1. Pythagorean triples

(a) It is a fact that if $m$ and $n$ are positive integers with $m > n$, then the three numbers:

\[
\begin{align*}
    a &= m^2 - n^2 \\
    b &= 2mn \\
    c &= m^2 + n^2
\end{align*}
\]

satisfy the equation $a^2 + b^2 = c^2$. Use algebra to prove this.

(b) Show that every odd integer $n \geq 3$ is the smallest element of a primitive Pythagorean triple.

(c) Show that 1 and 2 can never occur in Pythagorean triples but that every other positive integer occurs in at least one Pythagorean triple.

(d) Use these expressions to fill in the following table and discover new Pythagorean triples:

\[
\begin{array}{c|c|c|c|c}
 m & n & a & b & c \\
\hline
 2 & 1 & & & \\
 3 & 1 & & & \\
 3 & 2 & & & \\
 4 & 1 & & & \\
 4 & 2 & & & \\
 4 & 3 & & & \\
 5 & 1 & & & \\
 5 & 2 & & & \\
 5 & 3 & & & \\
 5 & 4 & & & \\
 6 & 1 & & & \\
 6 & 2 & & & \\
 6 & 3 & & & \\
 6 & 4 & & & \\
 6 & 5 & & & \\
 7 & 1 & & & \\
 7 & 2 & & & \\
 7 & 3 & & & \\
 7 & 4 & & & \\
 7 & 5 & & & \\
 7 & 6 & & & \\
\end{array}
\]

(e) For which values of $m$ and $n$ do you get primitive Pythagorean triples? Explain.
2. Iterating functions

For function defined as $f : \mathbb{N} \to \mathbb{N}$, consider what happens to a given integer if we iterate the function. That is, given an $n \in \mathbb{N}$, what is $f(n)$? And $f(f(n))$? And $f(f(f(n)))$? Etc. And what happens for different starting values of $n$?

For each of the functions below, consider the following questions:

(a) What happens to each integer $n$? (Yes, this is vaguely phrased. Explore.)

(b) Does the algorithm ever terminate in a fixed point or a loop? That is, are there values $x \in \mathbb{N}$ such that $f(x) = x$, creating a fixed point (and thus terminating the algorithm if we ever reach $x$)? Or, are there values $x_1, \ldots, x_k$ such that $f(x_i) = x_{i+1}$ for $i = 1, \ldots, k-1$, but $f(x_k) = x_1$, thus creating a loop (and in some sense also terminating the algorithm)?

(c) Given a starting value $n \in \mathbb{N}$, how many step of iteration are required to reach a termination point?

(d) Given a fixed length of the iteration (that is, a number of steps to reach termination), how many values of $n \in \mathbb{N}$ are there that take that many steps?

(e) What patterns and conjectures do you have about your observations from the previous questions?

Okay, here are the functions:

- $f(n) = \begin{cases} n - 1 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$

- $g(n) = \begin{cases} 2n & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$

- $h(n) = \begin{cases} n - 1 & \text{if } n \text{ is thordd} \\ \frac{n}{3} & \text{if } n \text{ is threen} \end{cases}$

- $k(n) = \begin{cases} \frac{n}{p} & \text{if } n \geq 2 \text{ and } p \text{ is the largest prime such that } p \mid n \\ 1 & \text{if } n = 1 \end{cases}$

- Make up your own function!