A non obvious estimate for the pressure

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Abstract

In Euler and Navier Stokes equations, the pressure is related to the velocity by the formula $p = R_i R_j u_i u_j$. We prove that if $u \in C^\alpha$ then $p \in C^{2\alpha}$.

1 Introduction

In Euler or Navier Stokes equation, the pressure is computed from the velocity by the formula

$$p = R_i R_j \; u_i u_j. \tag{1.1}$$

where R_i denotes the Riesz transform and repeated indexes are summed. Since the Riesz transforms are operators of order zero, it is generally understood that p would have the same regularity estimates as $u \otimes u$ or $|u|^2$. Therefore, if $u \in C^\alpha$, it is natural to obtain that also $p \in C^\alpha$. The purpose of this note is to show that if $\alpha \in (0, 1/2) \cup (1/2, 1)$, actually $p \in C^{2\alpha}$, which seems somewhat surprising.

The case $\alpha = 1/2$ is a borderline case because in that case one would expect p to be Lipschitz. It is well known that that kind spaces do not get along well with singular integrals.

Note that (1.1) arises from the following equivalent formula

$$\triangle p = \partial_i \partial_j u_i u_j. \tag{1.2}$$

Even thought the most interesting cases for Euler or Navier Stokes equation are in dimension 2 and 3, we will present the proof in arbitrary dimension n, since there is no difference in difficulty.

As a notational clarification, we denote by $[u]_{C^\alpha}$ the C^α seminorm given by

$$[u]_{C^\alpha} = \sup_{x,y \in \mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

The main result of this note is the following.

Theorem 1.1. *Assume $u \in C^\alpha$ for $\alpha \in (0, 1/2) \cup (1/2, 1)$ is a divergence free vector field, and p be given by the formula (1.1). Then if $\alpha \in (0, 1/2)$, we have for all $x, y \in \mathbb{R}^n$,*

$$|p(x) - p(y)| \leq C|x-y|^{2\alpha} [u]^2_{C^\alpha},$$

where C is a constant depending on n and α. In addition, if $\alpha \in (1/2, 1)$,

$$|\nabla p(x) - \nabla p(y)| \leq C|x-y|^{2\alpha - 1} [u]^2_{C^\alpha}.$$

I came up with these estimates by 2010. Since I could not find a good application for them, I did send them for publication. However, the result was cited at least in [1] and [2] as a personal communication.

The rest of the article consists of the proof of Theorem 1.1

1.1 Subtracting constants

We start by the following simple observation. Since $\operatorname{div} u = 0$, the value of $\partial_i \partial_j (u_i - A_i)(u_j - B_j)$ does not depend on A and B for any two constant vectors A and B. In particular, for any two points x_1 and x_2, we have

$$\partial_i \partial_j u_i(x) u_j(x) = \partial_i \partial_j (u_i(x) - u_i(x_1))(u_j(x) - u_j(x_2)). \tag{1.3}$$

1.2 The case $\alpha \in (0, 1/2)$.

Let $\Phi(y) = \frac{c_n}{|y|^{n-2}}$ be the fundamental solution of the Laplace equation, i.e. $\triangle \Phi = -\delta_0$.

For any two points x_1 and x_2, let $\varphi(y) = \Phi(y - x_1) - \Phi(y - x_2)$. We multiply both sides of (1.2) by φ and integrate by parts. We obtain

$$p(x_2) - p(x_1) = \int p(y) \triangle \varphi(y) \, \mathrm{d}y = \int (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) D^2 \varphi(y) \, \mathrm{d}y$$

We assume that u has an appropriate decay at infinity so that the tail of integral is integrable. Assuming $u \in L^2$ is sufficient. The estimates below do not depend on any norm of u except $[u]_{C^\alpha}$.

Note that $D^2 \varphi$ contains some singular part (delta functions) at $y = x_1$ and $y = x_2$. However, we have that $(u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2))$ vanishes for both $y = x_1$ and $y = x_2$, so we can ignore the singular part of $D^2 \varphi$.

Let us compute the second derivatives of φ. We start by computing $D^2 \Phi$. We have $D^2 \varphi(y) = D^2 \Phi(y - x_1) - D^2 \Phi(y - x_2)$, where

$$\partial_{ij} \Phi(y) = \frac{|y|^2 \delta_{ij} - 2 y_i y_j}{|y|^{n+2}}.$$

In particular $|D^2 \Phi(y)| \leq C |y|^{-n}$.

There is some cancellation between the two terms when y is far from x_1 and x_2. Let $\bar{x} = \frac{x_1 + x_2}{2}$ and $r = |x_1 + x_2|$. Then if $|y - \bar{x}| > 5r$, by mean value theorem we have

$$|D^2 \varphi(y)| \leq \frac{Cr}{|y - \bar{x}|^{n+1}}.$$

Therefore, we can estimate that part of the integral

$$\begin{aligned}
\int_{B_{5r}^c(\bar{x})} & (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) \partial_{ij} \varphi(y) \, \mathrm{d}y \leq \\
& \leq \|u\|_{C^\alpha}^2 \int_{B_{5r}^c(\bar{x})} |y - x_1|^\alpha |y - x_2|^\alpha \frac{Cr}{|y - \bar{x}|^{n+1}} \, \mathrm{d}y \\
& \leq [u]_{C^\alpha}^2 \int_{B_{5r}^c(\bar{x})} \frac{Cr}{|y - \bar{x}|^{n+1-2\alpha}} \, \mathrm{d}y \leq C [u]_{C^\alpha}^2 r^{2\alpha}
\end{aligned}$$

Now we estimate the part of the integral where y is close to $\bar{x}$.

$$\begin{aligned}
\int_{B_{5r}(\bar{x})} & (u_i(y) - u_i(x_1))(u_j(y) - u_j(x_2)) \partial_{ij} \varphi(y) \, \mathrm{d}y \leq \\
& \leq \int_{B_{5r}(\bar{x})} |u_i(y) - u_i(x_1)| |u_j(y) - u_j(x_2)| (|D^2 \Phi(y - x_1)| + |D^2 \Phi(y - x_2)|) \, \mathrm{d}y
\end{aligned}$$

Note that we bound both terms, from $|D^2\Phi(y-x_1)|$ and $|D^2\Phi(y-x_2)|$, in the same way. Let us bound the first term. We use that $|u_j(y)-u_j(x_2)| \leq C[u]_{C^\alpha} r^\alpha$ in $B_{5r}(\bar{x})$.

$$\leq Cr^\alpha \|u\|_{C_\alpha} \int_{B_{5r}(\bar{x})} (u_i(y)-u_i(x_1))|D^2\Phi(y-x_1)| \, dy$$
$$\leq Cr^\alpha \|u\|^2_{C_\alpha} \int_{B_{5r}(\bar{x})} |y-x_1|^\alpha \frac{1}{|y-x_1|^n} \, dy \leq C \|u\|^2_{C_\alpha} r^{2\alpha}$$

Adding the two parts of the integral together, we obtain

$$p(x_1) - p(x_2) \leq C \|u\|^2_{C_\alpha} r^{2\alpha}$$

which finishes the proof of the case $\alpha \in (0, 1/2)$.

1.3 The case $\alpha \in (1/2, 1)$

When $\alpha \in (1/2, 1)$, $2\alpha > 1$ and the estimate obtained ($p \in C^{2\alpha}$) is actually a Hölder continuity result for ∇p. The proof is slightly different because instead of estimating $p(x_1) - p(x_2)$ we have to estimate $|\nabla p(x_1) - \nabla p(x_2)|$. For that we note that

$$\nabla p(x_k) = \int (u_i(y)-u_i(x_k))(u_j(y)-u_j(x_k))\nabla\partial_{ij}\Phi(y-x_k) \, dy$$

The kernel $\nabla\partial_{ij}\Phi(y-x_k)$ has a singularity of the form $|y-x_k|^{-n-1}$ and some singular part at $y = x_k$ of order one (derivatives of Dirac delta functions). However, note that $|(u_i(y)-u_i(x_k))(u_j(y)-u_j(x_k))| \leq C|y-x_k|^{2\alpha}$ and $2\alpha > 1$, therefore the singular part of $\nabla\partial_{ij}\Phi(y-x_k)$ can be ignored and the integral above is convergent.

We write $|\nabla p(x_1) - \nabla p(x_2)|$ in integral form and divide the integral as above in the domains $|y-\bar{x}| < 5r$ and $|y-\bar{x}| \geq 5r$. Let us start with the first of these integrals.

$$|\int_{B_{5r}(\bar{x})} (u_i(y)-u_i(x_1))(u_j(y)-u_j(x_1))\nabla\partial_{ij}\Phi(y-x_1) - (u_i(y)-u_i(x_2))(u_j(y)-u_j(x_2))\nabla\partial_{ij}\Phi(y-x_2) \, dy| \leq$$
$$\leq 2\left|\int_{B_{5r}(\bar{x})} (u_i(y)-u_i(x_1))(u_j(y)-u_j(x_1))\nabla\partial_{ij}\Phi(y-x_1) \, dy\right|$$
$$\leq C[u]^2_{C^\alpha} \int_{B_{5r}(\bar{x})} |y-x_1|^{2\alpha} \frac{1}{|y-x_1|^{n+1}} \, dy \leq C[u]^2_{C^\alpha} r^{2\alpha-1}$$

Now we analyze the part of the integral where y is far from $\bar{x}$.

$$|\int_{B^c_{5r}(\bar{x})} (u_i(y)-u_i(x_1))(u_j(y)-u_j(x_1))\nabla\partial_{ij}\Phi(y-x_1) - (u_i(y)-u_i(x_2))(u_j(y)-u_j(x_2))\nabla\partial_{ij}\Phi(y-x_2) \, dy| \leq$$
$$\leq |\int_{B^c_{5r}(\bar{x})} (u_i(y)-u_i(x_1))(u_j(y)-u_j(x_1))\big(\nabla\partial_{ij}\Phi(y-x_1) - \nabla\partial_{ij}\Phi(y-x_2)\big)$$
$$+ \big((u_i(y)-u_i(x_1))(u_j(y)-u_j(x_1)) - (u_i(y)-u_i(x_1))(u_j(y)-u_j(x_2))\big)\nabla\partial_{ij}\Phi(y-x_2) \, dy \; |$$
$$+ \big((u_i(y)-u_i(x_1))(u_j(y)-u_j(x_2)) - (u_i(y)-u_i(x_2))(u_j(y)-u_j(x_2))\big)\nabla\partial_{ij}\Phi(y-x_2) \, dy \; |$$
$$\leq C[u]^2_{C^\alpha} \int_{B^c_{5r}(\bar{x})} |y-\bar{x}|^{2\alpha} \frac{r}{|y-\bar{x}|^{n+2}} + r^\alpha |y-\bar{x}|^\alpha \frac{1}{|y-\bar{x}|^{n+1}} \, dy$$
$$\leq C[u]^2_{C^\alpha} r^{2\alpha-1}$$

Adding the two parts of the integral together, we obtain

$$|\nabla p(x_1) - \nabla p(x_2)| \leq C[u]_{C^\alpha}^2 r^{2\alpha-1}$$

which finishes the proof of the case $\alpha \in (1/2, 1)$.

References

[1] Peter Constantin. Local formulas for the hydrodynamic pressure and applications. *arXiv preprint arXiv:1309.5789*, 2013.

[2] Camillo De Lellis and László Székelyhidi Jr. Dissipative euler flows and onsager's conjecture. *arXiv preprint arXiv:1205.3626*, 2012.