1. GCDs I
   Compute the following GCDs by finding the prime factorizations of each pair of integers.
   (a) G.C.D.(640, 288)
   (b) G.C.D.(1001, 9183)
   (c) G.C.D.(980, 343)

2. GCDs II
   Compute the following GCDs by using the Euclidean Algorithm.
   (a) G.C.D.(301, 430)
   (b) G.C.D.(1000, 625)
   (c) G.C.D.(377, 233)

3. GCDs III
   (a) Show that if \( n \) is odd and \( n \geq 3 \), then G.C.D.(\( n + 1, n - 1 \)) = 2.
   (b) Show that if \( p \) is prime, then G.C.D.(\( p^2 + 1, p \)) = 1.
   (c) Let \( a, b \in \mathbb{Z} \). Show that if there are \( m, n \in \mathbb{Z} \) such that \( ma + nb = 1 \), then G.C.D.(\( a, b \)) = 1.
   (d) Let \( a, b \in \mathbb{Z} \), and suppose G.C.D.(\( a, b \)) = \( d \). Show that there are \( m, n \in \mathbb{Z} \) such that \( ma + nb = d \).
   (e) Show that if \( mn \) is a perfect square and G.C.D.(\( m, n \)) = 1, then both \( m \) and \( n \) are perfect squares.

4. (The Locker Problem) A school with 500 students and 500 lockers (numbered 1–500) plays the following game. The students are lined up and numbered from 1 to 500. All lockers start out closed. Student number 1 goes into the school and opens every locker. Student number 2 goes in closes every 2nd locker. Student number 3 goes in and reverses the status of every 3rd locker. (That is, if the locker is already open they close it, and if it is closed, they open it.) Students 4 through 500 take turns going in and repeating Student #3’s behavior (which was also that of #1 and #2, though not so described). After Student #500 goes through, which lockers are open?

5. Divisors
   (a) Which positive integers have exactly 2 positive divisors?
   (b) Which positive integers have exactly 3 positive divisors?
   (c) Which positive integers have exactly 4 positive divisors? (Hint: There are two distinct types. Describe them in terms of their prime factorizations.)
   (d) Give a formula describing the number of positive divisors of an integer in terms of its prime factorization.
6. Diophantine Equations

(a) Diophantus was an Ancient Greek mathematician who lived in the 3rd Century BCE. It is said that his tombstone told the following account of his life:

“This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son. Alas! late-begotten and miserable child, when he had reached the measure of half his father’s life, the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life.”

How old was Diophantus when he died?

(b) A farmer takes his eggs to market. When he tries to divide them evenly into two baskets, he finds that there is one left over. When he tries to divide them evenly into three baskets, he finds that there is one left over. Similarly, when he tries to divide them evenly into four, five, or six baskets, he finds that there is one left over. Finally, when he tries to divide them into seven baskets, he finds that they divide evenly. What is the minimum number of eggs that the farmer took to market?