Differential Topology

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Office Hours Th 9:30-10:30 and by appointment.
In class exam on February 17.
Final, TBA.

Homework will be assigned on Fridays and due on the following Friday. (*) means that the problem might be pretty hard.

Suggested texts:

Milnor, Topology form the differentiable viewpoint
Guillemin and Pollak, Differential Topology
Hirsch, Differential Topology
Spivak, Differential Geometry vol 1.
Frankel, The Geometry of Physics
Kosinski, Differential Manifolds

Homework #1.

1. Of the flavours discussed in class: Topological, Smooth, Lipschitz, Complex, Translation, Rigid Motion: which of these structures can be put upon the 2-sphere, which on the torus, which on a klein bottle? (Which compact 2-manifolds have translation structures?)
2. View $S^1 \times R$ as a translation-manifold using the product structure. Show that it is not translation-diffeomorphic to R^2-{0}. (Construct uncountably many "translation-diffeomorphism" types of manifolds diffeomorphic – in the usual sense -- to the annulus.)

3. Show that in a (topologically) connected smooth manifold, any two points can be connected via a smooth path.

4. Show that on a connected smooth manifold, M, for any p and q, there a diffeomorphism f: M → M such that f(p) = f(q). (Hint: Use problem 2.)

5. Show that O(n), the group of (linear) isometries of R^n (with the usual Euclidean distance) is a compact smooth manifold. What is its dimension? Show that it has 2 components.

6. Prove that each component of O(2) is diffeomorphic to S^1, and of O(3) is

diffeomorphic to RP^3. (Some people like using quaternions for the second part.)

7. Show that if G is a Lie group, so is the component of the identity. Indeed, show that this is a normal subgroup. Show that any covering space of G is a Lie group as well.

8. Construct infinitely many simply connected compact 4-manifolds.

9. (*) Show that there is no compact 3-manifold (without boundary) with fundamental group $\mathbf{Z_2}^2$**. (Hint:** Show that any simply connected compact 3-manifold has the homotopy type of S^3. Then show that any one with fundamental group $\mathbf{Z_2}$ has the homotopy type of RP^3. Then solve the problem.)

10. Show that if M is a compact n-manifold (without boundary) then M does not embed in $\mathbf{R}^n$.

Homework #2.

1. What are the regular values of the map f: SO(3) → SO(3) given by $f(A) = A^2$.

2. Construct a smooth map $\mathbf{R} \to \mathbf{R}$ that has a dense set of critical values.

3. Let CP^n = {complex lines in $\mathbf{C}^{n+1}$} = S^{2n+1}/S^1 (where S^1 acts by rotation of each coordinate). Show that CP^n is a smooth manifold, by giving coordinate charts; indeed check that it is a complex analytic manifold.

4. Consider the image of $\{(x_0, \ldots x_n) \mid x_0^k + \ldots + x_n^k = 0)$ in CP^n. Show that it is also a smooth manifold. (*) Is it connected? Simply connected? (This last part is quite hard.)

5. Give a differentiable homeomorphism that is not a diffeomorphism.

6. Show that if M is a compact submanifold of Euclidean space, then there is a neighborhood of M so that each point in the neighborhood is uniquely of the form m+v, where v is a vector of norm $|v| < \varepsilon$ and v is normal to every curve passing through m.

7. Show that if M is compact and f: M → W is a smooth embedding, then if g is C^1 close to f (you might want to give a definition of this), then g is an embedding as well. Show by example that this fails with the C^0 topology on functions.

8. If f: M → M is a diffeomorphism, and f(p) = p, i.e. p is a fixed point, then the differential of f at p is a well defined m × m matrix (up to conjugacy). Show that the graph of f is transverse to the diagonal in M × M if and only if 1 is not a singular value of this matrix, i.e. if $I-Df_p$ is non-singular.

Homework #3.

1. Show that if M and N are compact smooth manifolds, then the smooth functions $C^\infty(M,N)$ are dense in $C^0(M,N)$. Go through the effort of checking that in the compact open topology, this is true even if N is non-compact.

2. Show that homotopic smooth maps are smoothly homotopic.

3. Suppose that f and g are homotopic smooth maps satisfying the condition of #8 of homework 2. Show that the number of fixed points of f and g are finite, and have the same parity.

4. Show that there is no f: $S^1 \times S^1 \rightarrow S^1$ so that f(x,x) = x and f(x,y) = f(y,x).

5. (*) Show a connected n-manifold had a connected codimension one submanifold that does not separate it if and only if its fundamental group has a homomorphism onto Z/2Z.

6. If M and the submanifold are assumed to be oriented, then the condition is that the fundamental group surjects to **Z**.

7. Show that for any compact oriented manifold there is a number g, so that any k disjoint oriented hypersurfaces must together separate M if k>g.

8. Suppose f: M → N is a smooth map between compact connected manifolds, show that the mod 2 degree of f is the same thing as the induced map on top homology.

9. Suppose that π is a finitely presented group and that it has a nontrivial homomorphism into U(n). Show that if we add on a new generator and a any new relation, the new group also has a nontrivial homomorphism to U(n).. (Hint: Consider separately the case where the weight of the new generator in the added relation is 1 versus other possibilities. Also, make use of the fact that U(n) is a compact connected Lie group.)

10. Prove that if A_i are quaternions, then any monic polynomial $x^n + \sum A_i x^i$ (this is not the most general polynomial over the quaternions) has a root. (Warning: the polynomial $x^2 + 1$ has infinitely many roots, so don't copy our proof of the fundamental theorem of algebra too slavishly.)

Homework #4.

(Some of these problems are easier if the spaces are compact, and some are false if they are not. You should try to figure out which are which. But if you find life easier assuming compactness, then you may.)

1. Given a linear map A in $GL_n\mathbf{Z}$, it induces a map, also called A, $A:T \to T$ from the torus $T = \mathbf{R}^n/\mathbf{Z}^n$. What is deg A?

2. Show that if G is a discrete group acting freely and properly discontinuously on a connected smooth manifold M, then M/G is orientable if and only if M is and the action is orientation preserving. Deduce that $\mathbf{RP}^n$ is orientable iff n is odd.

3. Suppose that the circle acts on a compact manifold M with no fixed points, show that M has a nowhere vanishing vector field.

4. Prove the same thing if M has a torus action with no fixed points.

5. Generalize to the case of finitely many fixed points (making use of the Poincare-Hopf theorem). Using this, compute the Euler characteristic of $U(n)/\mathbf{T}^n$ where the torus here is the group of diagonal unitary matrices.

6. Given a vector bundle $\pi: E \to X$, show that it is always possible to define positive definite inner products $< , >$ on each fiber that is compatible in the following sense. If s and t are smooth sections of E, (i.e. $s: X \to E$ is so that $\pi s = id$, and similarly for t) then $<s,t>$ is a smooth function on X.

7. Let $\pi: E \to X$ be a vector bundle. Show that if π has a nowhere vanishing section (i.e. $s: X \to E$ so that $\pi s = id$) then $E \cong F \times \mathbf{R}$ for some vector bundle F over X.

8. Prove that if the fiber dimension of E > dim X, then it is always possible to find a nonzero section.

9. Prove a stronger form of the Whitney embedding theorem than asserted in class: if $f: M \to N$ is a map and dim N > 2dim M, then f can continuously approximated by a smooth embedding.

10. Show that the immersion of the circle in the plane whose image looks like the number "8" cannot be approximated by an embedding.

11. Show that the dimension in the Whitney embedding theorem can be improved from 2n+1 to 2n if one is looking for an immersion (a smooth map whose differential is locally 1-1).

12. Show that the maps $z \to z^n$ where z lies in the unit circle, thought of as maps $\to$ $\mathbf{C}$ are immersions for $n = \pm 1, \pm 2, \pm 3\ldots$, and they are never regularly homotopic,

i.e. homotopic through maps that are also immersions. Can you find an immersion that is not homotopic to any on this list?

Homework #5.

1. Show that any bundle on M × [0,1] is equivalent to a vector bundle on M and then × [0,1]. Deduce that any vector bundle on a ball is trivial.

2. Show that $GL_n(\mathbf{R})$ contains the orthogonal group O(n) as a deformation retract, and that $GL_n(\mathbf{C})$ contains the unitary group U(n) as a deformation retract.

3. Show that real, respectively complex, vector bundles over S^n are in a 1-1 correspondence with $\pi_{n-1}(O(n))$ and $\pi_{n-1}(U(n))$.

4. Show that the classification of n-dimensional real (or complex) vector bundles over S^k is independent of k, if n>k (or >k/2 in the complex case). Classify bundles over the 1- and 2-spheres.

5. Show that every embedding of S^n in S^{n+2} (a "knot") has trivial normal bundle. Deduce that it has a neighborhood diffeomorphic to $S^n \times D^2$. Extend the projection map of the boundary of this neighborhood to S^1 to the complement (hint: use the first part of the next problem and Alexander duality) and deduce that every knot bounds an orientable surface in S^{n+2}. Generalize this to the situation when one has an oriented embedding of a disjoint union of several spheres.

6. Prove that $H^1(M ; \mathbf{Z}) \cong$ homotopy classes of maps $M \to S^1$. Interpret the Pontrjagin-Thom construction in this case and relate this to the Jordan separation theorem.

7. Let M be a triangulated n-manifold and ξ an oriented vector bundle over M. Find obstruction to finding a non-zero section of ξ that is an element of $H^n(M ; \mathbf{Z})$.

8. If M is a manifold with boundary, show that the sum of the indices of the zeroes of a vector field that is inwardly pointing and having isolated zeroes is χ(M).

9. Show that any Lie group G has χ(G) = 0.

10. Show that any translation manifold (remember those from week 1?) has trivial tangent bundle.

Remark: If M is a connected non-compact manifold with trivial tangent bundle, then it has a translation structure. For compact manifolds, this is not true with S^3 being a counterexample.

Homework #6.

1. What is the necessary and sufficient condition for two codimension 1 oriented submanifolds of an oriented manifold to be cobordant to disjoint submanifolds? Give a simple cohomological condition that is necessary and an example where it is insufficient.

2. Show that if $M = \partial W$ are compact manifolds with M being the boundary, and dim M is odd, then $\chi(M) = 2\chi(W)$. Deduce that RP^{2k} is not the boundary of any compact manifold.

3. Suppose that M and N are compact submanifolds of W and m+n=w, then if the submanifolds are transverse, show that the parity of the number of intersections only depends on the homotopy class of the embeddings of M and N. Moreover, if all these manifolds are oriented, show that it is possible to assign signs to the intersections to get an intersection number in **Z**.

4. If M and N are I dimensional oriented submanifolds of S^3, then the intersection number of N with a surface bounded by M (use a result of the previous homework) is independent of the surface, and the embedding of N in its homootpy class in the complement of M. Moreover identify this number with the degree of the map $M \times N \to S^2$ given as $(m,n) \to (m-n)/||m-n||$. As a corollary, show that this number is symmetric in M and N. This is called the linking number of M and N, denoted lk(M,N).

5. Show that the intersection of two transverse hypersurfaces in CP^2 of degrees m and n respectively has exactly mn points. If the intersection is not transverse, but has isolated intersection points, then it is possible to define multiplicities of these intersection points (using linking numbers of the intersections of the surfaces) with an ε-sphere and then the sum of the multiplicities of the intersection points.

6. If $f: S^3 \to S^2$ is a smooth map and p and q are regular values. Show that the integer $lk(f^{-1}(p), f^{-1}(q))$ is independent of p and q and only depends on the homotopy class of f. Show that it defines a homomorphism $\pi_3(S^2) \to \mathbf{Z}$.

7. Suppose that G is a finite group, then there is a number n_G so that for all sufficiently large genus, a surface of genus g has an effective G action if and only if $g = 1 \bmod n_G$. (In other words, this criterion holds with only finitely many exceptions.)

8. Show that for $G = Z_p$, $n_G = 1$ and for $G = Z_p \times Z_p$, $n_G = p$. (I don't remember whether this is right for all p, or all odd p, so figure this out!)

9. Let W be a compact manifold, and $f:W \to S^1$ be a function so that Df is nowhere zero, show that W is a fiber bundle over the circle. Using this, show that dim $H^1(M, \mathbf{R}) > 1$, then M fibers over the circle via infinitely many nonhomotopic primitive maps (if it fibers at all).

Remark: Of course, if f is a fiber bundle map, so is f^n for n>1 which gives new fibrations. By primitive, we mean that the map to the circle is indivisible in $H^1(W; \mathbf{Z})$, which corresponds to the fiber being connected.

Remark: It is possible for a manifold to fiber over the circle in more than one way via *homotopic* maps, but this subtle phenomenon is governed by complicated algebraic properties of the fundamental group. All known examples of this except in dimension 5 have a torsion free fundamental group (and for many torsion free fundamental groups, it is known to be impossible – aside from dimension 5). On the other hand, in dimension 5 there are examples of infinitely many homotopic fibrations with $\pi \cong \mathbf{Z}$.

Homework #7. (week 8, because of midterm)

1. Show that a Lie group with a left invariant measure is compact iff its measure is $<\infty$.
2. Show that any compact Lie group has a bi-invariant metric, i.e. one which is both left and right invariant.
3. Classify the bi-invariant metrics on a torus, T, and on SU(2).
4. Show that G = Aff($\mathbf{R}$) (= the affine group of the line = $\{x \to ax+b \mid a>0$ and b is real$\}$) does not possess a bi-invariant measure.
5. Show that if G is a compact Lie group acting smoothly on a smooth manifold M then M has a smooth G-invariant Riemannian metric. Use this to state a G-invariant tubular neighborhood theorem for such group actions and their invariant submanifolds. Also remark that the fixed set of a smooth G-action is a smooth submanifold.
6. Show that the obvious action of $SL_2(\mathbf{Z})$ on T^2 is not isometric for any Riemannian metric on the torus.
7. Show that the action of fractional linear transformations on the Riemann sphere does not preserve any smooth measure. (A smooth measure is one whose Radon-Nikodym derivative w.r.t. the measure defined by any coordinate chart is a smooth function.)
8. Let v be a tangent vector in T_eG of a Lie group. Consider the integral curve through the identity of the vector field obtained by left translation of v around G. Show that this defines a homomorphism of $\mathbf{R} \to G$. Check that the closure of this image is an abelian Lie group.
9. Show that if V and W are vector fields on a manifold, then the functional on $C^\infty(M)$ defined by VWf= V(W(f)) is not necessarily a vector field. But, [V,W] given by VW-WV is.
10. Show that [V,W] + [W,V] = 0 and that [V,[W,X]] + [W,[X,V]] + [X,[V,W]] = 0.

11. Classify the irreducible representations of SU(2) over **R**, and the complex irreducible representations of SO(3).

Homework 8 (due last day of class)

1. Prove that every orthogonal matrix of det = 1, is of the form e^A where A is skew symmetric. Use this to describe the tangent space to SO(n) at the identity, and the Lie bracket of left invariant vector fields.
2. If V is a vector field and ω is a k-form, then by putting V into the first coordinate of ω, we get a map i_V: $\Lambda^k \to \Lambda^{k-1}$. Compute i_V for a wedge of forms.
3. If V is a vector field, then we can integrate it to get diffeomorphisms $\varphi_V(t)$ t>0. If ω is a k-form, then we can consider lim $[\varphi_V(t)^*\omega - \omega]/t$ to obtain another k form. What is this operation for k=0?
4. What is this operation, if done to a vector field W, rather than a form?
5. Express this operation for forms in terms of d and i_V. (Hint use the derivation properties of all objects entering in this problem to reduce to functions and their differentials.)
6. Let W be an oriented Riemannian manifold, and M an oriented submanifold. Construct a closed w-m form, υ, that is nontrivial on the normal vector spaces to M, so that $\int_N \upsilon$ = int(N,M) for N a compact oriented submanifold of dimension w-m. What is the geometric interpretation of inegration with respect to the wedge of the forms associated to two submanifolds, $\upsilon_M \wedge \upsilon_{M'}$?
7. Show that if G is a compact Lie group and T is a torus, so that the normalizer has N(T)/T finite. (This is true exactly for the maximal torus, but that's not relevant.) Show that the Euler characteristic satisfies χ(G/N(T)) = 1. (Hint: Note that Euler characteristic is multiplicative in finite covers, so study G/T instead.)
8. Show that if G is a finite group acting freely on a manifold M, then H*(M/G) can be computed using invariant differential forms on M, and that therefore H*(M/G) is the G-invariants of H*(M).

(*) This last statement is true even if the action is not free.

9. Show that on a compact metric manifold M, there is an e>0, such that any nontrivial G action on M, for any nontrivial group, some orbit has diameter > e. (If you find it easier to assume that G is acting freely, then do so – but it's true in general.)
10. Give an example of a manifold for which the Morse inequalities are not sharp for any Morse function.

Final Exam.

This will be a take-home that will be individualized. During the last week of class, each person should email me the day of final week that they want to do the exam. I will email to them their exam at 11:59 am of that day, and the exam should be returned as a pdf to me by 3 pm.