ON MEAN FIELD GAMES

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I. INTRODUCTION

• New class of models for the average (Mean Field) behavior of “small” agents (Games) started in the early 2000’s by J-M. Lasry and P-L. Lions.

• Requires new mathematical theories.

• Numerous applications: economics, finance, social networks, crowd motions...

• Independent introduction of a particular class of MFG models by M. Huang, P.E. Caines and R.P. Malhamé in 2006.

• A research community in expansion: mathematics, economics, finance.

• Some written references but most of the existing mathematical material to be found in the Collège de France videotapes (4 × 18h) that can be downloaded...!
• Combination of Mean Field theories (classical in Physics and Mechanics) and the notion of Nash equilibria in Games theory.

• Nash equilibria for continua of “small” players : a single heterogeneous group of players (adaptations to several groups . . .).

• Interpretation in particular cases (but already general enough!) like process control of McKean-Vlasov . . .

• Each generic player is “rational” i.e. tries to optimize (control) a criterion that depends on the others (the whole group) and the optimal decision affects the behavior of the group (however, this interpretation is limited to some particular situations. . . ).

• Huge class of models : agents → particles, no dep. on the group are two extreme particular cases.
II. A REALLY SIMPLE EXAMPLE

• Simple example, not new but gives an idea of the general class of models (other “simple” exs later on).

• $E$ metric space, $N$ players ($1 \leq i \leq N$) choose a position $x_i \in E$ according to a criterion $F_i(X)$ where $X = (x_1, \ldots, x_N) \in E^N$.

• Nash equilibrium : $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_N)$ if for all $1 \leq i \leq N$ $\bar{x}_i$ min over $E$ of $F_i(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots \bar{x}_N)$.

• Usual difficulties with the notion

• $N \to \infty$ ? simpler ?

• Indistinguishable players :

$$F_i(X) = F(x_i, \{x_j\}_{j \neq i}), F \text{ sym. in } (x_j)_{j \neq i}$$
• Part of the mathematical theories is about $N \to \infty$:

$$F_i = F(x, m) \quad x \in E, \quad m \in \mathcal{P}(E)$$

where $x = x_i$, $m = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$

• “Thm” : Nash equilibria converge, as $N \to \infty$, to solutions of

$$(MFG) \quad \forall x \in \text{Supp} \ m, F(x, m) = \inf_{y \in E} F(y, m)$$

• Facts : i) general existence and stability results

ii) uniqueness if $(m \to F(\bullet, m))$ monotone

iii) If $F = \Phi'(m)$, then $(\min \Phi)$ yields one solution of MFG.
Example: $E = \mathbb{R}^d$, $F_i(X) = f(x_i) + g \left( \frac{\# \{ j : |x_i - x_j| < \varepsilon \}}{(N - 1) |B_\varepsilon|} \right)$

$g \uparrow$ aversion crowds, $g \downarrow$ like crowds

$$F(x, m) = f(x) + g(m * 1_{B_\varepsilon}(x)(|B_\varepsilon|^{-1}))$$

$$\varepsilon \to 0 \quad F(x, m) = f(x) + g(m(x))$$

(MFG) $\text{supp } m \subset \text{Arg min} \left( f(x) + g(m(x)) \right)$

$- g \uparrow$ uniqueness, $g \downarrow$ non uniqueness

$$\min \left\{ \int fm + \int G(m) / m \in \mathcal{P}(E) \right\}, \quad G = \int_0^Z f(s) ds$$

$- \text{explicit solution if } g \uparrow: m = g^{-1}(\lambda - f), \lambda \in \mathbb{R} \text{ s.t. } \int m = 1$
III. GENERAL STRUCTURE

- Particular case: dynamical problem, horizon $T$, continuous time and space, Brownian noises (both indep. and common), no intertemporal preference rate, control on drifts (Hamiltonian $H$), criterion dep. only on $m$

- $U(x, m, t)$ ($x \in \mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{R}^d)$ or $\mathcal{M}_+(\mathbb{R}^d)$, $t \in [0, T]$ and $H(x, p, m)$ (convex in $p \in \mathbb{R}^d$)

- MFG master equation

\[
\begin{aligned}
\frac{\partial U}{\partial t} - (\nu + \alpha) \Delta_x U + H(x, \nabla_x U, m) + \\
+ \langle \frac{\partial U}{\partial m}, -(\nu + \alpha) \Delta m + \text{div} \left( \frac{\partial H}{\partial p} m \right) \rangle + \\
- \alpha \frac{\partial U}{\partial m^2} (\nabla m, \nabla m) + 2\alpha \langle \frac{\partial}{\partial m} \nabla_x U, \nabla m \rangle = 0
\end{aligned}
\]

and $U \mid_{t=0} = U_0(x, m)$ (final cost)

- $\nu$ amount of ind. rand., $\alpha$ amount of common rand.
• $\infty$ $d$ problem!

• If $\nu = 0$ (ind) : Nash $N$ special case

using $x = x_i$, $m = \frac{1}{N-1} \sum_{j \neq i} \delta x_j$

• Aggregation/decentralization : IF $H(x, p, m) = H(x, p) + F'(m)$

and $U_0 = \Phi'_0(m)$, then $U = \frac{\partial \Phi}{\partial m}$ solves MFG if $\Phi$ solves $HJB$ on $\mathcal{P}(E)$ for the optimal control of a $SPDE$

• Particular case : many extensions and variants . . .
IV. TWO PARTICULAR CASES

• ∞ d problem in general but reductions to finite d in two cases

1. Indep. noises (α = 0)

int. along caract. in m yields

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u, m) & = 0 \\
\left. u \right|_{t=0} & = U_0(x, m(0)), \left. m \right|_{t=T} = \bar{m} \\
\frac{\partial m}{\partial t} + \nu \Delta m + \text{div} \left( \frac{\partial H}{\partial p} m \right) & = 0
\end{align*}
\]

(MFGi)

where \( \bar{m} \) is given

FORWARD — BACKWARD system !

contains as particular cases: HJB, heat, porous media, FP, V, B, Hartree, semilinear elliptic, barotropic Euler . . .
2. Finite state space \((i \leq i \leq k)\)

\[
\text{(MFGf)} \quad \frac{\partial U}{\partial t} + (F(x, U) \cdot \nabla) U = G(x, U), \quad U \big|_{t=0} = U_0
\]

(no common noise here to simplify . . .)

\[x \in \mathbb{R}^k, \quad U \to \mathbb{R}^k, \quad F \text{ and } G : \mathbb{R}^{2k} \to \mathbb{R}^k\]

non-conservative hyperbolic system

Example: If \(F = F(U) = H'(U), G \equiv 0\)

and if \(U_0 = \nabla \varphi_0 (\varphi_0 \to \mathbb{R})\) then

- solve \(HJ\)

\[
\frac{\partial \varphi}{\partial t} + H(\nabla \varphi) = 0, \quad \varphi \big|_{t=0} = \varphi_0
\]

- take \(U = \nabla \varphi\), “\(U\) solves” (MFGf) in this case
V. OVERVIEW AND PERSPECTIVES

Lots of questions, partial results exist, many open problems

- Existence/regularity:
  - (MFGi) “simple” if $H$ “smooth” in $m$ (or if $H$ almost linear ...), OK if monotone (Zoom 1)
  - (MFGf) OK if $(G, F)$ mon. on $\mathbb{R}^{2k}$ or small time (Zoom 2)

- Uniqueness: OK if “monotone” or $T$ small ...

- Non existence, non uniqueness, non regularity (!)

- Qualitative properties, stationary states and stability, comparison, cycles ...

- $N \to \infty$ (see above)

- Numerical methods (currently, 3 “general” methods and some particular cases)

- Variants: other noises, several populations ...

- random heterogeneity, partial info ...

- applications (MFG Labs ...)
- intertemporal preference rates ($\lambda \to \infty$ effective models)
- macroscopic limits
- ? Beyond MFG ? (fluctuations, LD, transitions)
- Two more S. examples:
  - at which time will the meeting start ?
  - the (mexican) wave
(MFGi) \[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) &= f(x, m) \\
u \big|_{t=0} &= U_0(x), \quad m \big|_{t=T} = \bar{m} \\
\frac{\partial m}{\partial t} + \nu \Delta m + \text{div} \left( \frac{\partial H}{\partial p} m \right) &= 0
\end{align*}
\]

- \( m \mapsto f(\bullet, m) \) smoothing operator
- \( \exists \) regular solution
- uniqueness if operator monotone or if \( T \) small
- \( f(m(x)) \uparrow: \exists ! \) regular solution \( \nu > 0 \)
- \( f(m(x)) \uparrow: \) if \( \nu = 0 \) \( m = f^{-1}(\frac{\partial u}{\partial t} + H(x, \nabla u)) \)

equation in \( m \) becomes quasilinear elliptic equation of second order \((x \in Q, t \in [0, T])\) with “elliptic” boundary conditions

\[
\begin{align*}
u \big|_{t=0} &= U_0(x), \quad \frac{\partial u}{\partial t} + H(\nabla u) = f(\bar{m}) \quad \text{if} \quad t = T
\end{align*}
\]
\[(MFGf) \begin{cases} \frac{\partial u}{\partial t} + (F(x, U) \cdot \nabla) U = G(x, U) & x \in \mathbb{R}^d \\ U \to \mathbb{R}^d, U\mid_{t=0} = U_0(x) \end{cases}\]

- shocks (discontinuities of $U$) in finite time in general
- well-posed problem on $[0, T_{\text{max}})$ ($T_{\text{max}} \leq +\infty$)
- $\exists \; !$ regular solution monotone in $x$ if $U_0$ monotone and $(G, F)$ monotone of $\mathbb{R}^{2,k}$ in $\mathbb{R}^{2k}(+\ldots)$
- $+$ change of unknown functions:

ex. : $\frac{\partial U}{\partial t} + (F(U) \cdot \nabla) U = 0$
then \( V = F(U) \) solves

\[
\frac{\partial V}{\partial t} + (V \cdot \nabla)V = 0
\]

max class of regularity

\[
\forall \delta > 0, \inf_{x \in \mathbb{R}^d} \text{dist}(\text{Sp}(DV_0(x)), (-\infty, \delta]) > 0
\]

\((V_0 = F(U_0) \) gives the maximum class of regularity \( \approx \) composed of 2 monotone applications\)

Rem. : gives new results of regularity for Hamilton-Jacobi equations of the first order.
VI. MEANINGFUL DATA

• MFG Labs

• Practical expertise and models mainly for “big” data involving “people”

• New models that include classical clustering models in M.L. (K-mean, EM . . .), then algorithms

• No need for euclidean structures or for “a priori” distances
• Why “PEOPLE”

Ex. 1 : Taxis

Ex. 2 : Movies and Fb

People that are “close” will say they like movies that are “close”

→ consistency distance - like on items/people
• Even for “pure data” models make sense: data points become agents . . . (in fact lots of terminology from Game Theory in M.L./Data Analysis)

• Clustering: classical K-Mean

set of points \( \{x_1, \ldots, x_N\} \) in \( \mathbb{R}^d (N >> 1) \)
Find $K$ points $y_1, \ldots, y_k$ s.t. $\exists$ partition $(A_1, \ldots, A_K)$ of \{1, $\ldots, N$\} for which

i) $|y_i - x_j| \leq |y_{i'} - x_j|$, $\forall j \in A_i$, $\forall i' \neq i$

ii) $y_i = (\#A_i)^{-1} \sum_{j \in A_i} j$
MFG INTERPRETATION:

INTRODUCE

- A GLOBAL CRITERION

\[ F(u_1, \ldots, u_k) \]

\[ \text{Ex} : \min(u_1, \ldots, u_k) \]

- \( K \) value functions \((u_1, \ldots, u_k)\)

- \( K \) "densities" \((m_1, \ldots, m_k)\)

\( f \) being the initial density of "data" (no need to restrict to "discrete" data)

SOLVE MFG: EXAMPLE

\[ \rho u_i - \nu \Delta u_i + \frac{1}{2}(\nabla u_i)^2 = F_i(x; m_i) \]

\[ \rho m_i - \nu \Delta m_i - \text{div}(\nabla u_i m_i) = \rho \frac{\partial F}{\partial u_i} f \]

ex. \( 1_{(u_i < \min_{j \neq i} u_j)} \)
BACK TO K-Mean

\[ F_i = \frac{1 + \rho}{2} |x - \frac{\int xm_i}{\int m_i}|^2 - \nu d \]

then indeed: \( u_i = \frac{1}{2} |x - y_i|^2, y_i = \frac{\int xm_i}{\int m_i} \) and

\[ \int m_i = \int f 1(u_i < \min_{j \neq i} u_j), \int xm_i = \int xf_i 1(u_i < \min_{j \neq i} u_j) \]

Next, this allows to

- create lots of new models
- smooth clustering if needed, clusters within clusters, overlaps...
- no need for distances, no need for euclidean structure (choose criterion $F$, class criteria $F_i \rightarrow u_i$ . . .)
- transposition to graphs easy (ODE’s, massively //)

Remark: 

$$- \Delta u + |\nabla u|^2 = e^u (+\Delta)e^{-u}$$

$$e^{u_i} \sum_j (e^{-u_j} - e^{-u_i}) = \sum_j (e^{u_i - u_j} - 1)$$

- social networks equilibria: “distance on items” $\longleftrightarrow$
  “distance on users” $\longleftrightarrow$ preferences $\longleftrightarrow$ “distances on items” . . .