1. Determine the splitting behavior of rational primes in the number system $\mathbb{Z}[^{\sqrt{-2}}]$ and $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\phi]$ where
\[ \phi = \frac{1 + \sqrt{5}}{2} \]
is the golden ratio. That is, determine exactly which primes split up and which remain inert (still prime) in the number system. For those that split apart determine which actually ramify in the sense that their prime factors are actually associates.

2. This one is very exploratory for now, but it’s fun and concrete. You probably know how to write a positive integer in a different base $b$ where $b$ is some positive integer at least 2. The fact that $\mathbb{Z}/b\mathbb{Z}$ has representatives
\[ \{0, 1, \ldots, b-1\} \]
is the basic reason we use these as digits in the base $b$ expansion (though for $b > 10$ we usually replaces 10 be $A$, 11 by $B$, etc. so we have actual single digits, which you have probably seen in hexadecimal where $b = 16$). If you have not done this kind of thing - practice with it. Write 389 in binary, write 1729 in base 7, etc.

Note that negatives don’t play so well here. After all, a binary expansion like
\[ d_0 + d_1 \cdot 2^1 + d_2 \cdot 2^2 + \cdots + d_k \cdot 2^k, \quad \text{where} \quad d_i \in \{0, 1\} \]
is always non-negative. But certainly every nonzero integer is an associate of such an expansion.

Imagine we tried to do this in $\mathbb{Z}[i]$. What base? Let’s try $b = 1 + i$. It is a fact that there are exactly two congruences classes modulo $1 + i$ and $\{0, 1\}$ are representatives. This is related to the fact that $N(1 + i) = 2$, but let’s just assume this for now. So $(1 + i)$-ary expansions look like
\[ d_0 + d_1 \cdot (1 + i)^1 + d_2 \cdot (1 + i)^2 + \cdots + d_k \cdot (1 + i)^k, \quad \text{where} \quad d_i \in \{0, 1\}. \]

Gobs of questions present themselves. Which Gaussians have $(1 + i)$-ary expansions? All of them? Probably not, judging by the $\mathbb{Z}$ situation. Which ones do? Which ones do not? Is every Gaussian an associate to such an expansion at least?

You should probably just start by computing some expansions. Write a string of binary digits like 101 and compute the Gaussian integer with this expansion. Could you figure out how to “go backwards” and get this binary expansion from the number you computed?