1. If you haven’t yet had to use the Euclidean algorithm in \(\mathbb{Z}[i]\), do so. We’re actually going to use it for something concrete. So just pick some pairs of elements of small norm (e.g. \(3+5i\) and \(7-i\)) and run the EA to see what you get and try to realize the GCD as a linear combination. [If you haven’t done any division algorithms in \(\mathbb{Z}[i]\), this will be plenty of practice with that.]

2. Which integers arise as norms of elements of \(\mathbb{Z}[i]\)? That is, which integers are sums of two squares? We know the answer for primes now, but what about non-primes? Is 2020 the sum of two squares?

3. Pick a prime \(p\) that splits in \(\mathbb{Z}[i]\), e.g. \(p = 5\). Note that each of its prime factors has norm \(p\) and thus the square of each such factor has norm \(p^2\). Use this fact to write down a Pythagorean triple with \(p\) as the “hypotenuse.” How far can you push this method? What about non-prime hypotenuses? Can you make a Pythagorean triple with 2020 as hypotenuse?

4. Show that every element of \(\mathbb{Z}[i]\) is the root of a monic polynomial with integer coefficients. Can you prove the following converse: every element of

\[
\mathbb{Q}(i) = \{x + yi \mid x, y \in \mathbb{Q}\}
\]

that is the root of such a polynomial must lie in \(\mathbb{Z}[i]\)?