + \fint_0^1 f.$$

Smooth $\check{F}$ by convolving with a smooth function and call the result $\tilde{F}$.

In [29], a smooth v is constructed such that $\|v\|_{L^\infty} \leq C\|b\|_{L^\infty}$, $|\nabla v(x,t)| \leq \|b\|_{L^\infty}/t$ and $\| \, |\nabla v(x,t)| \, dx \, dt\|_{\mathcal{C}} \leq C\|b\|_{L^\infty}$. Let $F = \tilde{F} + v$. See [28] and [29] for details; here we only remark that, if $I \ni x$ is the dyadic interval with $t < |I| \leq 2t$, then

$$\begin{aligned}
\left|\check{F}(x,t) - \fint_{x-t}^{x+t} f(y)\,dy\right| &\leq \left|\check{F}(x,t) - \fint_I f(y)\,dy\right| + \left|\fint_I f(y)\,dy - \fint_{x-t}^{x+t} f(y)\,dy\right| \\
&\leq \left|\check{F}(x,t) - \fint_I f(y) - b(y)\,dy\right| + \|b\|_{L^\infty} + C\|f\|_{BMO} \\
&= \left|\sum_{\omega \ni x,\, |\omega| > t} \alpha(\omega) - \sum_{\omega \in W} \alpha(\omega) \frac{|\omega \cap I|}{|I|}\right| + \|b\|_{L^\infty} + C\|f\|_{BMO} \\
&= \left|\sum_{\omega \subsetneq I} \alpha(\omega) \frac{|\omega|}{|I|}\right| + \|b\|_{L^\infty} + C\|f\|_{BMO} \leq C\|f\|_{BMO}.
\end{aligned}$$

So since $\tilde{F}$ is a convolution of $\check{F}$ with a smooth cutoff, we have that

$$\begin{aligned}
\left|F(x,t) - \fint_{x-t}^{x+t} f(y)\,dy\right| &\leq |v(x,t)| + |\tilde{F}(x,t) - \check{F}(x,t)| + \left|\check{F}(x,t) - \fint_{x-t}^{x+t} f(y)\,dy\right| \\
&\leq C\|b\|_{L^\infty} + C\|f\|_{BMO} \leq C\|f\|_{BMO}
\end{aligned}$$

as desired. □

Theorem 12.14. *Suppose that* $\operatorname{div} A\nabla u = 0$ *in* V *for some good Lipschitz domain, and assume* $N(\nabla u) \in L^1(\partial V)$.

If V *is bounded, or if* $V = \Omega$ *is a special Lipschitz domain, then*

$$\|\nu \cdot A\nabla u\|_{H^1(\partial V)} \leq C\|N(\nabla u)\|_{L^1(\partial V)}$$

in the sense that $\left|\int \eta \, \nu \cdot A\nabla u\right| \leq C\|N(\nabla u)\|_{L^1(\partial V)} \|\eta\|_{BMO(\partial V)}$ *for all* $\eta \in C_0^\infty(\mathbf{R}^2_+)$.

If ∂V is bounded but V is not, then $\nu \cdot A\nabla u - C_u$ is in H^1 for some constant C_u with $|C_u| \leq C\|N(\nabla u)\|_{L^1}/\sigma(\partial V)$.

Proof. By Lemma 3.3, $\nabla u \in L^2_{loc}(V)$, so $\nu \cdot A\nabla u$ exists in the weak sense. Let $f \in C_0^\infty(\partial V)$. To show that $\nu \cdot A\nabla u - C_u \in H^1(\partial V)$, it suffices to show that

$$\left|\int_{\partial V} f(\nu \cdot A\nabla u - C_u)\,d\sigma\right| \leq C\|f\|_{BMO}\|N(\nabla u)\|_{L^1}.$$

But if F is compactly supported, $\nabla F \in L^2$ and $\mathbf{Tr}\, F = f$ then

$$\int_{\partial V} f(\nu \cdot A\nabla u - C_u)\,d\sigma = \int_V \nabla F \cdot A\nabla u.$$

If V^C is bounded, let $C_u = \fint_{\partial V} \nu \cdot A\nabla u$. Then

$$\begin{aligned}\int_{\partial V} f(\nu \cdot A\nabla u - C_u)\,d\sigma &= \int_{\partial V} f\,\nu \cdot A\nabla u\,d\sigma - \int_{\partial V} f \fint_{\partial V} \nu \cdot A\nabla u\,d\sigma \\ &= \int_{\partial V} \left(f - \fint_{\partial V} f\right) \nu \cdot A\nabla u\,d\sigma\end{aligned}$$

and so in this case we replace f by $f - \fint_{\partial V} f\,d\sigma$.

Since f is Lipschitz and compactly supported, and if V^C is bounded then $\fint_{\partial V} f = 0$, the F of Lemma 12.13 is compactly supported and has bounded gradient, with $\|\nabla F\|_{\mathcal{C}} \leq C\|f\|_{BMO}$. So we need only show that

$$\left|\int_V \nabla F \cdot A\nabla u\right| \leq C\|\nabla F\|_{\mathcal{C}}\|N(\nabla u)\|_{L^1}.$$

We must review some basic theorems about Carleson measures. Let G, H be two functions. It is well known (see, for example, [27, Section II.2]) that

$$\int_{\mathbf{R}^2_+} |G|\,|H| \leq C\|G\|_{\mathcal{C}}\|NH\|_{L^1(\partial \mathbf{R}^2_+)}.$$

This clearly extends by a change of variables to special Lipschitz domains. If V is a good but not special Lipschitz domain, then the inequality holds if we integrate not over all of V but over the k_2 boundary cylinders $R_j \cap V$ of Definition 2.4.

If X is not in one of these boundary cylinders, then $\operatorname{dist}(X, \partial V) \geq \sigma(\partial V)/C$. Therefore, if $R > 0$ and $X_0 \in \partial V$ then

$$\begin{aligned}
\int_{B(X_0,R)\cap V} |G|\,|H| &\leq C\|NH\|_{L^1(\partial V)}\|G\|_{\mathcal{C}} + \int_{X\in B(X_0,R),\operatorname{dist}(X,\partial V)\geq\sigma(\partial V)/C} |G|\,|H| \\
&\leq C\|NH\|_{L^1(\partial V)}\|G\|_{\mathcal{C}} + C\frac{\|NH\|_{L^1(\partial V)}}{\sigma(\partial V)}\int_{B(X_0,R)\cap V} |G| \\
&\leq C\|NH\|_{L^1(\partial V)}\|G\|_{\mathcal{C}}.
\end{aligned}$$

Applying this to $G = \nabla F$ and $H = \nabla u$, and taking the limit as $R \to \infty$, we have that

$$\int_V |\nabla F|\,|\nabla u| \leq C\|f\|_{BMO}\|N(\nabla u)\|_{L^1}$$

as desired. □

We now move on to the regularity problem.

Theorem 12.15. *Suppose that* $\operatorname{div} A\nabla u = 0$ *in* V *and* $N(\nabla u) \in L^1(\partial V)$ *for some good Lipschitz domain* V. *Then* $f(X) = \lim_{Z\to X \text{ n.t.}} u(Z)$ *exists for almost every* $X \in \partial V$. *Furthermore,* $\partial_\tau f$ *exists in the weak sense and* $\|\partial_\tau f\|_{H^1} \leq C\|N(\nabla u)\|_{L^1}$.

Proof. If $N(\nabla u)(X)$ is finite, then $\lim_{Z\to X \text{ n.t.}} u(X)$ exists; we need only show $\partial_\tau u \in H^1$.

Recall the conjugate $\tilde{u}$ of Section 4.4. If V is simply connected, then $\tilde{u}$ exists and is continuous on V. In this case, define $v = u$, $\tilde{v} = \tilde{u}$.

If V is not simply connected, then $\bar{V}^C$ is a bounded, simply connected Lipschitz domain. Let $X \notin V$ with $\operatorname{dist}(X, \partial V) \geq \sigma(\partial V)/C$, and let R be large enough that $V^C \subset B(X, R)$, so $\partial B(X, R) \subset V$.

Let $C_1 = \int_{\partial B(X,R)} \nu \cdot A\nabla u\, d\sigma$. Let $v = u - C_1\Gamma_X$. Since $\int_{\partial B(X,R)} \nu \cdot A\nabla\Gamma_X\, d\sigma = 1$, and since $\int_{\partial U} \nu \cdot A\nabla v\, d\sigma = 0$ for any simply connected domain $U \subset V$, we must have that $\int_\omega \nu \cdot A\nabla v\, d\sigma = 0$ for any closed path $\omega \subset V$; thus, $\tilde{v}$ is well-defined and continuous on V.

But $\|\partial_\tau\Gamma_X\|_{H^1(\partial V)} \leq C$; so we need only show that $\|\partial_\tau v\|_{H^1} \leq C$.

We have that $\operatorname{div}\tilde{A}\nabla\tilde{v} = 0$ in V and $\|N(\nabla\tilde{v})\|_{L^1} \approx \|N(\nabla v)\|_{L^1}$. So by Theorem 12.14, $\nu \cdot \tilde{A}\nabla\tilde{v} \in H^1(\partial V)$ in the weak sense.

Thus, we need only show that $\partial_\tau v = -\nu \cdot \tilde{A}\nabla\tilde{v}$.

It suffices to prove that, if $I \subset \partial V$ is connected and has boundary points X_0, X_1, and $N(\nabla v)(X_i)$ is finite, then for the appropriate ordering of X_0, X_1,

$$v(X_1) - v(X_0) = -\int_I \nu \cdot \tilde{A}\nabla\tilde{v}.$$

In fact, we need only prove this for short intervals; thus, it suffices to prove this for $V = \Omega$ a special Lipschitz domain.

Let $X_i = \psi(x_i)$; then $x_0 < x_1$. Pick some $\epsilon > 0$. Let $0 < h < \epsilon/N(\nabla v)(X_i)$ for $i = 0, 1$. Let $U = \psi((x_0, x_1) \times (0, h)) \subset V$.

We may split ∂U into four parts: I, $I + h\mathbf{e}$, and two segments $\psi(\{x_i\} \times (0, h))$ of length h. Since U is bounded and simply connected, $\int_{\partial U} \nu \cdot \tilde{A}\nabla\tilde{v} = 0$. But by our choice of h, $\int_{\psi(\{x_i\}\times(0,h))} |\nabla\tilde{v}| \leq C\epsilon$.

So, letting ν be the outward unit normal to U,

$$\left|\int_I \nu \cdot \tilde{A}\nabla\tilde{v} + \int_{I+h\mathbf{e}} \nu \cdot \tilde{A}\nabla\tilde{v}\right| \leq C\epsilon.$$

Recall that the tangential derivative is given by

$$\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau.$$

So by (4.4), and recalling that we traverse $I + h\mathbf{e}$ from $X_1 + h\mathbf{e}$ to $X_0 + h\mathbf{e}$, we have that

$$\int_{I+h\mathbf{e}} \nu \cdot \tilde{A}\nabla\tilde{v} = -\int_{I+h\mathbf{e}} \tau \cdot \nabla v = v(X_1 + h\mathbf{e}) - v(X_0 + h\mathbf{e}).$$

But $|v(X_i + h\mathbf{e}) - v(X_i)| \leq hN(\nabla v)(X_i)$, so

$$\left|\int_I \nu \cdot \tilde{A}\nabla\tilde{v}\,d\sigma + v(X_1) - v(X_0)\right| < C\epsilon.$$

Letting $\epsilon \to 0$ completes the proof. □

CHAPTER 13
THE MAXIMUM PRINCIPLE ON BOUNDED DOMAINS

Theorem 13.1. *Suppose that* $u = f$ *on* ∂V *for some bounded good Lipschitz domain* V*, and that* $\operatorname{div} A\nabla u \equiv 0$ *in* V*. Then* $\|u\|_{L^\infty(V)} \leq C\|f\|_{L^\infty(\partial V)}$ *for some constant* C *depending on the ellipticity constants of* A *and the Lipschitz constants of* V*, but not on* $\sigma(\partial V)$*.*

This theorem is obviously not true on unbounded domains: consider the harmonic functions $u(x,t) = t$ in $\mathbf{R}^2_+$ or $v(X) = \log|X|$ in $\mathbf{R}^2 \setminus B(0,1)$.

For notational convenience, we will instead prove this for $\operatorname{div} A^T\nabla u = 0$. Throughout this section, let p be an exponent such that $(R)^A_p$, $(N)^A_p$, $(D)^{A^T}_q$ hold.

13.1 Green's function and a priori bounds

Let V be a bounded Lipschitz domain, and assume that $(R)^{A^T}_p$ holds in V. Choose some $X \in V$. $\Gamma_X(Y)$ is bounded and continuous with bounded gradient on ∂V, so $\partial_\tau \Gamma_X \in L^p$. Therefore, since ∂V is compact, there is some Φ_X with $\operatorname{div} A\nabla\Phi_X = 0$ in V and $\Phi_X = \Gamma_X$ on ∂V.

Define $G_X = \Gamma_X - \Phi_X$.

Theorem 13.2. *Suppose that* V *is a bounded Lipschitz domain, that* $(R)^A_p$, $(R)^{A^T}_p$, $(D)^A_q$ *hold in* V*. Suppose that* $\operatorname{div} A\nabla u = 0$ *in* V*,* $Nu \in L^q(\partial V)$*, and* $u = f$ *on* ∂V*.*

Then

$$u(X) = \int_{\partial V} f\,\nu \cdot A\nabla G_X\,d\sigma.$$

Proving this for $f = 0$ was the core of our proof of Theorem 12.2.

Proof. Let $R = \operatorname{dist}(X, \partial V)$. Recall that $|\nabla\Gamma_X(Y)| \leq \frac{C}{|X-Y|}$. So by (2.7),

$$\|\nabla\Gamma_X(Y)\|_{L^p(\partial V)} \leq CR^{1/p-1}.$$

But since $(R)_p^A$ holds in V, and $\tau\cdot\nabla\Gamma_X = \tau\cdot\nabla\Phi_X$, we must have that

$$\|N(\nabla\Phi_X)\|_{L^p(\partial V)} \leq CR^{1/p-1}. \tag{13.3}$$

Let $g\in L^q(\partial V)$ be such that $\|f-g\|_{L^q} < \epsilon$ and $\partial_\tau g\in L^\infty(\partial V)\supset L^p(\partial V)$. Let v solve $\operatorname{div} A\nabla v = 0$ in V, $v = g$ on ∂V. By Theorem 12.4, we may require both that $\|Nv\|_{L^q(\partial V)}\leq C\|g\|_{L^q(\partial V)}$ and that $\|N(\nabla v)\|_{L^p(\partial V)}\leq C\|\partial_\tau g\|_{L^p(\partial V)}$. By Theorem 12.2, $\|N(u-v)\|_{L^q(\partial V)}\leq C\|f-g\|_{L^q(\partial V)}\leq C\epsilon$.

Then

$$\begin{aligned}
&\left|u(X) - \int_{\partial V} f\,\nu\cdot A\nabla G_X\,d\sigma\right|\\
&\qquad\leq |u(X)-v(X)| + \left|v(X) - \int_{\partial V} g\,\nu\cdot A\nabla G_X\,d\sigma\right| + \left|\int_{\partial V}(f-g)\,\nu\cdot A\nabla G_X\,d\sigma\right|\\
&\qquad\leq C\|N(u-v)\|_{L^p(\partial V)}\operatorname{dist}(X,\partial V)^{-1/p}\\
&\qquad\qquad+\left|v(X) - \int_{\partial V} g\,\nu\cdot A\nabla G_X\,d\sigma\right| + \|f-g\|_{L^p}\|\nabla G_X\|_{L^q(\partial V)}\\
&\qquad\leq C\epsilon R^{-1/p} + \left|v(X) - \int_{\partial V} g\,\nu\cdot A\nabla G_X\,d\sigma\right|.
\end{aligned}$$

So we need only show that the theorem holds for v.

By Lemma 3.3, v is bounded and $\nabla v\in L^2(V)$. By (3.8), ∇v is bounded on $B(X,R/2)$. By Lemma 7.11, v is Hölder continuous on $\bar V$. We specified that $\partial_\tau v$ was bounded. So we may extend v to a bounded compactly supported function, with gradient in $L^2(\mathbf{R}^2)$, whose gradient is bounded on $B(X,R/2)$.

Let $\operatorname{supp} v\subset B(X,R_0)$. For any $\epsilon_1>0$, there is some smooth, compactly supported function η such that

$$|\eta(X)-v(X)| + \|\nabla\eta-\nabla v\|_{L^\infty(B(X,R/2))} + \|\nabla\eta-\nabla v\|_{L^2(B(X,R_0))} < \epsilon_1.$$

Then (2.9) applies to η, so

$$\left|v(X) + \int\nabla v\cdot A\nabla\Gamma_X\right| \leq |v(X)-\eta(X)| + \left|\int\nabla(v-\eta)\cdot A\nabla\Gamma_X\right| \leq C(R,R_0)\epsilon_1$$

since $\nabla\Gamma_X \in L^1(B(X,R/2))$ and $\nabla\Gamma_X \in L^2(B(X,R_0)\setminus B(X,R/2))$.

So (2.9) holds for v as well. Then

$$\begin{aligned}
v(X) &= -\int \nabla v\cdot A\nabla\Gamma_X = -\int_V A^T\nabla v\cdot\nabla\Gamma_X - \int_{V^C}\nabla v\cdot A\nabla\Gamma_X\\
&= -\int_{\partial V}\Gamma_X\,\nu\cdot A^T\nabla v\,d\sigma + \int_{\partial V} v\,\nu\cdot A\nabla\Gamma_X\,d\sigma\\
&= -\int_{\partial V}\Phi_X\,\nu\cdot A^T\nabla v\,d\sigma + \int_{\partial V} v\,\nu\cdot A\nabla\Gamma_X\,d\sigma\\
&= -\int_V \nabla\Phi_X\cdot A^T\nabla v + \int_{\partial V} v\,\nu\cdot A\nabla\Gamma_X\,d\sigma\\
&= -\int_{\partial V} v\,\nu\cdot A\nabla\Phi_X\,d\sigma + \int_{\partial V} v\,\nu\cdot A\nabla\Gamma_X\,d\sigma = \int_{\partial V} v\,\nu\cdot A\nabla G_X\,d\sigma
\end{aligned}$$

as desired. □

Thus, the maximum principle is equivalent to bounding $\|\nu\cdot A\nabla G_X\|_{L^1(\partial V)}$, with a bound independent of X and $\sigma(\partial V)$.

By (13.3) and Hölder's inequality, $\|\nabla G_X\|_{L^1}$ is bounded for $\operatorname{dist}(X,\partial V) > \frac{1}{C}\sigma(\partial V)$; we need only bound $\|\nu\cdot A\nabla G_X\|_{L^1(\partial V)}$ for X near the boundary.

13.2 Bounding the Green's function

Pick any $X\in V$ with X near ∂V. Let $R = \operatorname{dist}(X,\partial V)$, and let $X^*\in\partial V$ with $|X - X^*| = 0$. Assume that $R < \frac{1}{2}\min_i r_i$, where the r_i are as in Definition 2.4. Then $X^*\in B(X_i,r_i)\cap\Omega_i$ for some i. In particular, the domains $Q(X^*,r)$ of (2.6) exist for $r < r_i$.

If $R < r < 2R$, then $\operatorname{dist}(X,\partial Q(X^*,r)) \approx \operatorname{dist}(X,\partial V) \approx \sigma(\partial Q(X^*,r))$. As usual, we take a large enough that $V\cap\partial Q(X^*,r)\subset\gamma_{a,V}(\chi_-(X^*,r))\cup\gamma_{a,V}(\chi_+(X^*,r))$.

For convenience write $U_r = V\setminus\overline{Q(X^*,r)}$. We need to show that $\int_{\partial V}|\nu\cdot A\nabla G_X|\le C$. Begin by integrating only over $\partial V\setminus\partial U_r = \partial Q(X^*,r)\cap\partial V$. We have that

$$\begin{aligned}
\int_{\partial V\setminus\partial U_r}|\nabla G_X|\,d\sigma &\le \sigma(\partial V\setminus\partial U_r)^{1-1/p}\left(\int_{\partial V\setminus\partial U_r}|\nabla G_X|^p\,d\sigma\right)^{1/p}\\
&\le \sigma(\partial V\setminus\partial U_r)^{1-1/p}CR^{1/p-1}\le C
\end{aligned}$$

since $\sigma(\partial V\setminus\partial U_r)\le\sigma(Q(X^*,r))\approx CR$ and $\|\nabla G\|_{L^p(\partial V)}\le CR^{1/p-1}$.

Now, consider ∂U_r. Since $X \notin U_r$, we have that G_X satisfies $\operatorname{div} A\nabla G_X = 0$ in U_r; consequently, $\|\nabla G_X\|_{L^1(\partial U_r)} \leq C\|\tau \cdot \nabla G_X\|_{H^1(\partial U_r)}$. So we need only bound $\tau \cdot \nabla G_X$ in $H^1(\partial U_r)$.

But $G_X = 0$ on ∂V; therefore, we need only consider $\tau \cdot \nabla G_X$ on $\partial U_r \setminus \partial V = \partial Q(X^*, r) \setminus \partial V$.

Let $g = \tau \cdot \nabla G_X$ on ∂U_r. Then $\int g = 0$, and $\sigma(\operatorname{supp} g) \leq Cr$. So we need only show that $\|g\|_{L^\infty(\partial U_r \setminus \partial V)} \leq C/r$.

Now, $|g(Y)| \leq |\nabla \Gamma_X(Y)| + |\nabla \Phi_X(Y)|$. On $\partial Q(X^*, r)$,

$$|\nabla \Gamma_X(Y)| \leq \frac{C}{|X - Y|} \leq \frac{C}{\operatorname{dist}(X, \partial Q(X^*, r))} \leq \frac{C}{R}.$$

Since $\partial U_r \setminus \partial V \subset \gamma(\chi_+(X^*, r)) \cup \gamma(\chi_-(X^*, r))$, we have that

$$\|g\|_{L^\infty(\partial U_r \setminus \partial V)} \leq C/R + N(\nabla \Phi_X)(\chi_+(X^*, r)) + N(\nabla \Phi_X)(\chi_-(X^*, r)).$$

If $r \leq 2R$, this lets us bound $\|g\|_{H^1(\partial U_R)}$, which lets us conclude

$$\|\nu \cdot A\nabla G_X\|_{L^1(\partial V)} \leq C + RN(\nabla \Phi_X)(\chi_+(X, r)) + RN(\nabla \Phi_X)(\chi_-(X, r)).$$

As in Section 8.2, we take the average from $r = R$ up to $r = 2R \leq r_i$:

$$\begin{aligned}
\|\nu \cdot A\nabla G_X\|_{L^1(\partial V)} &\leq C + \fint_{\Delta(x^*,2R)} RN(\nabla \Phi_X)\, d\sigma + \fint_{\Delta(x^*,2R)} RN(\nabla \Phi_X)\, d\sigma \\
&\leq C + C\fint_{\Delta(x^*,2R)} RN(\nabla \Phi_X)\, d\sigma \\
&\leq C + C\left(\fint_{\Delta(x^*,2R)} R^p N(\nabla \Phi_X)^p\, d\sigma\right)^{1/p} \\
&\leq C + CR^{1-1/p}\left(\int_{\Delta(x^*,2R)} N(\nabla \Phi_X)^p\, d\sigma\right)^{1/p} \\
&\leq CR^{1-1/p}\|N(\nabla \Phi_X)\|_{L^p(\partial V)} \leq C
\end{aligned}$$

as desired.

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