THE FREYD-HELLER GROUP AND THE FAILURE OF BROWN REPRESENTABILITY

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Abstract. It is a classical result due to Brown ([4]) that any set valued contravariant functor on the homotopy category of connected based topological spaces taking coproducts to products and weak pushouts to weak pullbacks is representable.

This is however, false when we drop the assumption that our spaces and maps are based, or if we drop the assumption that the spaces under consideration are connected. We describe a construction, often called the "Freyd-Heller" group ([2]), that results in a counterexample in both cases.

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1. Introduction

1.1. **Overview.** The classical Brown representability theorem ([4]) states that any set valued contravariant functor on the homotopy category of connected based CW-complexes taking coproducts to products and weak pushout to weak pullbacks is representable. In particular, cohomology theories satisfy these properties, and consequently are represented by spaces. These spaces are the Eilenberg-Maclane spaces.

In fact, Brown himself proved a generalisation in [7]. We present a modern update below.

Theorem 1.1. *Let $\mathcal{C}$ be a category with coproducts, weak pushouts and weak sequential colimits. Assume further that $\mathcal{C}$ admits a strongly generating set G that is closed under finite coproducts and weak pushouts. Finally, assume that for each $X \in G$ the functor $\mathcal{C}(X, -)$ takes weak sequential colimits to strict colimits.*

Then, if $F : \mathcal{C}^{op} \to Set$ takes coproducts to products and weak pushouts to weak pullbacks, it is representable.

The homotopy category of pointed connected CW-complexes satisfies these assumptions. In particular mappings out of spheres suffice to detect isomorphisms, as here weak equivalences are in fact invertible and thus such a strongly generating set exists. Explicitly, we take the G of Theorem 1.1. to be the closure of the set of (based) spheres under finite coproducts and weak pushouts. (We say G is a strongly generating set if a morphism $f : Y \to Z$ inducing isomorphisms $\mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ for each $X \in G$ is sufficient to conclude that f is an isomorphism)

Interestingly, these assumptions fail to hold if we disregard either the connectedness or the based hypotheses. The point of failure is that there is no suitable generating set. (for more details, see [6])

We prove that in both the unbased and disconnected case, there exists functors F as above taking coproducts to products and weak pushouts to weak pullbacks that are not representable. It then also follows that neither of the two categories admit suitable strongly generating sets.

1.2. **Some Definitions and Brown Functors.**

Notation 1.2. Let *Set* denote the category of sets, and $\mathcal{H}$ the homotopy category of connected based CW-complexes, i.e. the category whose objects are connected based CW-complexes and morphisms are homotopy classes of based continuous maps.

Remark 1.3. When we say a CW complex is based, we require the basepoint to be a 0-cell. In particular, the CW complex is then nondegenerately based, i.e. the inclusion of the basepoint is a cofibration.

Explicitly, given a based map $f : (X, x) \to (Y, y)$ and a path α from y to $y' \in Y$, there exists a free (unbased) homotopy of f restricting to the path α on the basepoint.

Definition 1.4. An *idempotent* $f \in \mathcal{C}(X, X)$ in a (locally small) category $\mathcal{C}$ is a map f such that $f^2 = f$. It is further said to *split* if there exists an object Y and morphisms $r \in \mathcal{C}(X, Y), i \in \mathcal{C}(Y, X)$ such that

$$ir = f$$

$$ri = \mathrm{id}_Y$$

We then call r a *retraction* of X onto Y, and say Y is a *retract* of X.

Definition 1.5. A *diagram* in a category $\mathcal{C}$ is a set of objects in $\mathcal{C}$ and morphisms such that the composites of any two sequences $f_0, f_1, \ldots, f_k$ of composable morphisms with the same domain(f_0) and codomain(f_k) are equal.

Formally, for a small category J, a diagram of shape J in $\mathcal{C}$ is a functor $F : J \to \mathcal{C}$. The intuition is that J specifies the structure of the diagram and the functor F assigns a labelling to the objects and morphisms, defining the image.

We identify a diagram of shape J with it's image, for the sake of exposition.

Example 1.6. Consider a 'square' category, i.e.

$$\begin{tikzcd}
A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr] & B \arrow[loop right, "\mathrm{id}_B"] \arrow[d, "g"] \\
C \arrow[loop left, "\mathrm{id}_C"] \arrow[r, "v"'] & D \arrow[loop right, "\mathrm{id}_D"]
\end{tikzcd}$$

such that $vu = gf$.
Then a diagram of this shape in a category $\mathcal{C}$ is merely a commuting square.

Definition 1.7. A *cone* over a diagram in a category $\mathcal{C}$ with apex $A \in \mathcal{C}$ is a collection of morphisms from A to every object of the diagram, such that the 'big diagram' commutes.

Formally, a cone over a diagram $F : J \to \mathcal{C}$ with apex $A \in \mathcal{C}$ is a natural transformation $\bar{A} \to F$, where $\bar{A}$ is the constant functor, sending every object to A and every morphism to id_A.

Example 1.8. For example a cone over the discrete diagram $A \qquad B$ is simply

$$\begin{tikzcd}
& C \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}$$

Definition 1.9. Dually, a *co-cone* over a diagram in a category $\mathcal{C}$ with nadir $A \in \mathcal{C}$ is a collection of morphisms to A from every object of the diagram, such that the 'big diagram' commutes.

Formally, a co-cone over a diagram $F : J \to \mathcal{C}$ with nadir $A \in \mathcal{C}$ is a natural transformation $F \to \bar{A}$, where $\bar{A}$ is the constant functor, sending every object to A and every morphism to id_A.

Definition 1.10. The *limit* of a diagram $F : J \to \mathcal{C}$ is a universal cone (with say, apex A), in the sense that every other cone over the diagram factors uniquely through it.

Formally, the limit of a diagram $F : J \to \mathcal{C}$ is a cone $\alpha : \bar{A} \to F$, such that for any other cone $\beta : \bar{B} \to F$, there is a unique morphism $\lambda : B \to A$ in $\mathcal{C}$ such that $\beta = \alpha \circ \lambda$.

Example 1.11. In the category of sets, the limit of the discrete diagram $A \qquad B$ is the cartesian product, as a function $C \to A \times B$ is uniquely determined by a pair of functions $C \to A, C \to B$.

$$\begin{tikzcd}
& C \arrow[d, dashed, "!"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
A & & B
\end{tikzcd}$$

Definition 1.12. Dually, the *colimit* of a diagram $F : J \to \mathcal{C}$ is a universal co-cone (with say, nadir A), in the sense that every other co-cone under the diagram factors uniquely through it.

Formally, the colimit of a diagram $F : J \to \mathcal{C}$ is a co-cone $\alpha : F \to \bar{A}$, such that for any other co-cone $\beta : F \to \bar{B}$, there is a unique morphism $\lambda : A \to B$ in $\mathcal{C}$ such that $\beta = \lambda \circ \alpha$.

Definition 1.13. A weak limit over a diagram is a cone over a diagram such that any other cone over the diagram factors through it. Dually, a weak colimit under a diagram is a co-cone under a diagram such that every other co-cone under the diagram factors through it.

That is, we delete uniqueness from the definitons of limit and colimit.

Definition 1.14. We say a functor $F : \mathcal{C}^{op} \to Set$ is a Brown functor if it takes coproducts to products and weak-pushouts to weak-pullbacks.

Definition 1.15. A functor $F : \mathcal{C} \to Set$ is *representable* if there exists an object Y of $\mathcal{C}$ and isomorphisms $F(X) \to \mathcal{C}(Y, X)$ for every object X of $\mathcal{C}$ that are further natural in X, i.e. a natural isomorphism $F \cong \mathcal{C}(Y, -)$.

Notation 1.16. As is conventional, we identify the functors $\mathcal{C}^{op}(Y, -)$ and $\mathcal{C}(-, Y)$.

Examples 1.17. The primary diagrams we will be interested in limits and colimits of are,

(1) A discrete diagram is one where the only morphisms under consideration are the identities. The limit over a discrete diagram is called the product. Dually, the colimit under a diagram is called the coproduct.
(2) The limit of a diagram of the form $A \rightrightarrows B$ is called an equaliser. Dually, the colimit of such a diagram is called a coequaliser.
(3) The limit of a diagram of the form
$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$
is called a pullback. Dually, the colimit of a diagram of the shape
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$
is called a pushout.
(4) The limit of a diagram of the form $\ldots \longrightarrow X_1 \longrightarrow X_0$ is called a sequential limit. Dually, the colimit of a diagram of the form $X_0 \longrightarrow X_1 \longrightarrow \ldots$ is called a sequential colimit.

Analogously, we can define weak versions of the limits and colimits above.

Example 1.18. Reduced cohomology theories on based CW complexes are Brown functors.

- The "homotopy" axiom states that reduced cohomology descends to a functor on $\mathcal{H}$. The "additivity" axiom (sometimes called the "wedge" axiom) states that cohomology sends coproducts to products.
- Further for any reduced cohomology theory, we can derive the "Mayer-Vietoris" axiom, that will imply that reduced cohomology takes weak pushouts to weak pullbacks.

2. The Brown Representability Theorem

We now have enough to state the Brown Representability theorem.

Theorem 2.1. *(Brown Representability Theorem): Let $F : \mathcal{H}^{op} \to Set$ be a Brown functor. Then F is representable.(See [4])*

Theorem 2.1 will imply that all idempotents split in $\mathcal{H}$. We state the lemmas used to prove this fact for an arbitrary category, as we will use them later to show non-representability of a Brown functor in a different category.

Lemma 2.2. *Let $F : \mathcal{C}^{op} \to Set$ be a Brown functor, and $G : \mathcal{C}^{op} \to Set$ a retract (in the sense of Definition 1.2) of F(in the functor category $Set^{\mathcal{C}^{op}}$), then G is a Brown functor.*

Proof. As G is a retract of F, we have natural transformations $i : G \to F$ and $r : F \to G$ such that $ri = \mathrm{id}_G$. We must show two things:

(1) G sends weak pushouts to weak pullbacks: Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

be a weak pushout square, we must show it gets mapped to a weak pullback square.

Let α be a cone over the diagram with apex X (depicted with dotted arrows)

$$\begin{tikzcd} GA & GB \arrow[l, "Gf"'] \\ GC \arrow[u, "Gg"] & X \arrow[l, dashed, "\alpha_C"] \arrow[u, dashed, "\alpha_B"'] \end{tikzcd}$$

Applying i, we get a cone which factors through FD as follows.

$$\begin{tikzcd} FA & FB \arrow[l, "Ff"'] & \\ FC \arrow[u, "Fg"] & FD \arrow[l, "Fv"] \arrow[u, "Fu"'] & \\ & & X \arrow[ul, dashed, "\exists h"] \arrow[uul, bend right, "i_B\alpha_B"'] \arrow[ull, bend left, "i_C\alpha_C"] \end{tikzcd}$$

And then we apply r giving us the desired factorization.

$$\begin{tikzcd} GA & GB \arrow[l, "Gf"'] & \\ GC \arrow[u, "Gg"] & GD \arrow[l, "Gv"] \arrow[u, "Gu"'] & \\ & & X \arrow[ul, dashed, "r_D h"] \arrow[uul, bend right, "\alpha_B"'] \arrow[ull, bend left, "\alpha_C"] \end{tikzcd}$$

Hence, G does in fact send weak pushouts to weak pullbacks.

(2) G sends coproducts to products: The proof of existence of a factorization is same as for the previous case *mutatis mutandis*.

Let A_i be a collection of objects of $\mathcal{C}$, denote the corresponding discrete diagram by J. Let $\alpha : J \to \coprod A_i$ be the colimit co-cone. Let $\beta : X \to GJ$ be a cone over GJ, then as remarked there is a factorisation $\beta = (G\alpha)\lambda$ for some $\lambda : X \to G(\coprod A_i)$

To prove uniqueness, assume for the purpose of contradiction that there were two different factorisations $f, g : X \to G(\coprod A_i)$ as before. Then on applying i we have that $i_{G(\coprod A_i)}f, i_{G(\coprod A_i)}g$ are two factorisations of the cone $i\beta : X \to FJ$ through $F(\coprod A_i) = \prod F(A_i)$ (by hypothesis). By the uniqueness criterion of the universal property,

$$i_{G(\coprod A_i)}f = i_{G(\coprod A_i)}g$$

and hence

$$f = r_{F(\coprod A_i)} i_{G(\coprod A_i)} f = r_{F(\coprod A_i)} i_{G(\coprod A_i)} = g$$

□

We recall the (contravariant) Yoneda lemma. (Chapter 3, Section 2 of [3])

Theorem 2.3. *Let $\mathcal{C}$ be a locally small category and $F : \mathcal{C}^{op} \to Set$ be a functor. Then there is a bijection*

$$Set^{\mathcal{C}^{op}}(\mathcal{C}(-, X), F) \cong F(X)$$

between natural transformations $\mathcal{C}(-, X) \to F$ and elements of $F(X)$, that is further natural in both F and X.

Which has as a corollary,

Corollary 2.4. *The 'Covariant Yoneda Embedding' $y : \mathcal{C} \to Set^{\mathcal{C}^{op}}$ satisfying $y(X) = \mathcal{C}(-, X)$ is a fully-faithful embedding.*

Remark 2.5. Recall that a fully-faithful embedding is one that induces bijections on morphism *sets* (as the category is locally small).

We make one final observation,

Lemma 2.6. *Let $f \in \mathcal{C}(X, X)$ be an idempotent in $\mathcal{C}$. Then, on applying y, $f_* : \mathcal{C}(-, X) \to \mathcal{C}(-, X)$ is an idempotent in $Set^{\mathcal{C}^{op}}$, which splits.*

Proof. It is a direct check that f_* is an idempotent. To construct a splitting, we define a functor $R : \mathcal{C}^{op} \to Set$. On objects, define $R(X) = \text{Image}(f_*)$. For a morphism $g : Y \to Y'$ in $\mathcal{C}$, from the commuting square

$$\begin{tikzcd}
\mathcal{C}(Y', X) \arrow[r, "g^*"] \arrow[d, "f_*"'] & \mathcal{C}(Y, X) \arrow[d, "f_*"] \\
\mathcal{C}(Y', X) \arrow[r, "g^*"'] & \mathcal{C}(Y, X)
\end{tikzcd}$$

we observe that g^* maps $R(Y')$ into $R(Y)$, hence decends to a map $\bar{g} : R(Y') \to R(Y)$.

The assignment $R(g) = \bar{g}$ defines a functor $R : \mathcal{C}^{op} \to Set$. Also, the inclusion $R(Y) \hookrightarrow \mathcal{C}(Y, X)$ defines a natural transformation $\iota : R \to \mathcal{C}(-, X)$. Further, the restriction of the codomain of f to its image defines another natural transformation $\rho : \mathcal{C}(-, X) \to R$.

The identities

$$\iota\rho = f_*$$
$$\rho\iota = \mathrm{id}_F$$

hold as they hold on objects, thus providing the desired splitting for f_*. □

Consequently, if the retract of $\mathcal{C}(-, X)$ is representable, Corollary 2.4 implies that the splitting for f_* lifts to a splitting of f in $\mathcal{C}$.

We now have enough to prove a general fact about categories where the Brown Representability theorem holds. A counterexample to this will prove that there exists a Brown functor which is not representable.

Theorem 2.7. *Let $\mathcal{C}$ be a category in which all Brown functors are representable. Then all idempotents split in $\mathcal{C}$.*

Proof. Applying Lemma 2.6 to an arbitrary idempotent f of $\mathcal{C}$ we get a splitting for f_* in $Set^{\mathcal{C}^{op}}$. One observes that by Lemma 2.2 the retract of $\mathcal{C}(-, X)$ induced by f_* is a Brown functor.

Hence by the hypothesis of the Theorem the retract is representable, and thus f splits in $\mathcal{C}$ by Corollary 2.4. □

Corollary 2.8. *All idempotents in $\mathcal{H}$ split.*

3. Conjugacy actions on group valued functors

Theorem 2.7 implies that to show failure of Brown representability in the unbased or disconnected case, it suffices to locate an idempotent that doesn't split. Before proceeding further we introduce some notation.

Notation 3.1. Let $u\mathcal{H}$ denote the category whose objects are nonempty (unbased) connected CW-complexes and morphisms are homotopy classes of (unbased) continuous functions.

Let $\mathcal{H}_+$ denote the category whose objects are based CW-complexes and morphisms are homotopy classes of based continuous functions.

Definition 3.2. Define a functor $(-)_+ : u\mathcal{H} \to \mathcal{H}_+$ by defining $(X)_+$ to be X with a disjoint basepoint for objects X of $u\mathcal{H}$. For a function $f : X \to Y$ representing a homotopy class $[f]$ in $u\mathcal{H}$, define its image under $(-)_+$ to be the based homotopy class of the map $(X)_+ \to (Y)_+$ restricting to f on X. One can check that this assignment gives a well defined functor.

We desire idempotents in both these categories that do not split. In fact it suffices to find one in $u\mathcal{H}$, by the following lemma.

Lemma 3.3. *Let f be an idempotent in $u\mathcal{H}$ that does not split. Then, $(f)_+$ is an idempotent in $\mathcal{H}_+$ that does not split.*

Proof. We prove the contrapositive. Let f have source (and target) X. We use crucially that X is connected. Let $r : (X)_+ \to Y$ and $i : Y \to (X)_+$ be a splitting of $(f)_+$. Then note that as $X \times I$ is connected, any element in the homotopy class of r maps X into the same connected component of Y. Thus, r restricts to a map from X to a connected component Z of Y. Further, i restricts to a map of Z into X. This defines a splitting of f in $u\mathcal{H}$. □

We now describe a construction used to construct categories with non-split idempotents. To do so, we define conjugacy actions.

Definition 3.4. Denote by $\mathfrak{G}$ the category of groups, and let $\mathbb{A}$ be a category in which all idempotents split.

Then a *conjugacy action* on a functor $\pi : \mathbb{A} \to \mathfrak{G}$ is an assignment $f^\alpha : A \to B$ for every $f : A \to B$ in $\mathbb{A}$ and $\alpha \in \pi(B)$ such that:

$$f^1 = f \tag{3.5}$$

$$(f^\alpha)^\beta = f^{\alpha\beta} \tag{3.6}$$

$$g^\alpha \circ f = (g \circ f)^\alpha \tag{3.7}$$

$$g \circ f^\alpha = (g \circ f)^{\pi g(\alpha)} \tag{3.8}$$

$$\pi(f^\alpha) = \alpha^{-1}\pi(f)\alpha \tag{3.9}$$

Given such a conjugacy action on a group valued functor $\pi : \mathbb{A} \to \mathfrak{G}$, we can define a new category $\mathbb{A}_{/\sim}$ by identifying f and f^α for all $\alpha \in \pi(B)$. One can check that (3.8) and (3.9) ensure that we have a well defined category.

Definition 3.10. A *conjugacy idempotent* f is a morphism in $\mathbb{A}$ such that for some α, $f^2 = f^\alpha$, i.e. the equivalence class of f is an idempotent in $\mathbb{A}_{/\sim}$. We further say a conjugacy idempotent f in $\mathbb{A}$ *splits* if its equivalence class is a split idempotent in $\mathbb{A}_{/\sim}$.

We have one immediate example of a conjugacy action, when π is the identity functor on the category of groups and (as is forced by (3.10)) the conjugacy action is just the inner automorphism, i.e. $f^\alpha = \alpha^{-1} f \alpha$.

The other example that we will be primarily concerned with is the fundamental group functor $\pi_1 : \mathcal{H} \to \mathfrak{G}$ and the conjugacy action is defined as follows.

Definition 3.11. Fix the basepoint 0 for the unit interval $I = [0, 1]$. Let $f : X \to Y$ be a representative of a morphism in $\mathcal{H}$. Consider an element of the fundamental group of X, represented by a loop α at the basepoint of X. By the universal property of the coproduct, the maps f and α determine a map $X \vee I \to Y$. As X is nondegenerately based, $X \vee I$ is a retract of $X \times I$. Thus, we get a map $h : X \times I \to Y$ by pre-composing with r. Define $f^\alpha = h_{X\times\{1\}}$. The map f^α is defined uniquely up to based homotopy. One can check that this is in fact a conjugacy action.

Proposition 3.12. *$\mathcal{H}_{/\sim}$ is equivalent to $u\mathcal{H}$.*

Proof. Denote by $\mathfrak{H}$ the category of connected nondegenerately-based CW complexes and free homotopy classes of based maps. There is a canonical functor $\xi : \mathcal{H} \to \mathfrak{H}$.

Consider arbitrary $f : X \to Y$ in $\mathcal{H}$ and $\alpha \in \pi_1(Y)$, where $\pi_1(Y)$ is the fundamental group of Y. Then f^α is free-homotopic (homotopic in an unbased sense) to f. Then the map ξ descends to $\bar{\xi} : \mathcal{H}_{/\sim} \to \mathfrak{H}$. One further observes that $\bar{\xi}$ induces bijections on all morphism sets, and is in fact an isomorphism.

Thus $\mathcal{H}_{/\sim}$ is precisely the category of pointed connected CW-complexes and free-homotopy classes of continuous pointed maps between them.

Construction 3.13. Define a functor $F : u\mathcal{H} \to \mathcal{H}_{/\sim}$ as follows. For a connected CW complex X, fix a 0-cell x to be the basepoint and set $F(X) = (X, x)$. Now for a morphism $[f] : X \to Y$ in $u\mathcal{H}$, pick a representative f.

Let x and y be the chosen basepoints of $F(X)$ and $F(Y)$ respectively. As Y is a connected CW complex, there exists a path α from $f(x)$ to y. This induces a map $\psi : X \vee I \to Y$ restricting to f on X and α on I. The map ψ extends to a map $X \times I \to Y$ as the 0-cells are non-degenerate basepoints.

This is a homotopy between f and a based map $g : (X, x) \to (Y, y)$ which is unique up to free homotopy. Also, the homotopy class of g, $[g]$ is independent of the representative f.

The assignment $F([f]) = [g]$ defines the functor.

One further checks that this functor is fully-faithful. Finally, it is essentially surjective, i.e. every object of the codomain is isomorphic to an object in the image of the functor. To see this consider any other choice of basepoint y'. The same strategy as that of Construction 3.13. applied to a path between y and y' exhibits a based map $(Y, y) \to (Y, y')$ free homotopic to the identity. Consequently, this map induces an isomorphism between (Y, y) and (Y, y') in $\mathcal{H}_{/\sim}$.

Thus F is a fully-faithful essentially surjective functor, and is thus an equivalence. $\square$

Consequently, it suffices to find an idempotent that does not split in either of these categories, as the retract induced by the splitting extends to the isomorphism class. We now proceed to construct a non-splitting idempotent in $\mathcal{H}_{/\sim}$.

4. A Splitting Lemma

Our tool in finding non-split idempotents is the following. (Main Lemma of [2])

Theorem 4.1. *Let $\mathbb{A}$ be a category in which all idempotents split, and let $\pi : \mathbb{A} \to \mathfrak{G}$ have a conjugation action, then we have a commutative diagram*

$$\begin{tikzcd} \mathbb{A} \arrow[r, "\pi"] \arrow[d] & \mathfrak{G} \arrow[d] \\ \mathbb{A}_{/\sim} \arrow[r, "\bar{\pi}"'] & \mathfrak{G}_{/\sim} \end{tikzcd}$$

Here, the conjugacy action on $\mathfrak{G}$ is the inner automorphism action described earlier and the column maps are the natural projections. Then for an idempotent f in $\mathbb{A}_{/\sim}$, f splits in $\mathbb{A}_{/\sim}$ if and only if $\bar{\pi}(f)$ splits in $\mathfrak{G}_{/\sim}$.

Remark 4.2. In the rest of the section, we assume $\mathbb{A}$ is a category in which all idempotents split.

Proof. We follow the proof in [2].

One direction is clear, by functoriality a splitting in $\mathbb{A}_{/\sim}$ is taken to a splitting in $\mathfrak{G}_{/\sim}$. For the converse, we first note some lemmas.

Lemma 4.3. *An idempotent f in $\mathbb{A}_{/\sim}$ splits if and only if there exists f' in $\mathbb{A}$ such that f' represents f and $f'^2 = f'$.*

Proof. One direction is clear as a splitting in $\mathbb{A}$ (which exists by the assumption of Remark 4.2) induces a splitting in $\mathbb{A}_{/\sim}$. For the converse, note that if f splits in $\mathbb{A}_{/\sim}$, then we can find morphisms r, i in $\mathbb{A}$ and a representative $\bar{f}$ of f such that

$$ir = \bar{f}$$

$$ri = 1^\alpha$$

Then a choice of f' satisfying the desired conditions is $f' := r^{\alpha^{-1}} i$. □

Lemma 4.4. *If $f^2 = f^\alpha$ in $\mathbb{A}$ for some $\alpha \in \mathrm{Image}(\pi(f))$ then f splits in $\mathbb{A}_{/\sim}$.*

Proof. The image of a group homomorphism is a subgroup of the codomain. Thus, if $\alpha \in \mathrm{Image}(\pi(f))$, then $\alpha^{-1} \in \mathrm{Image}(\pi(f))$. Pick arbitrary $\beta \in \pi(f)^{-1}(\alpha^{-1})$. Then f^β is an idempotent in $\mathbb{A}$, so by Lemma 4.3, f splits in $\mathbb{A}_{/\sim}$. □

Lemma 4.5. *If $f^2 = f^\alpha$ in $\mathbb{A}$ and $\pi(f)(\alpha)$ is a fixed point of $\pi(f)$, (i.e. $\pi(f)^2(\alpha) = \pi(f)(\alpha)$) then f splits in $\mathbb{A}_{/\sim}$.*

Proof. Let $g = f^2$.

$$g^2 = f^2 ff = f^\alpha ff = (f^2)^{\pi(f)(\alpha)} f = (f^2)^{\pi(f)(\alpha\pi(f)(\alpha))} = g^{\pi(f)\pi(f)(\alpha^2)} = g^{\pi(g)(\alpha^2)}$$

Then

$$g^2 = f^2 ff = f^\alpha ff$$

By (3.7), $f^\alpha f = (f^2)^{\pi(f)(\alpha)}$. Hence,

$$f^\alpha ff = (f^2)^{\pi(f)(\alpha)} f$$

Applying (3.6) yields

$$(f^2)^{\pi(f)(\alpha)} f = (f^\alpha)^{\pi(f)(\alpha)} f = f^{\alpha \cdot \pi(f)(\alpha)} f$$

Now, we use (3.7) as before.

$$f^{\alpha \cdot \pi(f)(\alpha)} f = (f^2)^{\pi(f)(\alpha \cdot \pi(f)(\alpha))}$$

As $\pi(f)(\alpha)$ is a fixed point of $\pi(f)$, we can rewrite

$$\pi(f)(\alpha \cdot \pi(f)(\alpha)) = (\pi(f)\pi(f)(\alpha))^2 = \pi(f^2)(\alpha^2)$$

We conclude the formula

$$g^2 = g^{\pi(f)(\alpha \cdot \pi(f)(\alpha)} = g^{\pi(g)(\alpha^2)}$$

By Lemma 4.4, g splits in $\mathbb{A}_{/\sim}$. Being equal in $\mathbb{A}_{/\sim}$, so does f. □

Now suppose for arbitrary f, $\bar{\pi}(f)$ splits in $\mathfrak{G}_{/\sim}$ and identify f with a fixed representative in $\mathbb{A}$. Then, by Lemma 4.3 $\pi(f)^\alpha$ is an idempotent in $\mathfrak{G}$ for some α. Thus, $g = f^\alpha$ is such that $\pi(g)$ is an idempotent in $\mathfrak{G}$. Further

$$g^2 = (ff^\alpha)^\alpha = (f^2)^{\alpha\pi(f)(\alpha)} = g^{\beta\pi(f)(\alpha)}$$

where $f^2 = f^\beta$.

As $\pi(f)$ is an idempotent, the conditions of Lemma 4.5 hold, and thus f splits in $\mathbb{A}_{/\sim}$. This proves the other implication in the Splitting Lemma. □

5. A Right Inverse

In order to find a non-splitting idempotent f in $u\mathcal{H}$ it suffices to check that its image under $\bar{\pi}_1$ does not split. We desire a functor $K(-,1) : \mathfrak{G} \to \mathcal{H}$ that serves as a right inverse to the fundamental group functor π_1.

We recall some facts about simplicial sets, their geometric realisations and classifying spaces. For more details, the reader is referred to Section 5, Chapter 16 of [1].

Lemma 5.1. *There exists a functor $\mathfrak{G} \to \mathcal{H}$, that assigns to a group G its total space EG.*

Lemma 5.2. *For any group G, its total space EG is contractible.*

Lemma 5.3. *There is a functor $\mathfrak{G} \to \mathcal{H}$, sending a group G to its classifying space BG. This functor is denoted $K(-,1)$, and called the Eilenberg-Maclane functor.*

Lemma 5.4. *The natural projection $EG \to BG$ is a bundle with fiber G. Consequently the fundamental group of BG is G.*

Theorem 5.5. *The Eilenberg-Maclane functor $K(-,1) : \mathfrak{G} \to \mathcal{H}$ is a right inverse to the fundamental group functor.*

We will construct a right inverse $\bar{B}$ to $\bar{\pi}_1$. Then, suppose there was an idempotent f in $\mathfrak{G}_{/\sim}$ that did not split. Its image $\bar{B}f$ would be an idempotent in $\mathcal{H}_{/\sim}$ that did not split. This follows from the observation that a splitting of $\bar{B}f$ would taken to a splitting of f by $\bar{\pi}_1$.

Lemma 5.6. *Let X be a connected based CW complex and let Y be a $K(G,1)$ as above. Then every homomorphism $\pi_1(X) \to \pi_1(Y)$ is induced by a (based) map $X \to Y$ that is unique up to (based) homotopy.*

This is Proposition 1B.9. of [5]

Lemma 5.7. *For spaces X, Y and based maps $p : (X,x) \to (Y, p(x)), q : (X,x) \to (Y, q(x))$ such that there is a free homotopy $h : p \simeq q$, let α be the path-homotopy class of the path traced out by x under h. Then the following diagram commutes.*

$$\begin{tikzcd} & \pi_1(X,x) \arrow[dl, "p_*"'] \arrow[dr, "q_*"] & \\ \pi_1(Y,p(x)) \arrow[rr, "\gamma[\alpha]"'] & & \pi_1(Y,q(x)) \end{tikzcd}$$

The proof can be found in chapter 1, section 4 of [1].

Lemma 5.8. *There exists a functor $\bar{B}$ making the following diagram commute.*

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "K(-,1)"] \arrow[d] & \mathcal{H} \arrow[d] \\ \mathfrak{G}_{/\sim} \arrow[r, "\bar{B}"'] & \mathcal{H}_{/\sim} \end{tikzcd}$$

i.e. $K(-,1)$ takes inner automorphisms to maps free homotopic to the identity.

Proof. Fix an inner automorphism, say by α. Denote the resulting map on the group G by $\gamma[\alpha]$. Then $\gamma[\alpha](x) = \alpha x \alpha^{-1}$. Let l be a representative of α.

Let $\psi = K(\gamma[\alpha], 1)$. We wish to show that ψ is free homotopic to the identity. Define a map $X \wedge I \to X$ restricting to l on I and the identity on X. By nondegeneracy of basepoints, $X \wedge I$ is a retract of $X \times I$. Pre-composition gives a free homotopy from the identity funtion to a function f, also a based map. Then, by Lemma 5.7, f induces $\gamma[\alpha]$ on $\pi_1(K(G, 1)) = G$.

From Lemma 5.6. and the fact that $K(-, 1)$ is a right inverse to π_1, it follows that f and ψ are homotopic as based maps. But f is free homotopic to the identity, and thus so is ψ. □

Consequently, $\bar{B}$ is a right inverse to $\bar{\pi}_1$. Thus given a non split idempotent in $\mathfrak{G}_{/\sim}$, $\bar{B}f$ is an idempotent in $\mathcal{H}_{/\sim}$ that does not split.

6. Conjugacy Actions on Groups

We are thus led to look for idempotents in $\mathfrak{G}_{/\sim}$ that do not split. We describe the "Freyd-Heller" idempotent, as defined in [2].

Definition 6.1. A *conjugacy triple* is a three-tuple (F, f, α) of a group F, a group endomorphism f, and an element $\alpha \in F$ such that $f^2 = f^\alpha$. (We assume the standard conjugacy action via inner automorphisms)

A morphism ψ of conjugacy triples $\phi : (F, f, \alpha) \to (G, g, \beta)$ is a commuting square

$$\begin{tikzcd} F \arrow[r, "\psi"] \arrow[d, "f"'] & G \arrow[d, "g"] \\ F \arrow[r, "\psi"'] & G \end{tikzcd}$$

such that $\psi(\alpha) = \beta$.

Construction 6.2. We construct the initial conjugacy idempotent as follows.

Let the "*Freyd-Heller*" group F, be the group with generators $x_0, x_1, \dots$ subject to the relations $x_j x_i = x_i x_{j+1}$ for $0 \le i < j$. That is,

$$G :=< x_0, x_1, \dots | x_j x_i = x_i x_{j+1}, 0 \le i < j >$$

Define $f : F \to F$ by $f(x_i) = x_{i+1}$, and observe that

$$x_j x_i x_{j+1}^{-1} x_i^{-1} \mapsto x_{j+1} x_{i+1} x_{j+2}^{-1} x_{i+1}^{-1}$$

This is the identity, as it is a defining relation of F. Thus, the map f is well defined. Further

$$f^2(x_i) = x_{i+2} = x_0^{-1} x_{i+1} x_0 = x_0^{-1} f(x_i) x_0 = f^{x_0}(x_i)$$

Consequently $f^2 = f^{x_0}$. Thus (F, f, x_0) forms a conjugacy idempotent.

Theorem 6.3. *The conjugacy idempotent of Construction 6.2 is initial in the category of Definition 6.1.*

Proof. Let (G, g, β) be another conjugacy idempotent. Define $h : F \to G$ by $F(x_0) = \beta$ (which is forced).

It is forced by the commutative square criterion that we have $h(x_i) = g^i(\beta)$. One checks that this does in fact form a morphism of conjugacy triples. Thus, it is the unique morphism we desire. □

We wish to show that f is an idempotent in $\mathfrak{G}_{/\sim}$ that doesn't split. Our strategy is motivated by the following lemma.

Notation 6.4. For a group G and $a, b \in G$, denote the commutator by $[a, b] := aba^{-1}b^{-1}$. The normal subgroup generated by the commutators, the commutator subgroup is denoted $[G, G]$.

Lemma 6.5. *Let (G, g, β) be a conjugacy idempotent. Then, g splits in $\mathfrak{G}_{/\sim}$ if and only if β and $g(\beta)$ commute.*

Proof. One direction is immediate with the theory we have developed. If β and $g(\beta)$ commute then $g^2(\beta) = g^\beta(\beta) = g(\beta)$. Then by Lemma 4.5, we conclude that g splits in $\mathfrak{G}_{/\sim}$.

For the converse, let g split and choose representatives to get $r : G \to H, i : H \to G$ such that

$$ir \sim g$$
$$ri \sim \mathrm{id}_H$$

But as the conjugacy action is by inner automorphisms we have that $ri = \mathrm{id}_H$. As a consequence we have that

$$\mathrm{Image}(g) = \mathrm{Image}(g^2) \tag{6.6}$$

We can express g^3 in two ways,

$$g^3 = g(g^2) = g(g^\beta) = (g^2)^\beta$$
$$g^3 = g^2 g = g^\beta g = (g^2)^{g(\beta)}$$

Thus conjugation by $g(\beta)$ and β are the same on $\mathrm{Image}(g^2)$. We know this to be $\mathrm{Image}(g)$ by (6.6). Thus

$$g^\beta(\beta) = g^{g(\beta)}(\beta) = g(\beta)$$

Hence β and $g(\beta)$ commute, proving the converse. □

We use the specific form of the initial conjugacy idempotent of Construction 6.2 to restate Lemma 6.5.

Corollary 6.7. *For a conjugacy triple (G, g, β), g splits in $\mathfrak{G}_{/\sim}$ if and only if the kernel of the canonical morphism of conjugacy triples $F \to G$ contains the commutator subgroup $F' := [F, F]$*

Thus, to show that f doesn't split in $\mathfrak{G}_{/\sim}$, it suffices to show that F is non-abelian.

7. The First Canonical Representation

We seek a concrete representation of F, from which it will be evident that F is non-abelian. To do so, we define a *First Canonical Representation.*

Definition 7.1. For σ a continuous order preserving bijection of $\mathbb{R}$, its *support* is defined as $spt(\sigma) := \{x \in \mathbb{R} | s(x) \neq x\}$.

Construction 7.2. Let Σ denote the group of continuous order preserving bijections of $\mathbb{R}$. (Note that any continuous order preserving bijection of $\mathbb{R}$ is in fact also a homeomorphism).

Let $S \in \Sigma$ be translation by 1, i.e. $S(x) = x + 1$. Define

$$\Sigma^+ := \{\sigma \in \Sigma | spt(\sigma) \subseteq \mathbb{R}_{>0}\}$$

Observe that Σ^+ is a subgroup of Σ. Now consider conjugation by S, given by $\sigma \mapsto S^{-1}\sigma S$ on Σ.

This restricts to an endomorphism of Σ^+. That is, if $\sigma \in \Sigma^+$, then $S^{-1}\sigma S \in \Sigma^+$. Define $g : \Sigma^+ \to \Sigma^+$ by $g(T) = T^S$.

Pick a $Q \in \Sigma^+$ that agrees with S on $[1, \infty)$. Then $g^2 = g^Q$ and thus (Σ^+, g, Q) is a conjugacy triple.

Definition 7.3. Let (Σ^+, g, Q) be the conjugacy triple of Construction 7.2. We have a canonical map of triples $h : F \to \Sigma^+$. We call this the '*First Canonical Representation*'.

Construction 7.4. We can in fact compute h explicitly. Define $Q_k \in \Sigma^+$ as follows,

$$Q_k(x) = \begin{cases} x & x \leq k \\ 2x - k & k \leq x \leq k+1 \\ x+1 & x \geq k+1 \end{cases}$$

Then $Q_k^S = Q_{k+1}$ and consequently $h(x_k) = Q_k$.

Lemma 7.5. *F is non-abelian.*

Proof. Let Q_k be defined as in Construction 7.4. Then we directly compute

$$(Q_1 \circ Q_0)(1) = 3$$
$$(Q_1 \circ Q_0)(1) = 2$$

Hence Q_0 and Q_1 do not commute. Consequently, their pre-images x_0 and x_1 under the group homomorphism h of Construction 7.4 do not commute. Thus F is non-abelian. □

Theorem 7.6. *The endomorphism f of the conjugacy triple (F, f, α) of Construction 6.2 does not split in $\mathfrak{G}_{/\sim}$.*

Proof. By Lemma 7.5, F is non-abelian. Applying Corollary 6.7 to the conjugacy triple (F, f, α) itself, we see that the kernel of the canonical map (the identity) does not contain the commutator subgroup. Consequently, f does not split in $\mathfrak{G}_{/\sim}$. □

Lemma 7.7. *Let (F, f, α) be the conjugacy triple of Construction 6.2. Then $\bar{B}f$ is an idempotent in $u\mathcal{H}$ that does not split.*

Proof. The map f is an idempotent in $\mathfrak{G}_{/\sim}$, hence $\bar{B}f$ is an idempotent in $u\mathcal{H}$ (Here we identify $\mathcal{H}_{/\sim}$ with the equivalent category $u\mathcal{H}$). Assume for the purpose of contradiction that $\bar{B}f$ splits in $u\mathcal{H}$. Then there is a witness i, r to the splitting such that

$$ir = Bf$$
$$ri = \mathrm{id}$$

in $u\mathcal{H}$.

Then $\bar{\pi}_1(i), \bar{\pi}_1(r)$ is a witness to the splitting of f, a contradiction. Thus $\bar{B}f$ does not split in $u\mathcal{H}$. □

In conclusion, Theorem 2.7 and Lemma 7.7 imply that not every Brown functor $u\mathcal{H}^{op} \to Set$ is representable. Similarly, Lemma 7.7, Lemma 3.3 and Theorem 2.7 imply that not every Brown functor $\mathcal{H}_+^{op} \to Set$ is representable. Thus Brown representability fails to hold in these two categories.

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