I. BASICS.

Metric Spaces (definition).
Isometries, bilipschitz, uniform embedding (examples: subgroups in the word metric. It’s the idea of distortion), bi-Hölder.

Everything can be embedded in $L^\infty$
4 points that can’t be embedded in $L^2$

II. Criterion for embedding in $L^2$.

**Definition C.2.1** A kernel conditionally of negative type on $X$ is a continuous function $\Psi : X \times X \to \mathbb{R}$ with the following property:

(i) $\Psi(x, x) = 0$ for all $x$ in $X$;
(ii) $\Psi(x, y) = \Psi(y, x)$ for all $x, y$ in $X$;
(iii) for any $n$ in $\mathbb{N}$, any elements $x_1, \ldots, x_n$ in $X$, and any real numbers $c_1, \ldots, c_n$ with $\sum_{i=1}^n c_i = 0$, the following inequality holds:

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(x_i, x_j) \leq 0.$$

**Example C.2.2** (i) Let $\mathcal{H}$ be a real Hilbert space. The kernel

$$\Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}, \ (\xi, \eta) \mapsto \|\xi - \eta\|^2$$

is conditionally of negative type. Indeed, for $\xi_1, \ldots, \xi_n \in \mathcal{H}$ and $c_1, \ldots, c_n \in \mathbb{R}$ with $\sum_{i=1}^n c_i = 0$, we have

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(\xi_i, \xi_j) = \sum_{i=1}^n \sum_{j=1}^n c_i c_j \|\xi_i - \xi_j\|^2 = -2 \left\| \sum_{i=1}^n c_i \xi_i \right\|^2.$$

See appendix C to Bekka, de la Harpe and Vallette’s forthcoming book on Property T.
(We will see some connections to ideas that are part of the property T world later, e.g. expander graphs, and how they obstruct uniform embeddings, at least in Hilbert spaces.)

**III. A Problem:** When do $L^p$ spaces embed in one another? Either bi-Lipschitz or uniformly. (see e.g. Mendel-Naor)
Algorithm for diameter in $l^d_{\infty}$

$$\max_{p,q \in P} \|p - q\|_{\infty}$$

$$= \max_{p,q \in P} \max_{i=1...d'} |p_i - q_i|$$

$$= \max_{i=1...d'} \left( \max_{p,q \in P} |p_i - q_i| \right)$$

$$= \max_{i=1...d'} \left( \max_{p \in P} p_i - \min_{q \in P} q_i \right)$$

Running time: $O(d'n)$.
Using an isometric embedding of $L^1$ in $L^\infty$, one can improve the calculation of diameter from being $O(n^2)$ calculations to $O(n)$. (See Indyk’s web page and also Matousek’s book on Lectures on Discrete Geometry)

V. Bourgin’s theorem for finite metric spaces.

An $n$ point metric space can be bi-Lipschitz embedded in Euclidean space with $O(\log n)$ distortion.

Also, see the work of Bartal-Linial-Mendel-Naor on Metric Ramsey theorems, where one gets a significant part of the finite metric space to bilipschitz embed, but now with a bounded Lipschitz constant.

VI Uniform embedding in Hilbert space is related to rigidity problems in differential geometry and topology. One can talk about this forever. It is related to the Novikov and Baum-Connes conjectures. Maybe I’ll expand this another time.

VII. What dimension can you stick it into? Johnson-Lindenstrauss phenomenon. (Barvinok’s AMS book on convex geometry is a good start, and I think it’s where I stole the following from).

Once it’s in Euclidean space then you can compress a great deal without it costing that much extra distortion.

(3.6) **Lemma.** Let $x \in \mathbb{R}^n$ be a non-zero vector and let $\mu_{n,k}$ be the invariant probability measure on the Grassmannian $G_k(\mathbb{R}^n)$ of $k$-dimensional subspaces in $\mathbb{R}^n$. For $L \in G_k(\mathbb{R}^n)$, let $x_L$ be the orthogonal projection of $x$ onto $L$.

Then, for any $0 < \epsilon < 1$,

$$\mu_{n,k} \left\{ L \in G_k(\mathbb{R}^n) : \sqrt{\frac{n}{k}} \| x_L \| \geq (1 - \epsilon)^{-1}\| x \| \right\} \leq e^{-\epsilon^2 k/4} + e^{-\epsilon^2 n/4} \quad \text{and}$$

$$\mu_{n,k} \left\{ L \in G_k(\mathbb{R}^n) : \sqrt{\frac{n}{k}} \| x_L \| \leq (1 - \epsilon)\| x \| \right\} \leq e^{-\epsilon^2 k/4} + e^{-\epsilon^2 n/4}.$$ 

Application: Often can lower a great deal with small distortion. Example $10^9$ points you can get 10% distortion down into a 17000 dimensional Euclidean space.

This is used a lot in “dimension reduction” for data compression.

VIII. Bilipschitz cones: $cM$ is bilipshitz to $cN$ iff $M$ and $N$ are topologically h-cobordant. Same is true for “coarse version”. (This is due to Jonathan Block and the note-taker.)
Note the key here is the interaction between small and large scale.

**IX.** Back to $\mathbb{R}^n$, the doubling condition for covering balls of radius $2r$ by balls of radius $r$.

Assouad’s theorem. If $(X,d)$ is doubling, then $d^s$ – the fractalization of $d$ – by any $s<1$ bilipschitz embeds.

Note fractalization kills rectifiable curves. We’ll be very interested in rectifiable curves and path spaces. For various reasons (essentially related to Rademacher theorems, to Poincare inequalities, etc.)

**X.** Rademacher theorem. Lipschitz in Euclidean space is a.e. differentiable. So global condition implies infinitesimal structure.

Rademacher theorem in Banach spaces. Example mapping to $L^1$ via characteristic functions shows it can go wrong. Ok for other $L^p$ norms.

**XI** Pansu’s theorem and application to non-embedding.

**XII** Poincare inequality. Nonnegative Ricci curvature implies it. Note too that it’s a linkage of different scales statement.

**Neumann-Poincaré inequality.** We say that $M^n$ satisfies a uniform Neumann-Poincaré inequality if there exists $C_N < \infty$ such that for all $p \in M^n$, $r > 0$ and $f \in W^{2,1}_{loc}(M)$

$$\int_{B_r(p)} (f - \mathcal{A})^2 \leq C_N r^2 \int_{B_r(p)} |\nabla f|^2,$$

where $\mathcal{A} = \frac{1}{\text{Vol}(B_r(p))} \int_{B_r(p)} f$.

but, we will sometimes and on some spaces need a more general one involving other $L^p$ norms.

$$\int_B |u - u_B|^p \, dx \leq C(n, p)(\text{diam}(B))^p \int_B |\nabla u|^p \, dx,$$

**Overall theme:** Interaction between different scales, bilipschitz geometry of metric spaces, and the associated measures they support, differentiability theory of Lipschitz maps, Poincare inequalities and Sobolev spaces and embeddings.