THE COHOMOLOGY OF LIE GROUPS

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Abstract. We follow the computations in [2], [5], and [8] to deduce the cohomology rings of various Lie groups ($SU(n)$, $U(n)$, $Sp(n)$ with $\mathbb{Z}$-coefficients and $SO(n)$, $\mathrm{Spin}(n)$, G_2, F_4, E_6, E_7, E_8 with field coefficients) using the Serre spectral sequence.

Contents

1. The cohomology of classical Lie groups

1.1. The Leray-Serre spectral sequence.

Theorem 1.1. *Let R be a commutative ring with 1. Let $F \hookrightarrow E \to B$ be a Serre fibration, and suppose F is connected and B is simply-connected. Then there is a first-quadrant cohomological spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, where d_r is of bidegree $(r, 1-r)$, such that $E_2^{p,q} \cong H^p(B; H^q(F; R))$ and the spectral sequence converges to $H^*(E; R)$ as an algebra. Moreover, the differential d_r satisfies the Leibniz rule.*

Remark 1.2. When R is a field or with other additional hypotheses, we can identify $H^p(B; H^q(F; R))$ with $H^p(B; R) \otimes H^q(F; R)$.

An analogous spectral sequence exists for homology. See [7] for a construction of these spectral sequences; [9] is also an excellent reference.

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1.2. **The cohomology of $SU(n)$, $U(n)$, and $Sp(n)$.** By well-known decompositions, Lie groups deformation retract onto their maximal compact subgroups. Thus it suffices to restrict attention to compact Lie groups.

Notation 1.3. Unless otherwise stated, the subscripts on generators refer to their grading in the cohomology ring, i.e., $x_i \in H^i$. Also, we suppress coefficients in the case of integral coefficients or when the coefficients are clear from the context.

Proposition 1.4.

(1) $H^*(SU(n)) \cong \Lambda[x_3, x_5, \ldots, x_{2n-1}]$.
(2) $H^*(U(n)) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}]$.
(3) $H^*(Sp(n)) \cong \Lambda[x_3, x_7, \ldots, x_{4n-1}]$.

Proof. For (1), induct on n. If $n = 2$, then $SU(2) \cong S^3$, and so $H^*(SU(2)) \cong \Lambda[x_3]$. Assume the result for $n-1$ and consider the spectral sequence for the fibre bundle $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$. Because of the multiplicative structure, it suffices to check d_r on the generators. By lacunary considerations, the spectral sequence collapses on the second page. There is no extension problem in this case and the result follows.

Parts (2) and (3) are similar. For (2), $U(1) \cong S^1$, so $H^*(U(1)) = \Lambda[x_1]$, and we can use the fibre bundle $U(n-1) \hookrightarrow U(n) \to S^{2n-1}$. For (3), $Sp(1) \cong S^3$, so $H^*(Sp(1)) = \Lambda[x_3]$, and we can use the fibre bundle $Sp(n-1) \hookrightarrow Sp(n) \to S^{4n-1}$. Both these spectral sequences collapse on the second page. □

Remark 1.5. The above calculations can be condensed into the Gysin exact sequence, which in turn can be proved using the Serre spectral sequence.

Remark 1.6. For many of these calculations, the results are exterior algebras on odd degree generators which are free graded commutative algebras, and therefore there are no extension problems.

FIGURE 1. The $E_2 = E_\infty$ page of the spectral sequence associated with $SU(n-1) \hookrightarrow SU(n) \to S^{2n-1}$

$2n-3$	x_{2n-3}			ux_{2n-3}	
$\vdots$					
$\vdots$					
5	x_5			ux_5	
4					
3	x_3			ux_3	
2					
1					
0	1			u	
	0	$\cdots$	$\cdots$	$2n-1$	

1.3. **Characteristic classes.** Let BG denote the classifying space of a group G. For a simple explicit construction for matrix Lie groups, see theorem 6.16 of chapter II in [8]. For more general groups, universal principal G-bundles can be constructed using Milnor's join construction. Classifying spaces are unique up to homotopy equivalence.

Theorem 1.7.

(1) $H^*(BSU(n)) = \mathbb{Z}[c_2, c_3, \ldots, c_n]$ *where* $c_i \in H^{2i}$.
(2) $H^*(BU(n)) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$, *where* $c_i \in H^{2i}$.
(3) $H^*(BSp(n)) = \mathbb{Z}[q_1, q_2, \ldots, q_n]$, *where* $q_i \in H^{4i}$.

The element c_i *is called the* i-th Chern class *and the element* q_i *is called the* i-th symplectic Pontryagin class.

Proof. To prove, say (2), consider first the case $n = 1$. Since $U(1) \cong S^1$, we have $BU(1) \cong BS^1 \cong \mathbb{C}P^\infty$, and we know that $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[c_1]$ where $c_1 \in H^2$.

If $n \geq 1$, consider the spectral sequence of the universal $U(n)$-bundle. (In this case, the total space can be thought of as an infinite Grassmannian manifold.) We see that $H^*(BU(n))$ must be $\mathbb{Z}[c_1, c_2, \ldots, c_n]$, where $c_i \in H^{2i}$. $\square$

It is possible to adapt this proof for the case of orthogonal groups. However, we will take this opportunity to illustrate how the Weyl group can be used to deduce facts about the cohomology of Lie groups and related spaces. (This approach will be useful when we derive the Wu formula later – see proposition 1.17.)

Let $G = O(n)$ and let Z_2^n be the subgroup consisting of diagonal matrices in G. It is clear that $H^*(BZ_2^n; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_n]$ for $t_i \in H^1$. Define the Weyl group $W = N_G(Z_2^n)/Z_2^n$. If we think of BZ_2^n as EZ_2^n/Z_2^n, then the Weyl group acts on BZ_2^n by $(nZ_2^n) \cdot (aZ_2^n) = anZ_2^n$.

Lemma 1.8. *Let* $\rho^* : H^*(BG) \to H^*(BZ_2^n)$ *be the homomorphism induced by the map* $\rho : BZ_2^n \to BG$. *For* $\varphi \in W$, $\rho \circ (\varphi \cdot) = \rho$. *In particular,* $\operatorname{im} \rho^* \subseteq H^*(BZ_2^n)^W$.

Proof. We have $\rho(aZ_2^n) = aG$. Let $\varphi \in nZ_2^n \in W$. Then

$$\rho(\varphi \cdot aZ_2^n) = \rho(anZ_2^n) = anG = aG = \rho(aZ_2^n).$$

So $\rho \circ (\varphi \cdot) = \rho$, which implies $(\varphi \cdot)^* \circ \rho^* = \rho^*$. So $\operatorname{im} \rho^*$ is fixed by $(\varphi \cdot)^*$, i.e., $\operatorname{im} \rho^* \subseteq H^*(BZ_2^n)^W$. $\square$

In this case, it turns out that the action of W on $H^*(BZ_2^n)$ is permutation of the elements $\{t_1, \ldots, t_n\}$ in $H^*(BZ_2^n; \mathbb{F}_2) = \mathbb{F}_2[t_1, \ldots, t_n]$

Theorem 1.9. *Consider the fibration* $O(n)/Z_2^n \xrightarrow{i} BZ_2^n \xrightarrow{\rho} BO(n)$ *with* $\mathbb{F}_2$ *coefficients. Then*

(1) $\rho^* : H^*(BO(n)) \to H^*(BZ_2^n) = \mathbb{F}_2[t_1, \ldots, t_n]$ *is a monomorphism.*
(2) $\operatorname{im} \rho^* = H^*(BZ_2^n)^W = \mathbb{F}_2[\sigma_1, \ldots, \sigma_n]$ *where* σ_i *is the* i*-th elementary symmetric polynomial in the variables* $\{t_1, \ldots, t_n\}$.
(3) If we put $\rho^*(w_i) = \sigma_i$, *then* $H^*(BO(n); \mathbb{F}_2) = \mathbb{F}_2[w_1, w_2, \ldots, w_n]$ *where* $w_i \in H^i$. *The element* w_i *is called the* i-th Stiefel-Whitney class.

Proof. Assume that the spectral sequence for the fibration

$$O(n)/Z_2^n \xrightarrow{i} BZ_2^n \xrightarrow{\rho} BO(n)$$

collapses. Then (1) is immediate. For (2), $\operatorname{im} \rho^* \subseteq H^*(BZ_2^n)^W$ is the previous lemma and the other direction is supplied by the fact that $\operatorname{im} \rho^*$ is a direct summand of $H^*(BZ_2^n)^W$ if the spectral sequence collapses. Finally, (3) follows from (1) and (2).

The remainder of this proof will be devoted to showing that the spectral sequence does indeed collapse. This happens iff i^* is epic.

First, we claim that $H^*(O(n)/Z_2^n;\mathbb{F}_2)$ is generated by $H^1(O(n)/Z_2^n;\mathbb{F}_2)$ which has dimension $n-1$. Induct on n. For the case $n=1$, $O(1)/Z_2 \cong *$. Now assume that it is true for n, and consider the fibration

$$O(n)/Z_2^n \xrightarrow{i} O(n+1)/Z_2^{n+1} \xrightarrow{p} O(n+1)/(O(n)\times Z_2) \cong \mathbb{R}P^n.$$

Since the cohomology of the fibre and base spaces are generated by H^1, it is enough to show that the spectral sequence for *this* fibration collapses, which happens iff i^* is epic. By induction, $H^1(O(n)/Z_2^n;\mathbb{F}_2)$ generates $H^*(O(n)/Z_2^n;\mathbb{F}_2)$. Thus it is enough to show that i^* surjects onto $H^1(O(n)/Z_2^n;\mathbb{F}_2)$.

Consider the following Serre exact sequence:

$$H^1(\mathbb{R}P^n;\mathbb{F}_2) \xrightarrow{p^*} H^1(O(n+1)/Z_2^{n+1};\mathbb{F}_2) \xrightarrow{i^*} H^1(O(n)/Z_2^n;\mathbb{F}_2).$$

To show that i^* is epic, it is enough to that $\dim_{\mathbb{F}_2} H^1(O(n+1)/Z_2^{n+1};\mathbb{F}_2) = n$, since $\dim_{\mathbb{F}_2} H^1(\mathbb{R}P^n;\mathbb{F}_2) = 1$ and $\dim_{\mathbb{F}_2} H^1(O(n)/Z_2^n;\mathbb{F}_2) = n-1$. To show *this*, consider another Serre exact sequence:

$$H^1(BO(n+1);\mathbb{F}_2) \to H^1(BZ_2^{n+1};\mathbb{F}_2) \to H^1(O(n+1)/Z_2^{n+1};\mathbb{F}_2).$$

Since $\dim_{\mathbb{F}_2} H^1(BO(n+1);\mathbb{F}_2) = 1$ and $\dim_{\mathbb{F}_2} H^1(BZ_2^{n+1};\mathbb{F}_2) = n+1$, we have $\dim_{\mathbb{F}_2} H^1(O(n+1)/Z_2^{n+1};\mathbb{F}_2) = n$ as wanted.

The upshot of all this is that it is enough to show that the original i^* surjects onto $H^1(O(n)/Z_2^n;\mathbb{F}_2)$.

Consider the following commutative diagram, where SZ_2^n is the group of diagonal matrices in $SO(n)$:

$$\begin{tikzcd}
0 = H^1(BSO(n);\mathbb{F}_2) \arrow[r] & H^1(BSZ_2^n;\mathbb{F}_2) \arrow[r, "i_0^*"] & H^1(O(n)/Z_2^n;\mathbb{F}_2) \\
 & H^1(BZ_2^n;\mathbb{F}_2) \arrow[u, "p^*"] \arrow[ur, "i^*"] &
\end{tikzcd}$$

Note that i_0^* is injective. Since $\dim_{\mathbb{F}_2} H^1(BSZ_2^n) = \dim_{\mathbb{F}_2} H^1(O(n)/Z_2^n) = n-1$, i_0^* is an isomorphism. Since p^* is an epimorphism, i^* is also an epimorphism as wanted. $\square$

1.4. **The cohomology of $SO(n)$ and $\mathrm{Spin}(n)$.** We will compute the cohomology ring of the real Stiefel manifold $V_k(\mathbb{R}^n)$ with field coefficients, following the approach in [5].

Lemma 1.10. *If $1 \le k \le n$, then $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected and*

$$\pi_{n-k}(V_k(\mathbb{R}^n)) = \begin{cases} \mathbb{Z} & \text{if } k=1 \text{ or } n-k \text{ is even, and } k<n \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k=n \text{ or } n-k \text{ is odd.} \end{cases}$$

Proof. Since $V_1(\mathbb{R}^n) \cong S^{n-1}$, we have $\pi_{n-1}(V_1(\mathbb{R}^n)) \cong \mathbb{Z}$. Also, as $V_n(\mathbb{R}^n) \cong O(n)$, we have $\pi_0(V_n(\mathbb{R}^n)) \cong \mathbb{Z}/2\mathbb{Z}$. So we can assume $1 < k < n$.

First consider the case where $k = 2$. The long exact sequence of the fibration $S^{n-2} \xrightarrow{i} V_2(\mathbb{R}^n) \xrightarrow{p} S^{n-1}$ is

$$\pi_{n-1}(S^{n-1}) \xrightarrow{\partial} \pi_{n-2}(S^{n-2}) \xrightarrow{i_*} \pi_{n-2}(V_2(\mathbb{R}^n)) \to \pi_{n-2}(S^{n-1}) = 0.$$

Similarly, the long exact sequence associated with $S^{n-k} \to V_k(\mathbb{R}^n) \to S^{n-1}$ proves that $V_k(\mathbb{R}^n)$ is $(n-k-1)$-connected.

We need to compute ∂. Let B^r_+ be the upper hemisphere of S^r and let S^{r-1} be the equator of S^r. Let $* = \{1, 0, 0, \dots, 0\} \in S^{r-1}$ be the basepoint. We can think of $V_2(\mathbb{R}^n)$ as $\{A \in M_{2\times n} \mid AA^t = I_2\}$. Define $\mu : B^{n-1}_+ \to V_2(\mathbb{R}^n)$ as $\mu(x) = (\mu_{ij}(x))$, where $\mu_{ij}(x) = \delta_{ij} - 2x_ix_j$ for $1 \le i \le 2$ and $1 \le j \le n$. Let $p : V_2(\mathbb{R}^n) \to S^{n-1}$ be the map $p(a_{ij}) = a_{1j}$. Then $(p \circ \mu)(S^{n-2}) = \{*\}$, and $p \circ \mu|_{B^{n-1}_+/S^{n-2}} : B^{n-1}_+/S^{n-2} \cong S^{n-1} \to S^{n-1}/\{*\} \cong S^{n-1}$ is a homeomorphism, and $[p \circ \mu]$ is a generator of $\pi_{n-1}(S^{n-1})$.

Define $\tau = \mu|_{S^{n-2}} : S^{n-2} \to p^{-1}\{*\} = S^{n-2}$. Then $[\tau]$ is a generator of $\operatorname{im}\partial$ in $\pi_{n-2}(S^{n-2})$. Moreover, $\tau(S^{n-3}) = \{*\} \in S^{n-2}$, so $\tau|_{B^{n-2}_+/S^{n-3}}$ is a map from B^{n-2}_+/S^{n-3} to $S^{n-2} \setminus \{*\}$. We can also check that $\tau(y) = \tau(-y)$. So τ represents the sum of the identity map and the antipode map. The degree of the antipode map on S^{n-2} is $(-1)^{n-1}$, so $\deg\tau = \pm(1 + (-1)^{n-1})$. This shows that $\partial = 0$ if n is even and $\partial = \pm 2$ if n is odd, whence the conclusion of the lemma.

Now consider the case $k = 3$. We have the following commuting diagram:

$$\begin{tikzcd}
S^{n-3} = V_1(\mathbb{R}^{n-2}) \arrow[r] \arrow[d, no head] & V_2(\mathbb{R}^{n-1}) \arrow[r] \arrow[d] & V_1(\mathbb{R}^{n-1}) = S^{n-2} \arrow[d, "i"] \\
S^{n-3} = V_1(\mathbb{R}^{n-2}) \arrow[r] & V_3(\mathbb{R}^n) \arrow[r] & V_2(\mathbb{R}^n)
\end{tikzcd}$$

This yields the following commutative diagram:

$$\begin{tikzcd}
\pi_{n-2}(S^{n-2}) \arrow[d, "i_*"'] \arrow[dr, "\partial"] & & & \\
\pi_{n-2}(V_2(\mathbb{R}^n)) \arrow[r, "\partial'"'] & \pi_{n-3}(S^{n-3}) \arrow[r] & \pi_{n-3}(V_3(\mathbb{R}^n)) \arrow[r] & \pi_{n-3}(V_2(\mathbb{R}^n)) = 0
\end{tikzcd}$$

Since i_* is epimorphic, $\operatorname{im}\partial = \operatorname{im}\partial'$, so the conclusions follow.

Finally suppose $k > 3$. The long exact sequence associated to the fibration $V_{k-2}(\mathbb{R}^{n-2}) \hookrightarrow V_k(\mathbb{R}^n) \to V_2(\mathbb{R}^n)$ shows that $\pi_i(V_k(\mathbb{R}^n)) \cong \pi_i(V_{k-2}(\mathbb{R}^{n-2}))$ for $i < n-3$, and so we can argue by induction. □

Remark 1.11. On the level of cohomology, this lemma can also be proven using the Gysin sequence: the relevant transgression is the cup product with the Euler class of the sphere, which is 0 or 2 depending on the parity of n.

Observe that $V_2(\mathbb{R}^2) \cong O(2) \cong S^0 \times S^1$, so $H^*(V_2(\mathbb{R}^2)) \cong H^*(S^0) \otimes H^*(S^1)$.

Lemma 1.12. *Suppose $n \ge 3$. If n is even, then $H^*(V_2(\mathbb{R}^n); \mathbb{Z}) = \Lambda[x_{n-2}, x_{n-1}]$. If n is odd, then $H^*(V_2(\mathbb{R}^n); \mathbb{Z}) = \Lambda[x_{n-1}, x_{2n-3}]/(2x_{n-1}, x_{n-1}x_{n-3})$.*

Proof. Consider the spectral sequence for the fibration $S^{n-2} \to V_2(\mathbb{R}^n) \to S^{n-1}$. If n is even, then we have

$$H^{n-2}(V_2(\mathbb{R}^n)) \cong H_{n-2}(V_2(\mathbb{R}^n)) \cong \pi_{n-2}(V_2(\mathbb{R}^n)) \cong \mathbb{Z}$$

by the universal coefficient theorem, the Hurewicz isomorphism, and the previous lemma respectively. Therefore $E_{n-1}^{0,n-2}$ survives forever, so $d_{n-1} = 0$. If n is odd, then the Hurewicz theorem and the previous lemma shows that

$$H_{n-2}(V_2(\mathbb{R}^n)) \cong \pi_{n-2}(V_2(\mathbb{R}^n)) \cong \mathbb{Z}/2\mathbb{Z}.$$

By the universal coefficient theorem,

$$H^{n-1}(V_2(\mathbb{R}^n)) \cong \mathrm{Hom}(0, \mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

So $d_{n-1} = 2$, and the lemma follows. □

We now split the calculation in two cases depending on the characteristic of the field.

Theorem 1.13. *Let $\mathbb{F}$ be a field of characteristic $\neq 2$. Then the cohomology ring $H^*(V_k(\mathbb{R}^n); \mathbb{F})$ is*

$$\Lambda[\{x_{4i-1} \mid n-k < 2i < n\}] \otimes \Lambda[\{y_{n-1} \mid n \text{ even}\}] \otimes \Lambda[\{x_{n-k} \mid n-k \text{ even}\}].$$

Proof. The theorem is true for $k = 1$. The case $k = 2$ follows from the previous lemma and the universal coefficient theorem. Assume $k > 2$, and we can induct on k. If $n-k$ is even, use the fibration $S^{n-k} \hookrightarrow V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$. The differential $d_{n-k+1} : E_{n-k+1}^{0,n-k} \to E_{n-k+1}^{n-k+1,0}$ is zero since $H^{n-k+1}(V_{k-1}(\mathbb{R}^n))$ is zero by the induction hypothesis, so the spectral sequence collapses. If $n-k$ is odd, use the fibration $V_2(\mathbb{R}^{n-k+2}) \hookrightarrow V_k(\mathbb{R}^n) \to V_{k-2}(\mathbb{R}^n)$. The spectral sequence collapses because $H^*(V_2(\mathbb{R}^{n-k+2}))$ is isomorphic to $\Lambda[x_{2(n-k)+1}]$ and the differential $d_{2n-2k+2} : E_{2n-2k+2}^{0,2(n-k)+1} \to E_{2(n-k)+2}^{2(n-k)+2,0}$ is zero since $H^{2(n-k)+2}(V_{k-2}(\mathbb{R}^n)) = 0$ by the induction hypothesis. The result follows from the Leray-Hirsch theorem. □

Corollary 1.14. $H^*(SO(n); \mathbb{F}) = \Lambda[\{x_{4i-1} \mid 0 < 2i < n\}] \otimes \Lambda[\{y_{n-1} \mid n \text{ even}\}]$.

Theorem 1.15. *$H^*(V_k(\mathbb{R}^n); \mathbb{F}_2)$ has a simple system of generators consisting of $\{x_i \mid n-k \leq i < n\}$.*

Proof. The base case $k = 1$ is clear since $V_1(\mathbb{R}^n) \cong S^{n-1}$. Assume the theorem for $k-1$. Consider the spectral sequence for $S^{n-k} \hookrightarrow V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$. The only possibly nonzero differential is $d_{n-k+1} : E_{n-k+1}^{0,n-k} \to E_{n-k+1}^{n-k+1,0}$. If $d_{n-k+1} \neq 0$, then $H^{n-k}(V_k(\mathbb{R}^n)) = 0$. But by the previous lemma, together with Hurewicz theorem and the universal coefficient theorem, $H^{n-k}(V_k(\mathbb{R}^n)) \neq 0$. This is a contradiction, so the theorem holds for k as well.

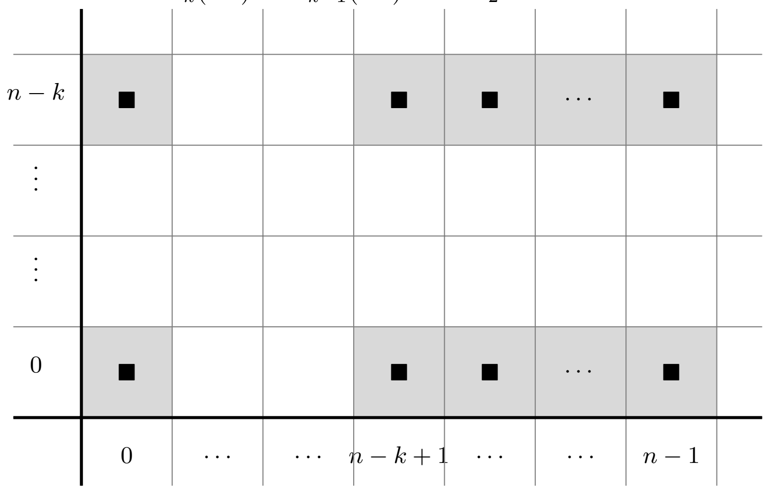
□

Figuring out the relations in the cohomology ring of $SO(n)$ with mod two coefficients requires a little more work.

Definition 1.16. There is a stable cohomology operation Sq^j of degree j and type $(\mathbb{F}_2, \mathbb{F}_2)$ called the *Steenrod squaring operation* satisfying the following axioms:

(1) Let $x \in H^j$. Then $\mathrm{Sq}^j(x) = x^2$.
(2) $\mathrm{Sq}^0 = \mathrm{id}$, $\mathrm{Sq}^j = 0$ for $j < 0$, and $\mathrm{Sq}^j(x) = 0$ if $\deg x < j$.
(3) Sq^1 is the Bockstein homomorphism.

Figure 2. The $E_2 = E_\infty$ page of the spectral sequence associated with $S^{n-k} \hookrightarrow V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$ with $\mathbb{F}_2$ coefficients



(4) (Cartan formula) $\mathrm{Sq}^j(xy) = \sum_{s+t=j} \mathrm{Sq}^s(x)\,\mathrm{Sq}^t(y)$.

(5) (Adem relations) $\mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_t^{\lfloor \frac{a}{2} \rfloor} \binom{b-t-1}{a-2t} \mathrm{Sq}^{a+b-t}\,\mathrm{Sq}^t$ for $a < 2b$.

Proposition 1.17 (Wu formula)**.** *Let w_i be the i-th Stiefel-Whitney class, c_i the i-th Chern class, and q_i the i-th symplectic Pontryagin class. Then*

$$\mathrm{Sq}^j\, w_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} w_{i+j-k} w_k \quad \text{for } 0 \le j \le i$$

$$\mathrm{Sq}^{2j}\, c_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} c_{i+j-k} c_k \quad \text{for } 0 \le j \le i$$

$$\mathrm{Sq}^{4j}\, q_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} q_{i+j-k} q_k \quad \text{for } 0 \le j \le i.$$

Proof. We will prove the Wu formula for the Stiefel-Whitney classes only; the other versions of the formula are similar. Recall that $\rho^* : H^*(BO(n); \mathbb{F}_2) \to H^*(BZ_2^n; \mathbb{F}_2)$ is a monomorphism. Thus it suffices to show the equality

$$\mathrm{Sq}^j\, \sigma_i = \sum_{k=0}^{j} \binom{i-k-1}{j-k} \sigma_{i+j-k}\sigma_k,$$

where σ_i is the i-th elementary symmetric polynomial in $\{x_1, \ldots, x_n\}$.

Define polynomials $\sigma_{a,b}(x_1, \ldots, x_n)$ by

$$\prod_{i=1}^{n} (1 + x_i t + x_i^2 u) = \sum_{a,b} \sigma_{a,b} t^a u^b.$$

Observe that $\sigma_{a,b}$ is a symmetric polynomial with term $x_1 \cdots x_a x_{a+1}^2 \cdots x_{a+b}^2$ and that $\sigma_{a,0} = \sigma_a$ and $\sigma_{0,b} = \sigma_b^2$. Using $\mathrm{Sq}^0 = \mathrm{id}$, $\mathrm{Sq}^1\, x_i = x_i^2$, $\mathrm{Sq}^j\, x_i = 0$ for

$j > 1$, and the Cartan formula, we have

$$\operatorname{Sq}^j \sigma_i = \operatorname{Sq}^j \sum x_{k_1} \cdots x_{k_i} = \sum_{\epsilon_1 + \cdots + \epsilon_i = j} \operatorname{Sq}^{\epsilon_1} x_{k_1} \cdots \operatorname{Sq}^{\epsilon_i} x_{k_i} = \sigma_{i-j,j}.$$

Thus it suffices to prove that for $0 \le b \le i$,

$$\sigma_{a,b} = \sum_{k=0}^{b} \binom{a+b-k-1}{b-k} \sigma_{a+2b-k}\sigma_k$$

by induction on n.

This is trivial for $n = 1$. Suppose $n \ge 1$, and denote by $\sigma'_{a,b}$ and σ'_i the corresponding versions of $\sigma_{a,b}$ and σ_i for the $n+1$ variables $\{x_1, \dots, x_n, t\}$. We have $\sigma'_i = \sigma_i + \sigma_{i-1}t$ and $\sigma'_{a,b} = \sigma_{a,b} + \sigma_{a-1,b}t + \sigma_{a,b-1}t^2$. Now,

$$\begin{aligned}
&\sum_{k=0}^{b} \binom{a+b-k-1}{b-k} \sigma'_{a+2b-k}\sigma'_k \\
&= \sum_{k=0}^{b} \binom{a+b-k-1}{b-k} (\sigma_{a+2b-k} + \sigma_{a+2b-k-1}t)(\sigma_k + \sigma_{k-1}t) \\
&= \sum_{k=0}^{b} \binom{a+b-k+1}{b-k} \sigma_{a+2b-k}\sigma_k \\
&\quad + \sum_{k=0}^{b-1} \left(\binom{a+b-k-1}{b-k} + \binom{a+b-k-2}{b-k-1} \right) \sigma_{a+2b-k-1}\sigma_k t \\
&\quad + \sigma_{a+b-1}\sigma_b t + \sum_{k=0}^{b-1} \binom{a+b-k-2}{b-k-1} \sigma_{a+2(b-1)-k}\sigma_k t^2 \\
&= \sigma_{a+b} + \sum_{k=0}^{b} \binom{a+b-k-2}{b-k} \sigma_{a+2b-k-1}\sigma_k t + \sigma_{a,b-1}t^2 \\
&= \sigma_{a,b} + \sigma_{a-1,b}t + \sigma_{a,b-1}t^2 \\
&= \sigma'_{a,b}
\end{aligned}$$

□

Theorem 1.18. *$H^*(SO(n); \mathbb{F}_2) = \mathbb{F}_2[x_1, x_3, \dots, x_{2m-1}]/(x_i^{a_i})$ where $m = \lfloor \frac{n}{2} \rfloor$ and a_i is the smallest power of two such that $ia_i \ge n$.*

Proof. The above theorem tells us that $H^*(SO(n); \mathbb{F}_2)$ has a simple system of generators $\{x_1, x_2, \dots, x_{n-1}\}$. Thus it remains to show that $x_i^2 = x_{2i}$ if $2i < n$ and zero otherwise.

Let σ denote the cohomology suspension. By considering the transgression in the spectral sequence for the universal bundle $SO(n) \to ESO(n) \to BSO(n)$, we see that x_i is universally transgressive and that $x_i = \sigma(w_{i+1})$. Recalling that the suspension commutes with stable cohomology operations and that decomposables suspend to zero, we can use the Wu formula to obtain

$$\begin{aligned}
\operatorname{Sq}^j x_i &= \operatorname{Sq}^j(\sigma(w_{i+1})) = \sigma(\operatorname{Sq}^j w_{i+1}) \\
&= \sigma(\tbinom{i}{j} w_{i+j+1} + \tbinom{i-1}{j-1} w_{i+j}w_1 + \cdots + \tbinom{i-j}{0} w_{i+1}w_j) = \tbinom{i}{j} x_{i+j}.
\end{aligned}$$

Taking $i = j$ yields the result. □

Remark 1.19. The homomorphism on cohomology induced by the quotient map $SO(n) \to V_k(\mathbb{R}^n)$ shows that similar relations hold between the generators of $V_k(\mathbb{R}^n)$.

We now consider the group $\mathrm{Spin}(n)$, the universal (double) cover of $SO(n)$.

Theorem 1.20.

(1) If p is an odd prime, then $H^*(\mathrm{Spin}(n); \mathbb{F}_p) \cong H^*(SO(n); \mathbb{F}_p)$.
(2) If $p = 2$, let s be the integer such that $2^{s-1} < n \leq 2^s$. Then $H^*(\mathrm{Spin}(n); \mathbb{F}_2)$ has a simple system of generators $\{u_i \mid 1 \leq i < n, i \neq 2^r\} \cup \{u\}$ where u has degree $2^s - 1$. These elements satisfy the following relations: $\mathrm{Sq}^j u_i = \binom{i}{j} u_{i+j}$ if $i \geq j$, $\mathrm{Sq}^j u_i = 0$ if $i < j$, and $u^2 = 0$.

The following proof is mostly due to Borel [2].

Proof. For (1), consider the fibration $SO(n) \to \mathrm{Spin}(n) \to BZ_2 \cong \mathbb{R}P^\infty$. Since $H^*(\mathbb{R}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p$, we have $H^*(SO(n); \mathbb{F}_p) \cong H^*(\mathrm{Spin}(n); \mathbb{F}_p)$.

Now consider the mod 2 cohomology of $\mathrm{Spin}(n)$. As the quotient of $SO(n)$ by $SO(2)$, the homogeneous space $V_{n-2}(\mathbb{R}^n)$ is also the quotient of $\mathrm{Spin}(n)$ by S^1, so we can study the fibration $S^1 \to \mathrm{Spin}(n) \to V_{n-2}(\mathbb{R}^n)$.

Recall that $H^*(V_{n-2}(\mathbb{R}^n))$ has a simple system of generators $\{x_2, x_3, \ldots, x_{n-1}\}$. The ideal generated by x_2 as a vector space has a basis consisting of the monomials $x_{i_1} x_{i_2} \cdots x_{i_k}$ where $i_1 < i_2 < \cdots < i_k$ and at least one of the i_j is a power of two. The $n - s - 1$ elements x_i whose degree is not a power of two form a simple system of generators of the subalgebra K of $H^*(V_{n-2}(\mathbb{R}^n))$ that is complementary to (x_2) as a vector space. The height of x_2 is equal to $2^s - 1$, and the annihilator of x_2, denoted $\mathrm{Ann}(x_2)$ is the ideal generated by the element $x^* = x_2^{2^{s-1}-1}$ of degree $2^s - 2$.

Consider the spectral sequence $E_2 \cong H^*(V_{n-2}(\mathbb{R}^n)) \otimes H^*(S^1) \Rightarrow H^*(\mathrm{Spin}(n))$. Let y be the nonzero element of $H^1(S^1)$. As $\mathrm{Spin}(n)$ is simply-connected, we have $E_\infty^{0,1} = 0$, so we must have $d_2 y = x_2$. This determines the differential d_2 completely. The d_2-cocycles form the subalgebra $C(E_2) = \mathrm{Ann}(h_2) \otimes y + H^*(V_{n-2}(\mathbb{R}^n)) \otimes 1$.

As vector spaces, $\mathrm{Ann}(h_2) \otimes y + K \otimes 1$ is complementary to $d_2(E_2)$ in $C(E_2)$ and survives forever in the spectral sequence. This subalgebra admits a simple system of generators of the $n - s - 1$ elements $u_i = x_i$ whose degrees are not powers of two together with the element $u = x^* \otimes y$ of degree $2^s - 1$. Thus $H^*(\mathrm{Spin}(n); \mathbb{F}_2)$ has the desired simple system of generators.

The relations on the u_i's come from the relations on the x_i's in $H^*(V_{n-2}(\mathbb{R}^n))$. Finally by considering the spectral sequence for $\mathrm{Spin}(n-1) \hookrightarrow \mathrm{Spin}(n) \to S^{n-1}$, we also deduce that $u^2 = 0$. See figures 4 and 5.

□

Remark 1.21. The cohomology of $\mathrm{Spin}(n)$ can also be obtained from the fibrations $\mathrm{Spin}(n-1) \to \mathrm{Spin}(n) \to S^{n-1}$ and $\mathrm{Spin}(n) \to SO(n) \to \mathbb{R}P^\infty$ alone. See [8].

For a different approach, May and Zabrodsky has demonstrated how to compute $H^*(\mathrm{Spin}(n))$ using the Eilenberg-Moore spectral sequence. See [6].

FIGURE 3. The E_2 page of the spectral sequence associated with $S^1 \hookrightarrow \mathrm{Spin}(n) \to V_{n-2}(\mathbb{R}^n)$ with $\mathbb{F}_2$ coefficients

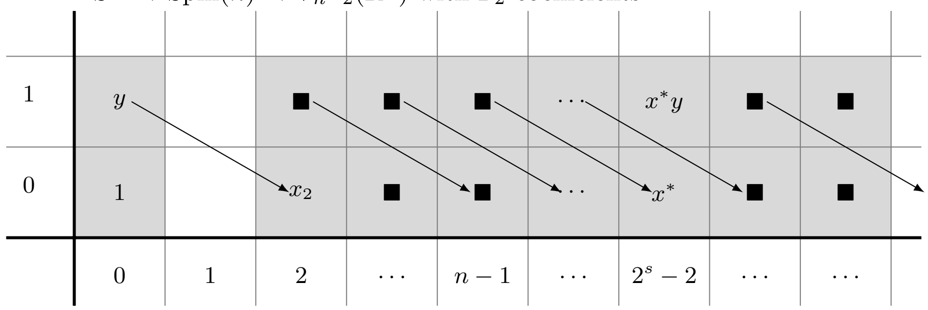


FIGURE 4. The E_2 page of the spectral sequence associated with $\mathrm{Spin}(n-1) \hookrightarrow \mathrm{Spin}(n) \to S^{n-1}$. Case $n - 1 < 2^s$. Observe that $\deg u = 2^s - 1$ and $u^2 = 0$ by induction.

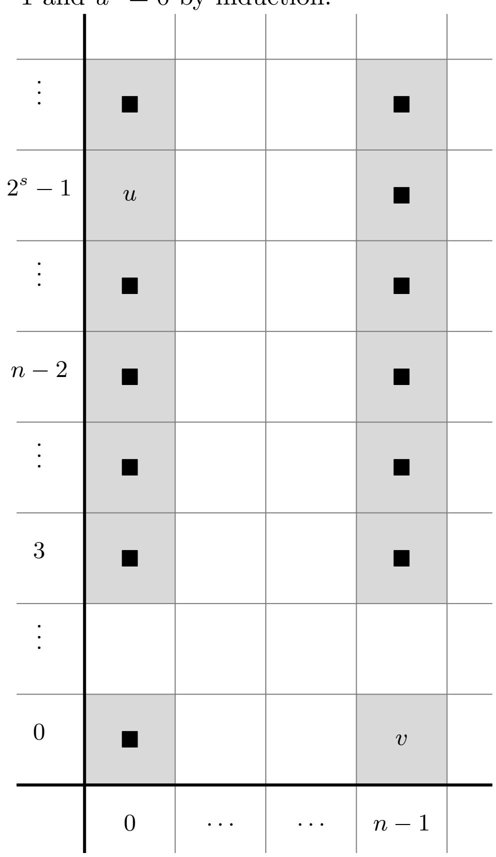
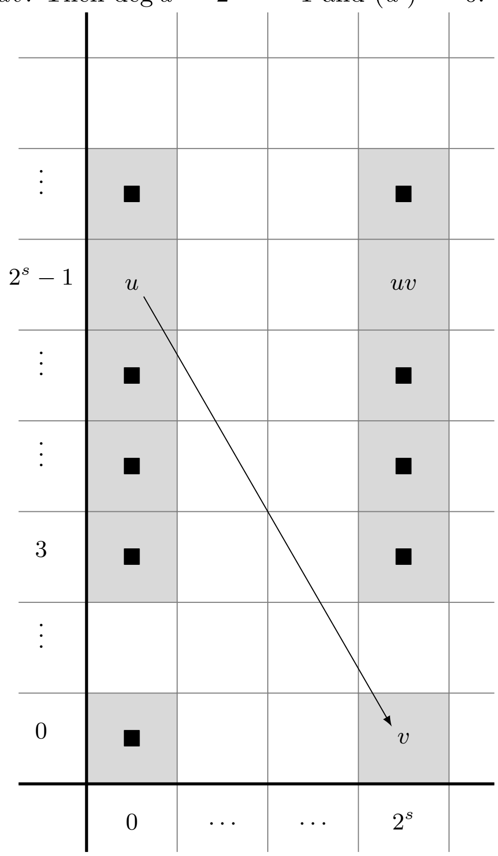


FIGURE 5. The E_2 page of the spectral sequence associated with $\mathrm{Spin}(n-1) \hookrightarrow \mathrm{Spin}(n) \to S^{n-1}$. Case $n-1 = 2^s$. In this case, write $u' = uv$. Then $\deg u' = 2^{s+1} - 1$ and $(u')^2 = 0$.



2. The cohomology of exceptional Lie groups

2.1. **The structure of Hopf algebras.**

Theorem 2.1 (Hopf-Borel). *Let k be a field of characteristic p where p is zero or a prime. Let A be a monogenic (connected) Hopf algebra generated by a homogeneous element x of positive degree. Then*

1. *If $p \neq 2$ and $\deg x$ is odd, then $A \cong \Lambda[x]$.*
2. *If $p \neq 2$ and $\deg x$ is even, then $A \cong k[x]/(x^h)$ or $k[x]$, where h is a power of p.*
3. *If $p = 2$, then $A \cong k[x]/(x^h)$ or $k[x]$, where h is a power of two.*

Moreover, a commutative Hopf algebra H of finite type over a perfect field is isomorphic as an algebra to a tensor product of monogenic Hopf algebras.

The following proof is found in [7].

Proof. Let A be a monogenic Hopf algebra with coproduct Δ that is generated by x. If $\deg x$ is odd, then by commutativity $x^2 = -x^2$, so $x^2 = 0$ if $p \neq 2$. Now suppose x has even degree and that its height is h. Since x is the element of least nonzero degree in A, x is primitive, and it follows that

$$0 = \Delta(x^h) = \Delta(x)^h = (1 \otimes x + x \otimes 1)^h = \sum_{i=1}^{h-1} \binom{h}{i} x^i \otimes x^{h-i}.$$

If $p = 0$, the right hand side is never zero, which is a contradiction, so x is free in A. Otherwise, note that $\binom{h}{i}$ is congruent to zero mod p for all $0 < i < h$ iff $h = p^s$. Thus we have shown the first part of the theorem (there are no other relations).

For the second part of the theorem, we argue by induction on the number of generators of H. If H is monogenic, then this is the first part of this theorem. Suppose the theorem holds for Hopf algebras with less than or equal to n generators. Order the generators $\{x_1, x_2, \ldots\}$ of the algebra H in nondecreasing degree. Let A_n denote the subalgebra of H generated by 1 and $\{x_1, x_2, \ldots, x_n\}$. The coproduct Δ is closed on A_n, so A_n is a sub-Hopf algebra of H and similarly, A_{n-1} is a sub-Hopf algebra of A_n.

Let H_2 be a sub-Hopf algebra of H and define $H_1//H_2$ to be the quotient $H_1/(H_2^+ \cdot H_1)$ where H_2^+ is the degree positive part of H_2. Let $B_n = A_n//A_{n-1}$. This quotient is a monogenic Hopf algebra generated by, say, $\bar{x}_n$.

Suppose we can choose $x_n \in A_n$ such that the quotient map from A_n to B_n maps x_n to $\bar{x}_n$ and such that the height of x_n is the same as that of $\bar{x}_n$. Let $\eta : B_n \to A_n$ by the algebra monomorphism satisfying $\eta(\bar{x}_n) = x_n$.

Consider the composite

$$A_{n-1} \otimes B_n \xrightarrow{\text{inc}\otimes\eta} A_n \otimes A_n \xrightarrow{\mu} A_n \xrightarrow{\Delta} A_n \otimes A_n \xrightarrow{1\otimes\text{proj}} A_n \otimes B_n.$$

Applied to an element $a \otimes \bar{x}_n$, we get $a \otimes \bar{x}_n +$(other terms not equal to $-a \otimes \bar{x}_n$), so the composite is a monomorphism, and hence $\phi = \mu \circ (\text{inc}\otimes\eta)$ is injective. On the other hand, since A_{n-1} and $x_n = \eta(\bar{x}_n)$ generate A_n, ϕ is surjective and therefore an isomorphism. Thus the inductive step holds and the result follows.

It remains to show that we can choose $x_n \in A_n$ appropriately. The interesting case is when $p = 2$ or $\deg x_n$ is even. In this case, $B_n \cong k[\bar{x}_n]/(\bar{x}_n^{p^s})$ for some $s > 0$. We need to show that there is a representative for x_n such that $x_n^{p^s} = 0$. Let

$\zeta : H \to H$ be the Frobenius map, i.e., $\zeta(a) = a^p$, and let $Z = \zeta^s(A_{n-1})$. Consider the diagram

$$\begin{tikzcd} A_{n-1} \arrow[r] \arrow[d, "q_{n-1}"] & A_n \arrow[r, "\text{proj}"] \arrow[d, "q_n"] & B_n \arrow[d, no head] \\ A_{n-1}//Z \arrow[r] & A_n//Z \arrow[r, "\pi"] & B_n \end{tikzcd}$$

Note that this makes sense because k is perfect; otherwise, Z need not be a sub-Hopf algebra of A_{n-1}. Also, we have used the fact that $(A_n//Z)//(A_{n-1}//Z) \cong A_n//A_{n-1}$.

Since π is onto, there exists $y \in A_n//Z$ such that $\pi(y) = \bar{x}_n$. We will show that $y^{p^s} = 0$. We have $\Delta(y) = y \otimes 1 + 1 \otimes y + \sum_j a'_j \otimes a''_j$. Since $\pi(a'_j) = \pi(a''_j) = 0$, we have $a'_j, a''_j \in A_{n-1}//Z$. Hence $\Delta(y^{p^s}) = y^{p^s} \otimes 1 + 1 \otimes y^{p^s}$, i.e., y^{p^s} is primitive. On the other hand, there are no primitives in B_n or in $A_{n-1}//Z$ with degree $p^s \deg \bar{x}_n$. By the exact sequence of primitives $0 \to P(A_{n-1}//Z) \to P(A_n//Z) \to P(B_n)$, the only primitive element in $A_n//Z$ is zero. So $y^{p^s} = 0$.

Let $w \in A_n$ be such that $q_n(w) = y$. Then $q_n(w^{p^s}) = y^{p^s} = 0$, so $w^{p^s} \in Z$. As $Z = \zeta^s(A_{n-1})$, there exists $v \in A_{n-1}$ such that $v^{p^s} = w^{p^s}$. Then $(w - v)^{p^s} = 0$ and $q_n(w - v) = y$. Thus we can take $w - v$ as our representative of x_n, and it has the same height as $\bar{x}_n$. □

2.2. **The rational cohomology.** The cohomology ring $H^*(X; R)$ of a H-space of finite type with a R-free multiplication μ (for example, any compact Lie group) is a commutative Hopf algebra whose product is the cup product and whose coproduct is given by $\mu^* : H^*(X; R) \to H^*(X \times X; R) \cong H^*(X; R) \otimes H^*(X; R)$. The structure theorem of Hopf algebras gives us:

Definition 2.2. Let G be a compact Lie group and let k be a field of characteristic zero. Then $H^*(G; k) \cong \Lambda[x_{2m_1+1}, x_{2m_2+1}, \ldots, x_{2m_l+1}]$. We call the sequence $(2m_1 + 1, 2m_2 + 1, \ldots, 2m_l + 1)$ in ascending order the *(rational) type* of G.

To deduce the type $(2m_1 + 1, \ldots, 2m_l + 1)$ of an exceptional Lie group, we use certain facts about the root system of the Lie algebra. Consideration of the infinitesimal diagrams of Lie groups also gives us a couple of theorems that relate the type to these data. For more details see [8].

Fact 2.3. We have $\dim E_6 = 78$, $\dim E_7 = 133$, $\dim E_8 = 248$, $\dim F_4 = 52$, and $\dim G_2 = 14$.

Knowing the dimension of the group G, we can figure out the sum of the m_i's, i.e., $\sum_{i=1}^{l} m_i = \frac{1}{2}(\dim G - l)$.

Fact 2.4. Let G be an exceptional Lie group. Let $\{\alpha_1, \ldots, \alpha_l\}$ be the simple roots of the Lie algebra associated with G. We can express the sum of positive roots in G as a linear combination of simple roots, namely:

Group	Sum of positive roots
E_6	$16\alpha_1 + 22\alpha_2 + 30\alpha_3 + 42\alpha_4 + 30\alpha_5 + 16\alpha_6$
E_7	$34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7$
E_8	$92\alpha_1 + 136\alpha_2 + 182\alpha_3 + 270\alpha_4 + 220\alpha_5 + 168\alpha_6 + 114\alpha_7 + 58\alpha_8$
F_4	$16\alpha_1 + 30\alpha_2 + 42\alpha_3 + 22\alpha_4$
G_2	$10\alpha_1 + 6\alpha_2$

Fact 2.5. We can also express the dominant root as a linear combination of simple roots:

Group	Dominant root
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$3\alpha + 12\alpha_2$

Theorem 2.6. *Let G be a compact simply-connected simple Lie group. Then $H^3(G;\mathbb{Q}) \cong \mathbb{Z}$. This implies that $m_1 = 1$ and $m_i > 1$ for $i > 1$.*

Proof. For the classical simple Lie groups, the theorem is easily deduced from the results in the previous section. For the Lie groups G_2 and F_4, one can (and we will) consider the fibrations $S^3 \to G_2 \to V_2(\mathbb{R}^7)$ and $\mathrm{Spin}(9) \to F_4 \to \mathbb{O}P^2$. Finally, there exist homomorphisms $\mathrm{Spin}(10) \to E_6$, $\mathrm{Spin}(12) \to E_7$, and $\mathrm{Spin}(16) \to E_8$ that induce isomorphisms on H^3. □

Remark 2.7. Indeed, one can show, without even invoking the classification theorem, that $\pi_2(G) = 0$ and that $\pi_3(G) \cong \mathbb{Z}$.

Theorem 2.8. *Let G be a compact connected Lie group, and suppose it has type $(2m_1 + 1, \ldots, 2m_l + 1)$. If the sum of the positive roots in terms of simple roots is $d_1\alpha_1 + \cdots + d_l\alpha_l$, and the dominant root is $a_1\alpha_1 + \cdots + a_l\alpha_l$, then we have*

$$\prod_{i=1}^{l} d_i = l! \prod_{i=1}^{l} a_i m_i.$$

Proof sketch. There is a Morse-theoretic formula for the Poincaré series of ΩG, expressed in terms of parameters of the roots of the Lie algebra associated with G, namely the coefficients $\{a_i\}$ and $\{d_i\}$. The equation above is obtained by equating this expression of the Poincaré series evaluated at $t = 1$ with that obtained from the type of G. □

Using the previous theorems and the data for the Lie groups above, we can compute the sum and product of the m_i's. In each case, there is a unique possible combination for the m_i's, and we obtain the following theorem.

Theorem 2.9. *The types of the exceptional Lie groups are:*

Group	*Type*
E_6	$(3, 9, 11, 15, 17, 23)$
E_7	$(3, 11, 15, 19, 23, 27, 35)$
E_8	$(3, 15, 23, 27, 35, 39, 47, 59)$
F_4	$(3, 11, 15, 23)$
G_2	$(3, 11)$

Remark 2.10. Slightly more is true: if G has type $(2m_1+1, \ldots, 2m_l+1)$, there exists a map $f : \prod S^{2m_i+1} \to G$ that induces the isomorphism on rational cohomology.

This gives us the rational cohomology of the exceptional Lie groups. Another approach starting from de Rham's theorems can be found in [3].

In the subsequent sections, we will analyse the mod p cohomology of these Lie groups. The following theorem is useful:

Theorem 2.11. *Let $(2m_1 + 1, \ldots, 2m_l + 1)$ be the type of a compact, simply-connected, simple Lie group G. If $m_l < \min(pm_2, p^2 - 1)$, then $H^*(G)$ has no p-torsion and $H^*(G; \mathbb{F}_p) \cong \Lambda[x_3, x_{2m_2+1}, \ldots, x_{2m_l+1}]$.*

Remarks on the proof. The proof of this theorem is broken into two parts, depending on the height of the generator y_2 of $H^2(\Omega G; \mathbb{F}_p)$. In either case, we perform calculations that are similar to those in section 2.5 to obtain the mod p cohomology of G. They coincide with the rational cohomology, from which we infer the non-existence of p-torsion. The inequality in the hypothesis is used to show that there are very few transgressions that need to be considered, so the calculations are easier. As such, we do not prove this theorem here. □

Remark 2.12. It turns out that the converse is also true. So G_2 only has 2-torsion; F_4, E_6, and E_7 have 2- and 3-torsion; and E_8 has 2-, 3-, and 5-torsion.

2.3. **The cohomology of G_2.** For this section and the next, more information can be found in [2].

The compact form of the group G_2 can be thought of as the automorphism group of the octonion algebra $\mathbb{O}$. Let $\{e_i\}_{i=1}^7$ be the standard vector space basis for $\operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$, and let $f_1, f_2 \in \operatorname{Im}\mathbb{O}$ be two orthonormal purely imaginary octonions. Let $f_4 = f_1 f_2$, and choose f_3 to be orthonormal to f_1, f_2, f_4. Define $f_5 = f_2 f_3$, $f_6 = f_3 f_4$, and $f_7 = f_4 f_5$. The collection $\{f_i\}_{i=1}^7$ forms a vector space basis for the octonion algebra, so the map sending e_i to f_i defines an automorphism of the octonions.

The group G_2 acts transitively on f_1, f_2 – two orthonormal vectors in $\mathbb{R}^7$, i.e., $V_2(\mathbb{R}^7)$. The isotropy group fixing f_1, f_2 corresponds to the unit vectors f_3 in the four-dimensional space orthogonal to f_1, f_2, f_4, i.e. S^3. Thus we have the following:

$$G_2/S^3 \cong V_2(\mathbb{R}^7).$$

On the other hand, the octonion algebra can be related to the special orthogonal group:

Theorem 2.13. *Let $A, B, C \in SO(8)$ satisfy $A(x)B(y) = C(xy)$ for all $x, y \in \mathbb{O}$. Then given exactly one of A, B, and C, the other two exist and are unique up to sign.*

Consider now the subgroup H of $SO(8) \times SO(8)$ consisting of pairs (B, C) such that there exists $A \in SO(8)$ satisfying $A(x)B(y) = C(xy)$ and $B = C$. Observe that $A(1) = 1$ since $A(1)B(x) = B(x)$. Thus the map $H \to SO(7)$ sending (B, C) to the restriction of A onto $\operatorname{Im}\mathbb{O}$ is a double cover of $SO(7)$, i.e., $H \cong \operatorname{Spin}(7)$. Indeed, H is generated by (L_z, L_z) where L_z is the isometry corresponding to multiplication on the left by $z \in \mathbb{O}$. This shows that H acts transitively on $S^7 \subset \mathbb{O}$ since L_z acts transitively on unit vectors. If $(B, C) \in H$ is in the stabilizer of $1 \in S^7$, then

$B(1) = 1$, so by the relation $A(x)B(y) = C(xy)$, we have $A(x) = B(x)$ for all $x \in \mathbb{O}$, so we have $B(x)B(y) = B(xy)$, i.e., $B \in G_2$. Thus we have

$$\mathrm{Spin}(7)/G_2 \cong S^7.$$

Theorem 2.14.
(1) $H^*(G_2;\mathbb{Z}) \cong \mathbb{Z}[x_3, x_{11}]/(x_3^4, x_{11}^2, x_3^2 x_{11}, 2x_3^2)$.
(2) $H^*(G_2;\mathbb{F}_2) \cong \mathbb{F}_2[x_3]/(x_3^4) \otimes \Lambda[x_5]$.

Proof. Consider the spectral sequence for the fibration $S^3 \to G_2 \to V_2(\mathbb{R}^7)$. Recall that $H^*(S^3) = \Lambda[x_3]$ and that

$$H^k(V_2(\mathbb{R}^7);\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 11 \\ \mathbb{Z}/2\mathbb{Z} & k = 6 \\ 0 & \text{otherwise.} \end{cases}$$

By lacunary reasons, the spectral sequence $H^*(V_2(\mathbb{R}^7)) \otimes H^*(S^3) \Rightarrow H^*(G_2)$ collapses on the second page, and we obtain the result in (1), modulo showing $x_3^2 = x_6$. This will be shown once we show the same in (2).

For $\mathbb{F}_2$ coefficients, we use the same fibration, but with the observation that $H^*(V_2(\mathbb{R}^7);\mathbb{F}_2) \cong \Lambda[x_5, x_6]$. The spectral sequence collapses and therefore we have $H^*(G_2;\mathbb{F}_2) \cong \Delta[x_3, x_5, x_6]$.

To show that $x_3^2 = x_6$, we can use the fibration $G_2 \xrightarrow{i} \mathrm{Spin}(7) \to S^7$. The spectral sequence for the mod two cohomology collapses too on the second page too, so i^* is an epimorphism. By counting dimensions, we see that i^* is an isomorphism for dimensions ≤ 6, Since $x_3^2 = x_6$ holds in $H^*(\mathrm{Spin}(7);\mathbb{F}_2)$, it holds in $H^*(G_2;\mathbb{F}_2)$ as well.

FIGURE 6. The $E_2 = E_\infty$ page of the spectral sequence associated with $S^3 \hookrightarrow G_2 \to V_2(\mathbb{R}^7)$ with integer coefficients

3	$\mathbb{Z}$			$\mathbb{Z}/(2)$			$\mathbb{Z}$
2							
1							
0	$\mathbb{Z}$			$\mathbb{Z}/(2)$			$\mathbb{Z}$
	0	$\cdots$	$\cdots$	6	$\cdots$	$\cdots$	11

$\square$

FIGURE 7. The $E_2 = E_\infty$ page of the spectral sequence associated with $S^3 \hookrightarrow G_2 \to V_2(\mathbb{R}^7)$ with $\mathbb{F}_2$ coefficients

3	$\mathbb{F}_2$			$\mathbb{F}_2$	$\mathbb{F}_2$			$\mathbb{F}_2$
2								
1								
0	$\mathbb{F}_2$			$\mathbb{F}_2$	$\mathbb{F}_2$			$\mathbb{F}_2$
	0	$\cdots$	$\cdots$	5	6	$\cdots$	$\cdots$	11

FIGURE 8. The $E_2 = E_\infty$ page of the spectral sequence associated with $G_2 \hookrightarrow \mathrm{Spin}(7) \to S^7$ with $\mathbb{F}_2$ coefficients

6	x_6			x_6x_7
5	x_5			x_5x_7
4				
3	x_3			x_3x_7
2				
1				
0	1			x_7
	0	$\cdots$	$\cdots$	7

2.4. **The cohomology of F_4.**

Theorem 2.15. $H^*(F_4;\mathbb{F}_3) \cong \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda[x_3, x_7, x_{11}, x_{15}]$.

Proof. The group F_4 acts transitively on the Cayley plane $\mathbb{O}P^2$ with isotropy group Spin(9). Consider the spectral sequence for the fibration $\mathrm{Spin}(9) \hookrightarrow F_4 \to \mathbb{O}P^2$. Recall that we have $H^*(\mathrm{Spin}(9);\mathbb{F}_3) \cong \Lambda[x_3, x_7, x_{11}, x_{15}]$ and $H^*(\mathbb{O}P^2) \cong \mathbb{F}_3[x_8]/(x_8^3)$. The elements x_3 and x_{11} are cocycles for all the differentials, and we have $x_7 = P^1 x_3$ and $x_{15} = P^1 x_{11}$ (see [1]), so x_7 and x_{15} do not transgress either and Spin(9) is totally nonhomologous to zero mod 3 in F_4. Hence

$$H^*(F_4;\mathbb{F}_3) \cong H^*(\mathbb{O}P^2;\mathbb{F}_3) \otimes H^*(\mathrm{Spin}(9);\mathbb{F}_3) \cong \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda[x_3, x_7, x_{11}, x_{15}].$$

□

Before we compute the mod 2 cohomology of F_4, we will need a few preliminary calculations. Fix a collection of embedded subgroups $F_4 \supset \mathrm{Spin}(9) \supset \mathrm{Spin}(7) \supset G_2$ such that $\mathrm{Spin}(9)/\mathrm{Spin}(7) \cong S^{15}$ and $\mathrm{Spin}(7)/G_2 \cong S^7$. See [2] for more details.

Lemma 2.16.

(1) $H^*(\mathrm{Spin}(9)/G_2;\mathbb{F}_2) \cong \Lambda[x_7, x_{15}]$.
(2) $H^*(F_4/\mathrm{Spin}(8);\mathbb{F}_2) \cong \mathbb{F}_2[x_8]/(x_8^3) \otimes \Lambda[y_8]$.
(3) $H^*(F_4/G_2;\mathbb{F}_2) \cong \Lambda[x_{15}, x_{23}]$.

Proof.

(1) Consider the fibration $\mathrm{Spin}(7)/G_2 \hookrightarrow \mathrm{Spin}(9)/G_2 \to \mathrm{Spin}(9)/\mathrm{Spin}(7)$. This is the same as $S^7 \hookrightarrow \mathrm{Spin}(9)/G_2 \to S^{15}$. The spectral sequence collapses, and the result follows.

(2) Consider the fibration $\mathrm{Spin}(9)/\mathrm{Spin}(8) \hookrightarrow F_4/\mathrm{Spin}(8) \to F_4/\mathrm{Spin}(9)$. This is the same as $S^8 \hookrightarrow F_4/\mathrm{Spin}(8) \to \mathbb{O}P^2$. The spectral sequence collapses, so the result follows.

(3) Consider the fibration $\mathrm{Spin}(9)/G_2 \hookrightarrow F_4/G_2 \to F_4/\mathrm{Spin}(9) \cong \mathbb{O}P^2$. Let x_7 be the generator for $H^7(\mathrm{Spin}(9)/G_2)$ and let x_8 be a generator for $H^8(\mathbb{O}P^2)$. It suffices to show that the transgression $\tau(x_7) = x_8$. In this case, the only nontrivial differential would be d_8 and $H^*(F_4/G_2) \cong \Lambda[x_{15}, x_{23}]$ as wanted.

So suppose that $\tau(x_7) = 0$. Consider the fibration

$$G_2 \hookrightarrow \mathrm{Spin}(9) \xrightarrow{p_1} \mathrm{Spin}(9)/G_2.$$

We know the cohomology of all three spaces, so the spectral sequence collapses on the second page, and in particular p_1^* is an isomorphism on H^7. Next consider the fibration

$$\mathrm{Spin}(9)/\mathrm{Spin}(8) \cong S^8 \hookrightarrow F_4/\mathrm{Spin}(8) \xrightarrow{p_2} F_4/\mathrm{Spin}(9) \cong \mathbb{O}P^2.$$

Again, the spectral sequence collapses and in particular $p_2^*(x_8) = x_8$.

Finally consider the fibration $\mathrm{Spin}(8) \hookrightarrow F_4 \to F_4/\mathrm{Spin}(8)$. The only possible nonzero differential is d_8. Let i^* be the map induced by the inclusion of Spin(8) into Spin(9). By naturality, the following diagram commutes:

$$\begin{tikzcd}
H^7(\mathrm{Spin}(8)) \arrow[r, shift left, "\tau"] & H^8(F_4/\mathrm{Spin}(8)) \arrow[l, shift left, "\sigma"] \\
H^7(\mathrm{Spin}(9)) \arrow[u, "i^*"] \arrow[r, shift left, "\tau"] & H^8(F_4/\mathrm{Spin}(9)) \arrow[l, shift left, "\sigma"] \arrow[u, hook, "p_2^*"'] \\
H^7(\mathrm{Spin}(9)/G_2) \arrow[u, hook, "p_1^*", "\sim"'] \arrow[r, shift left, "\tau"] & H^8(F_4/\mathrm{Spin}(9)) \arrow[l, shift left, "\sigma"] \arrow[u, no head]
\end{tikzcd}$$

Since the transgression on the third rung is zero, all the horizontal arrows in the diagram are zero, so x_8 survives forever in the spectral sequence $\mathrm{Spin}(8) \hookrightarrow F_4 \to F_4/\mathrm{Spin}(8)$. Clearly, x_8^2 survives as well so $H^*(F_4; \mathbb{F}_3)$ would be $\mathbb{F}_3[x_8]/(x_8^3) \otimes \Delta[x_3, x_5, x_6, x_7]$ which contradicts the dimension of F_4. So we have $\tau(x_7) = x_8$ in the original spectral sequence.

□

Theorem 2.17. $H^*(F_4; \mathbb{F}_2) \cong \mathbb{F}_2[x_3]/(x_3^4) \otimes \Lambda[x_5, x_{15}, x_{23}]$

Proof. Consider the spectral sequence $G_2 \hookrightarrow F_4 \to F_4/G_2$. We know the cohomology rings of G_2 and of F_4/G_2. The spectral sequence collapses by lacunary considerations, and so the result follows.

□

FIGURE 9. The $E_2 = E_\infty$ page of the spectral sequence associated with $G_2 \hookrightarrow F_4 \to F_4/G_2$ with $\mathbb{F}_2$ coefficients

6	x_3^2			$x_{15}x_3^2$			$x_{23}x_3^2$	
5	x_5			$x_{15}x_5$			$x_{23}x_5$	
4								
3	x_3			$x_{15}x_3$			$x_{23}x_3$	
2								
1								
0	1			x_{15}			x_{23}	
	0	$\cdots$	$\cdots$	15	$\cdots$	$\cdots$	23	

2.5. **The cohomology of E_6, E_7, and E_8.**

Theorem 2.18. *Let G be a simply-connected compact Lie group. Then we have $H^*(\Omega G;\mathbb{F}_p) \cong \bigotimes \mathbb{F}_p[y_{2n_j}]/(y_{2n_j}^{s_j})$, where s_j is a power of p (or infinity). If the type of G is $(2m_1+1,\dots,2m_l+1)$, then the integers n_j and s_j satisfy*

$$\prod_i (1-t^{2m_i})^{-1} = \prod_j (1-t^{2s_jn_j})(1-t^{2n_j})^{-1}.$$

Proof. Consider the spectral sequence for the path-loop fibration $\Omega G \to PG \to G$ with rational coefficients. This shows that $H^*(\Omega G;\mathbb{Q})$ is a polynomial algebra whose elements are of even degree, namely $2m_1 = 2,\dots,2m_l$. Using the Morse series, we can show that the Poincaré series of ΩG is independent of the field. Hopf's theorem gives us another form for the cohomology ring of ΩG since ΩG is a H-space, and equating the Poincaré series for ΩG gives us the above relation. □

Definition 2.19. Let p be an odd prime. There is a stable cohomology operation $P^j : H^i(X;\mathbb{F}_p) \to H^{i+2j(p-1)}(X;\mathbb{F}_p)$ for each positive integer j called the *reduced p-th power operations* satisfying the following axioms:

(1) Let $x \in H^{2j}$. Then $P^j(x) = x^p$.
(2) $P^0 = \mathrm{id}$ and $P^j(x) = 0$ if $\deg x < j$.

(3) (Cartan formula) $P^j(xy) = \sum_{s+t=j} P^s x P^t y$.
(4) (Adem relations)

$$P^a P^b = \sum_{t=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t \quad \text{if } a < pb$$

$$P^a \beta P^b = \sum_{t=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t} P^t + \sum_{t=0}^{\lfloor \frac{a-1}{p} \rfloor} (-1)^{a+t+1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t} \beta P^t \quad \text{if } a \leq pb,$$

where β is the Bockstein operation.

Theorem 2.20. *Let y_2 be a generator of $H^2(\Omega G; \mathbb{F}_p)$.*
(1) $y_2^p \neq 0$ if $(G, p) = (E_6, 3)$ or $(E_8, 5)$.
(2) $y_2^{p^2} \neq 0$ if $(G, p) = (E_7, 3)$ or $(E_8, 3)$.
(3) $y_2^4 \neq 0$ if $(G, p) = (E_6, 2)$.

Proof. There are maps $\mathrm{Spin}(10) \to E_6$, $\mathrm{Spin}(12) \to E_7$, and $\mathrm{Spin}(16) \to E_8$ that induce isomorphisms $H^2(\Omega G) \xrightarrow{\sim} H^2(\Omega \,\mathrm{Spin}(n))$. So, it suffices to show the result for these loop spaces on these spin groups. Recall that if $p \neq 2$, then $H^*(\mathrm{Spin}(2m); \mathbb{F}_p) = \Lambda[x_3, x_7, \ldots, x_{4m-5}, y_{2m-1}]$.

Let $p = 3$, and let $n = 10, 12$, or 16. Consider the spectral sequence for the fibration $\Omega \,\mathrm{Spin}(n) \to * \to \mathrm{Spin}(n)$. Clearly, $\tau(y_2) = x_3 \in H^3(\mathrm{Spin}(n))$. Since $\sigma(x_7) = \sigma(P^1 x_3) = P^1 y_2 = y_2^3$, $y_2^3 \neq 0$.

If $n = 12$ or 16, then $\tau(y_2^9) = \tau(P^3(y_2^3)) = P^3(\tau(y_2^3)) = P^3(x_7) = x_{19}$ also, so $y_2^9 \neq 0$.

Now let $p = 5$ and $n = 16$. We have $\tau(y_2^5) = \tau(P^1 y_2) = P^1 \tau(y_2) = P^1 x_3 = x_{11}$, so $y_2^5 \neq 0$.

Finally let $p = 2$. In the spectral sequence for $\Omega \,\mathrm{Spin}(10) \to * \to \mathrm{Spin}(10)$ with $\mathbb{F}_2$ coefficients, we have $\tau(y_2^4) = x_9$, so $y_2^4 \neq 0$. □

As G is 2-connected and $\pi_3(G) = \mathbb{Z}$, there is a map $G \to K(\mathbb{Z}, 3)$. We can convert this map via the mapping cocylinder to obtain a fibration $\tilde{G} \to G \to K(\mathbb{Z}, 3)$. In light of this, we will also need to know the cohomology of $K(\mathbb{Z}, 3)$.

Fact 2.21. The cohomology rings of $K(\mathbb{Z}, 3)$ in $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$ are as follows:

$$H^*(\mathbb{Z}, 3; \mathbb{F}_2) \cong \mathbb{F}_2[u_3, u_5, u_9, \ldots, u_{2^i+1}, \ldots]$$
$$H^*(\mathbb{Z}, 3; \mathbb{F}_3) \cong \Lambda[u_3, u_7, u_{19}, \ldots, u_{2\cdot 3^i+1}, \ldots] \otimes \mathbb{F}_3[u_8, u_{20}, u_{56}, \ldots, u_{2\cdot 3^i+2}, \ldots]$$
$$H^*(\mathbb{Z}, 3; \mathbb{F}_5) \cong \Lambda[u_3, u_{11}, u_{51}, \ldots, u_{2\cdot 5^i+1}, \ldots] \otimes \mathbb{F}_5[u_{12}, u_{52}, u_{252}, \ldots, u_{2\cdot 5^i+2}, \ldots].$$

Proof sketch. We know that $H^*(\mathbb{Z}, 2; \mathbb{F}_p) \cong H^*(\mathbb{C}P^\infty; \mathbb{F}_p) \cong \mathbb{F}_p[u_2]$. Consider the spectral sequence associated with $K(\mathbb{Z}, 2) \to * \to K(\mathbb{Z}, 3)$, first in the $\mathbb{F}_2$ case. We will prove that $H^*(\mathbb{Z}, 3; \mathbb{F}_2) \cong \mathbb{F}_2[u_3, \mathrm{Sq}^2 u_3, \mathrm{Sq}^4 \mathrm{Sq}^2 u_3, \mathrm{Sq}^8 \mathrm{Sq}^4 \mathrm{Sq}^2 u_3, \ldots]$. Set $u_3 = \tau(u_2)$. It is easily verified that

$$\tau(u_2^{2^i}) = \tau \,\mathrm{Sq}^{2^i} \mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2 u_2 = \mathrm{Sq}^{2^i} \mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2 u_3.$$

The other differentials are worked out similarly.

In the case of $\mathbb{F}_p$ coefficients where p is odd, we shall show

$$H^*(\mathbb{Z},3;\mathbb{F}_p) \cong \Lambda[u_3, P^1u_3, P^pu_3, P^{p^2}P^pP^1u_3,\ldots] \otimes \mathbb{F}_p[\beta P^1u_3, \beta P^pP^1u_3,\ldots].$$

Again set $u_3 = \tau(u_2)$, and we have as above $\tau(u_2^{p^i}) = P^{p^{i-1}}P^{p^{i-2}}\cdots P^1u_3$. We also see that the other differentials are

$$d_{2p^i(p-1)+1}(P^{p^{i-1}}\cdots P^1u_3 \otimes u_2^{p^i(p-1)}) = \beta P^{p^i}\cdots P^1u_3.$$

□

Remark 2.22. In general, the cohomology of a space of type $K(\mathbb{Z},n)$ is given in terms of admissible sequences of Steenrod operations. This can be proved inductively.

Consider again the fibration $\tilde{G} \to G \to K(\mathbb{Z},3)$. We can continue to extend this map to the left, i.e., $K(\mathbb{Z},1) \to \Omega\tilde{G} \to \Omega G \to K(\mathbb{Z},2) \to \tilde{G} \to G \to K(\mathbb{Z},3)$, so that every consecutive triple forms a Serre fibration. Thus the general outline of computing the cohomology ring of G is to first obtain an estimate of $H^*(\Omega G)$, and use that to compute $H^*(\Omega\tilde{G})$. This will allow us to compute partially $H^*(\tilde{G})$ and finally $H^*(G)$. Counting dimensions along with Hopf's theorem will in most cases allow us to recover the rest of the cohomology ring. See [8].

Proposition 2.23. $H^*(E_6;\mathbb{F}_3) \cong \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda[x_3,x_7,x_9,x_{11},x_{15},x_{17}]$.

Proof. The type of E_6 is $(3,9,11,15,17,23)$. Using the relation in theorem 2.18, we have

$$\prod_j(1-t^{2n_j}) = \prod_j(1-t^{2s_jn_j})\cdot(1-t^2)(1-t^8)(1-t^{10})(1-t^{14})(1-t^{16})(1-t^{22}).$$

Equating coefficients, we see that $2n_1 = 2$, $2n_2 = 8$, $2n_3 = 10$, $2n_4 = 14$, and $2n_5 = 16$. Moreover, since we know that $y_2^3 \neq 0$, $s_1 \geq 9$ holds, and that in general $s_j \geq 3$, the terms from $\prod_j(1-t^{2n_js_j})$ do not interfere for degrees < 18. Thus $H^*(\Omega E_6) = \mathbb{F}_3[y_2,y_8,y_{10},y_{14},y_{16}]$ for degrees < 18.

Now consider the spectral sequence for $K(\mathbb{Z},1) \to \Omega\tilde{E}_6 \to \Omega E_6$. (Recall that S^1 is a $K(\mathbb{Z},1)$.) This shows us that $H^*(\Omega\tilde{E}_6) \cong \mathbb{F}_3[y_8,y_{10},y_{14},y_{16}]$ for degrees < 18.

Next consider the spectral sequence $\Omega\tilde{E}_6 \to * \to \tilde{E}_6$. This shows that $H^(\tilde{E}_6)$ is isomorphic to $\Lambda[x_9,x_{11},x_{15},x_{17}]$ for degrees < 18.

Finally consider the spectral sequence for $\tilde{E}_6 \to E_6 \to K(\mathbb{Z},3)$. Since the image for a transgression must be a p-th power of a generator by Hopf's theorem, there are no trangressions for this spectral sequence, and hence for degrees < 18 we have $H^*(E_6) \cong \mathbb{F}_3[x_8] \otimes \Lambda[x_3,x_7,x_9,x_{11},x_{15},x_{17}]$. By dimensional reasons, since $\dim E_6 = 78$, we have $H^*(E_6;\mathbb{F}_3) \cong \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda[x_3,x_7,x_9,x_{11},x_{15},x_{17}]$ as wanted.

□

Remark 2.24. For the mod p cohomology of G (p odd), passing from the cohomology of ΩG to $\Omega\tilde{G}$ to $\tilde{G}$ is entirely mechanical. The only part of the calculation that needs to be considered in detail is the possible transgressions in $\tilde{G} \to G \to K(\mathbb{Z},3)$.

1	u		■			■		■		■		■
0	1		y_2		■	y_8		y_{10}		y_{14}		y_{16}
	0	…	2	…	…	8	…	10	…	14	…	16

FIGURE 10. The E_2 page of the spectral sequence associated with $K(\mathbb{Z},1) \to \Omega\tilde{E}_6 \to \Omega E_6$ with $\mathbb{F}_3$ coefficients

Figure 11. The spectral sequence associated with $\Omega\tilde{E}_6 \to * \to \tilde{E}_6$

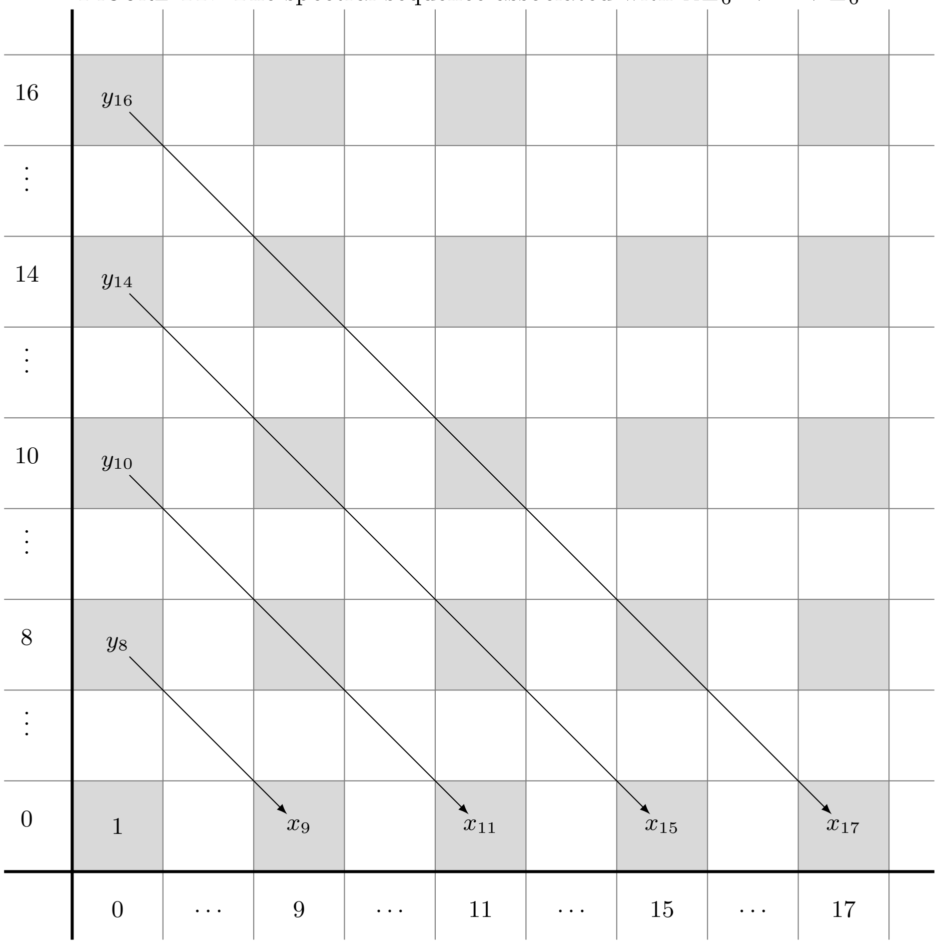


FIGURE 12. The $E_2 = E_\infty$ page of the spectral sequence associated with $\tilde{E}_6 \to E_6 \to K(\mathbb{Z}, 3)$

17	x_{17}		■		■	■
$\vdots$						
15	x_{15}		■		■	■
$\vdots$						
11	x_{11}		■		■	■
$\vdots$						
9	x_9		■		■	■
$\vdots$						
0	1		u_3		u_7	u_8
	0	$\cdots$	3	$\cdots$	7	8

Proposition 2.25. $H^*(E_7;\mathbb{F}_3) \cong \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda[x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}]$.

Proof. Using the same method as before, we can reconstruct

$$H^*(\Omega E_7) \cong \mathbb{F}_3[y_2, y_{10}, y_{14}, y_{18}, y_{22}, y_{26}] \text{ for degrees} < 30.$$

This implies that for degrees < 30,

$$H^*(\Omega \tilde{E}_7) \cong \mathbb{F}_3[y_{10}, y_{14}, y_{18}, y_{22}, y_{26}]$$
$$H^*(\tilde{E}_7) \cong \Lambda[x_{11}, x_{15}, x_{19}, x_{23}, x_{27}].$$

In the spectral sequence $\tilde{E}_7 \to E_7 \to K(\mathbb{Z}, 3)$, the only possible nontrivial transgressions are $\tau(x_{19}) = u_{20}$ and $\tau(x_{23}) = u_3^8$.

Hence $H^*(E_7) \cong \Lambda[u_3, u_7, x_{11}, x_{15}, x_{19}, x_{27}] \otimes A \otimes B \otimes C$, where

$$A = \begin{cases} \mathbb{F}_3[u_8]/(u_8^3) & \text{if } \tau(x_{23}) = u_8^3 \\ \Lambda[x_{23}] \otimes \mathbb{F}_3[u_8]/(u_8^f) \text{ where } f \geq 9 \text{ is a power of three,} & \text{if } \tau(x_{23}) = 0 \end{cases}$$

$$B = \begin{cases} \mathbb{F}_3 & \text{if } \tau(x_{19}) = u_{20} \\ \Lambda[x_{19}] \otimes \mathbb{F}_3[u_{20}]/(u_{20}^g) \text{ where } g \geq 3 \text{ is a power of three,} & \text{if } \tau(x_{19}) = 0 \end{cases}$$

$$C = \Lambda[x_\alpha] \otimes \mathbb{F}_3[x_\beta]/(x_\beta^{h_\beta}) \text{ where } \deg x_\alpha, x_\beta \geq 30 \text{ and } h_\beta \text{ is a power of three.}$$

Since the total dimension of E_7 is 133, we must have $A = \mathbb{F}_3[u_8]/(u_8^3)$, $B = \mathbb{F}_3$, and $C = \Lambda[x_{35}]$. Relabelling, we obtain

$$H^*(E_7) = \mathbb{F}_3[x_8]/(x_8^3) \otimes \Lambda[x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}].$$

□

Proposition 2.26.

$$H^*(E_8;\mathbb{F}_3) \cong \mathbb{F}_3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda[x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}].$$

Proof. Reasoning as above, we have for degrees < 42,

$$H^*(\Omega E_8) \cong \mathbb{F}_3[y_2, y_{14}, y_{22}, y_{26}, y_{34}, y_{38}]$$
$$H^*(\Omega \tilde{E}_8) \cong \mathbb{F}_3[y_{14}, y_{22}, y_{26}, y_{34}, y_{38}]$$
$$H^*(\tilde{E}_8) \cong \Lambda[x_{15}, x_{23}, x_{27}, x_{35}, x_{39}].$$

The only possible transgressive elements are x_{19} and x_{23}. From this we infer that $H^*(E_8) \cong \Lambda[x_3, x_7, x_{15}, x_{27}, x_{35}, x_{39}] \otimes A \otimes B \otimes C$, where again

$$A = \begin{cases} \mathbb{F}_3[u_8]/(u_8^3) & \text{if } \tau(x_{23}) = u_8^3 \\ \Lambda[x_{23}] \otimes \mathbb{F}_3[u_8]/(u_8^f) \text{ where } f \geq 9 \text{ is a power of three,} & \text{if } \tau(x_{23}) = 0 \end{cases}$$

$$B = \begin{cases} \mathbb{F}_3 & \text{if } \tau(x_{19}) = u_{20} \\ \Lambda[x_{19}] \otimes \mathbb{F}_3[u_{20}]/(u_{20}^g) \text{ where } g \geq 3 \text{ is a power of three,} & \text{if } \tau(x_{19}) = 0 \end{cases}$$

$$C = \Lambda[x_\alpha] \otimes \mathbb{F}_3[x_\beta]/(x_\beta^{h_\beta}) \text{ where } \deg x_\alpha, x_\beta \geq 30 \text{ and } h_\beta \text{ is a power of three.}$$

From dimensional reasons alone we can conclude that $A = \mathbb{F}_3[u_8]/(u_8^3)$. Next, since $\sigma(u_{20}) = \sigma(P^3 u_8) = P^3(\sigma(u_8)) = 0$, the transgression $\tau(x_{19}) = 0$, and so $B = \Lambda[x_{19}] \otimes \mathbb{F}_3[u_{20}]/(u_{20}^3)$. Hence $C = \Lambda[x_{47}]$, and our result is obtained. □

Proposition 2.27.

$$H^*(E_8;\mathbb{F}_5) \cong \mathbb{F}_5[x_{12}]/(x_{12}^5) \otimes \Lambda[x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}].$$

Proof. Reasoning as above, we have for degrees < 50,

$$H^*(\Omega E_8) \cong \mathbb{F}_5[y_2, y_{14}, y_{22}, y_{26}, y_{34}, y_{38}, y_{46}]$$
$$H^*(\Omega \tilde{E}_8) \cong \mathbb{F}_3[y_{14}, y_{22}, y_{26}, y_{34}, y_{38}, y_{46}]$$
$$H^*(\tilde{E}_8) \cong \Lambda[x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}].$$

By lacunary reasons, there are no nontrivial transgressions, so for degrees less than 50, we have $H^*(E_8) \cong \mathbb{F}_5[x_{12}] \otimes \Lambda[x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}]$. Counting dimensions, we must have

$$H^*(E_8; \mathbb{F}_5) \cong \mathbb{F}_5[x_{12}]/(x_{12}^5) \otimes \Lambda[x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}].$$

□

We now consider the mod two cohomology of these exceptional groups.

Proposition 2.28. $H^*(E_6; \mathbb{F}_2) = \mathbb{F}_2[x_3]/(x_3^4) \otimes \Lambda[x_5, x_9, x_{15}, x_{17}, x_{23}]$.

Proof. Assume for now that $y_2^8 \neq 0$. Then as before, we have for degrees < 28

$$H^*(\Omega E_6) \cong \mathbb{F}_2[y_2] \otimes \Delta[y_8, y_{10}, y_{14}, y_{16}, y'_{16}, y'_{20}, y_{22}]$$
$$H^*(\Omega \tilde{E}_6) \cong \Delta[y_8, y_{10}, y_{14}, y_{16}, y'_{16}, y'_{20}, y_{22}].$$

Note that it does not matter for this whether y_8^2 or y_{10}^2 is zero or not.

In the fibration $\Omega\tilde{E}_6 \to * \to \tilde{E}_6$, the elements y_8, y_{10}, y_{14} are transgressive. If both x_{16} and x'_{16} were transgressive as well, then $H^*(\tilde{E}_6) \cong \mathbb{F}_2[x_9, x_{11}, x_{15}, x_{17}, x'_{17}]$ for degrees < 21. Using the Adem relation $\mathrm{Sq}^9 = \mathrm{Sq}^2 \mathrm{Sq}^7 + \mathrm{Sq}^8 \mathrm{Sq}^1$, we see that $x_9^2 = \mathrm{Sq}^9 x_9 = 0$, which is a contradiction. So at least one of y_{16}, y'_{16} is not transgressive, say $d_9(y_{16}) = x_9 y_8$. On the other hand, y'_{16} is transgressive, so we have $H^*(\tilde{E}_6) \cong \Lambda[x_9] \otimes \mathbb{F}_2[x_{11}, x_{15}, x'_{17}]$ for degrees < 21.

Now consider the fibration $\tilde{E}_6 \to E_6 \to K(\mathbb{Z}, 3)$. The possible transgressions are $\tau(x_9) = u_5^2$, $\tau(x_{11}) = u_3^4$, and $\tau(x'_{17}) = u_9^2$. Hence for degrees < 20, $H^*(E_6)$ is isomorphic to $\Delta[u_3, u_5, u_3^2, u_9, u_{17}, x_{15}] \otimes \Delta[\{x_\alpha\}]$, where $\{x_\alpha\}$ is some combination of $\{x_9, u_5^2\}$, $\{x_{11}, u_3^4\}$, $\{x'_{17}, u_9^2\}$, and $\{x_\beta \mid \deg x_\beta \geq 20\}$.

By dimensional reasons, we have $\Delta[\{x_\alpha\}] = \Delta[x_{11}, u_3^4]$ or $\Delta[x_{23}]$. Since $\{x_9, u_5^2\}$ is not possible, we must have $\tau(x_9) = u_5^2$, so $u_5^2 = 0$ in $H^*(E_6)$. Observe that $u_5 = \mathrm{Sq}^2 u_3$ and $u_9 = \mathrm{Sq}^4 u_5$, because these relations hold from $H^*(\mathbb{Z}, 3)$. Using the Cartan formula and the Adem relation $\mathrm{Sq}^1 \mathrm{Sq}^{2i} = \mathrm{Sq}^{2i+1}$, we obtain

$$u_3^4 = (\mathrm{Sq}^3 u_3)^2 = (\mathrm{Sq}^1 \mathrm{Sq}^2 u_3)^2 = (\mathrm{Sq}^1 u_5)^2 = \mathrm{Sq}^2 u_5^2 = 0$$
$$u_9^2 = (\mathrm{Sq}^4 u_5)^2 = \mathrm{Sq}^8 u_5^2 = 0.$$

So $\Delta[\{x_\alpha\}] \neq \Delta[x_{11}, u_3^4]$. Therefore by relabelling, we get that $H^*(E_6)$ is isomorphic to $\mathbb{F}_2(x_3^4) \otimes \Lambda[x_5, x_9] \otimes \Delta[x_{15}, x_{17}, x_{23}]$. No other relations are possible, so

$$H^*(E_6; \mathbb{F}_2) \cong \mathbb{F}_2[x_3]/(x_3^4) \otimes \Lambda[x_5, x_9, x_{15}, x_{17}, x_{23}].$$

□

We still need to show that $y_2^8 \neq 0$.

Lemma 2.29. *If y_2 is the generator of $H^2(\Omega E_6; \mathbb{F}_2)$, then $y_2^8 \neq 0$. Since the inclusion maps $E_6 \subset E_7 \subset E_8$ induce isomorphisms on $H^2(\Omega G; \mathbb{F}_2)$, $y_2^8 \neq 0$ in $H^*(\Omega E_7; \mathbb{F}_2)$ and $H^*(\Omega E_8; \mathbb{F}_2)$ as well.*

Proof. Suppose for sake of contradiction that $y_2^8 = 0$. Then we have

$$H^*(\Omega E_6) \cong \mathbb{F}_2[y_2]/(y_2^8) \otimes \Delta[y_8, y_{10}, y_{14}, y_{16}, y'_{16}, y''_{16}, y'_{20}, y_{22}] \quad (\deg < 28)$$
$$H^*(\Omega \tilde{E}_6) \cong \Delta[y_8, y_{10}, y_{14}, y_{15}, y_{16}, y'_{16}, y''_{16}, y'_{20}, y_{22}] \quad (\deg < 28)$$
$$H^*(\tilde{E}_6) \cong \Lambda[x_9] \otimes \mathbb{F}_2[x_{11}, x_{15}, x_{16}, x'_{17}, x''_{17}] \quad (\deg < 21).$$

By dimensional reasons, $\tau(x_{16}) = u_{17}$ in the spectral sequence associated with $\tilde{E}_6 \to E_6 \xrightarrow{j} K(\mathbb{Z}, 3)$; in fact, all possible transgressions are realized. So relabelling, we have

$$H^*(E_6) \cong \mathbb{F}_2[x_3]/(x_3^4) \otimes \Lambda[x_5, x_9, x_{15}, x_{17}, x_{23}].$$

Observe that in this case

$$\mathrm{Sq}^8\, x_9 = j^*(\mathrm{Sq}^8\, u_9) = j^*(u_{17}) = j^*(\tau(x_{16})) = \tau(j^*(x_{16})) = 0.$$

We will show that this is false by deriving part of the Steenrod squares on $H^*(E_6; \mathbb{F}_2)$ using the Wu formula for Chern classes and the isomorphism $\pi_3(SU(6)) \cong \pi_3(E_6)$.

Consider the spectral sequence for the universal E_6-bundle: $E_6 \to * \to BE_6$ with $\mathbb{F}_2$ coefficients. For degrees < 16, we have $H^*(BE_6) \cong \mathbb{F}_2[y_4, y_6, y_7, y_{10}]$. Specifically, we have $\mathrm{Sq}^8\, y_{10} = \mathrm{Sq}^8\, \tau(x_9) = \tau(\mathrm{Sq}^8\, x_9) = 0$ modulo decomposables, say $\mathrm{Sq}^8\, y_{10} = P(y_4, y_6, y_7, y_{10})$ for some polynomial P.

Now consider the inclusion $SU(6) \to E_6$. This map induces an isomorphism $\pi_3(SU(6)) \to \pi_3(E_6)$. By the long exact sequence of the fibration $E_6 \to * \to BE_6$, $\pi_4(BSU(6)) \to \pi_4(BE_6)$ is an isomorphism. By the Hurewicz isomorphism and the universal coefficients theorem, the map $\rho^* : H^*(BE_6) \to H^*(BSU(6))$ is an isomorphism of H^4, i.e., $\rho^*(y_4) = c_2$ where c_2 is the first Chern class. Now applying Sq^2, Sq^4, and Sq^8 successively and using the Wu formula, we have $\rho^*(y_6) = c_3$, $\rho^*(y_{10}) = c_5 + c_3 c_2$, and $\rho^*(\mathrm{Sq}^8\, y_{10}) = c_6 c_3 + c_5 c_4 +$ (higher order terms). Also, $\rho^*(y_7) = 0$ since $H^7(BSU(6)) = 0$.

We have

$$\begin{aligned} P(c_2, c_3, 0, c_5 + c_3 c_2) &= \rho^*(P(y_4, y_6, y_7, y_{10})) = \rho^*(\mathrm{Sq}^8\, y_{10}) \\ &= c_6 c_3 + c_5 c_4 + \text{ (higher order terms)}. \end{aligned}$$

But this is absurd. So $\mathrm{Sq}^8\, x_9 \neq 0$ and $y_2^8 \neq 0$ as wanted. □

Proposition 2.30. $H^*(E_7; \mathbb{F}_2) \cong \mathbb{F}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda[x_{15}, x_{17}, x_{23}, x_{27}]$.

Proof. As before, we have for degrees < 29,

$$H^*(\Omega E_7) \cong \mathbb{F}_2[y_2] \otimes \Delta[y_{10}, y_{14}, y_{18}, y'_{20}, y_{22}, y_{26}, y'_{28}]$$
$$H^*(\Omega \tilde{E}_7) \cong \Delta[y_{10}, y_{14}, y_{18}, y'_{20}, y_{22}, y_{26}, y'_{28}].$$

For this range of degrees, there are two possibilities for $H^*(\tilde{E}_7)$:

$$H^*(\tilde{E}_7) \cong \begin{cases} \Lambda[x_{11}] \otimes \mathbb{F}_2[x_{15}, x_{19}, x_{23}, x_{27}] & \text{if } d_{11} y'_{20} = x_{11} y_{10} \\ \mathbb{F}_2[x_{11}, x_{15}, x_{19}, x'_{21}, x_{23}, x_{27}] & \text{if } \tau(y'_{20}) = x'_{21}. \end{cases}$$

Using the Adem relation $\mathrm{Sq}^4\, \mathrm{Sq}^7 = \mathrm{Sq}^{11} + \mathrm{Sq}^9\, \mathrm{Sq}^2$, we see that

$$x_{11}^2 = \mathrm{Sq}^{11}\, x_{11} = \mathrm{Sq}^4\, \mathrm{Sq}^7\, x_{11} + \mathrm{Sq}^9\, \mathrm{Sq}^2\, x_{11} = 0,$$

so we have $H^*(\tilde{E}_7) \cong \Lambda[x_{11}] \otimes \mathbb{F}_2[x_{15}, x_{19}, x_{23}, x_{27}$ for degrees < 29.

There are three possible transgressions for the fibration $\tilde{E}_7 \to E_7 \to K(\mathbb{Z}, 3)$, namely $\tau(x_{11}) = u_3^4$, $\tau(x_{19}) = u_5^4$, and $\tau(x_{23}) = u_3^8$. Moreover, if $\tau(x_{11}) = u_3^4$, then

$\tau(x_{23}) = 0$. By dimensional reasons, x_{19} must transgress to u_5^4 and exactly one of x_{11} and x_{23} transgresses. We thus have two possibilities for degrees < 29:

$$H^*(E_7) \cong \begin{cases} \mathbb{F}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Delta[x_{15}, x_{17}, x_{23}, x_{27}] \\ \mathbb{F}_2[x_3, x_5, x_9]/(x_3^8, x_5^4, x_9^4) \otimes \Delta[x_{11}, x_{15}, x_{17}, x_{27}]. \end{cases}$$

We claim that the second possibility does not occur. Assume the contrary and consider the spectral sequence associated to $E_6 \hookrightarrow E_7 \to E_7/E_6$. We thus have $\Delta[x_5^2, x_{11}, x_3^4, x_9^2, x_{27}] \subseteq H^*(E_7/E_6)$. But

$$\deg x_3^4 x_5^2 x_9^2 x_{11} x_{27} = 78 > 55 = \dim E_7/E_6.$$

This is a contradiction.

No other relations are possible, so

$$H^*(E_7; \mathbb{F}_2) \cong \mathbb{F}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda[x_{15}, x_{17}, x_{23}, x_{27}].$$

□

Proposition 2.31.

$$H^*(E_8; \mathbb{F}_2) = \mathbb{F}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda[x_{17}, x_{23}, x_{27}, x_{29}].$$

Proof. As before, we have for degrees < 32,

$$H^*(\Omega E_8) \cong \mathbb{F}_2[y_2] \otimes \Delta[y_{14}, y_{22}, y_{26}, y'_{28}]$$

$$H^*(\Omega \tilde{E}_8) \cong \Delta[y_{14}, y_{22}, y_{26}, y'_{28}]$$

$$H^*(\tilde{E}_8) \cong \begin{cases} \mathbb{F}_2[x_{15}, x_{23}, x_{27}, x_{29}] & \text{if } \tau(y'_{28}) = x_{29} \\ \Lambda[x_{15}] \otimes \mathbb{F}_2[x_{23}, x_{27}] & \text{if } d_{15} y_{28'} = x_{15} y_{14}. \end{cases}$$

We now consider the fibration $\tilde{E}_8 \to E_8 \to K(\mathbb{Z}, 3)$. The only possible transgression is $\tau(x_{23}) = u_3^8$, but $\sigma(u_3^8) = \sigma(\mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\, u_3) = \mathrm{Sq}^{12}\,\mathrm{Sq}^6\,\mathrm{Sq}^3\,\sigma(u_3) = 0$. So $\tau(x_{23}) = 0$.

Suppose that $H^*(\tilde{E}_8) \cong \Lambda[x_{15}] \otimes \mathbb{F}_2[x_{23}, x_{27}]$. Then for degrees < 32,

$$H^*(E_8) \cong \Delta[u_3, u_3^2, u_3^4, u_3^8, u_5, u_5^2, u_5^4, u_9, u_9^2, u_{17}, x_{15}, x_{23}, x_{27}].$$

For dimensional reasons this is a contradiction: there is no way to complete the cohomology ring with additional generators. Therefore $H^*(\tilde{E}_8) \cong \mathbb{F}_2[x_{15}, x_{23}, x_{29}]$ and hence

$$H^*(E_8; \mathbb{F}_2) = \mathbb{F}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda[x_{17}, x_{23}, x_{27}, x_{29}].$$

□

Remark 2.32. The Steenrod squares in $H^*(E_6)$, $H^*(E_7)$, and $H^*(E_8)$ are more fully worked out in [8].

Now that we have calculated the cohomology rings of all the exceptional Lie groups with field coefficients, we can make an observation:

Proposition 2.33. *In the chain of subgroups $G_2 \subset F_4 \subset E_6 \subset E_7 \subset E_8$, every group is totally nonhomologous to zero modulo two in the group that contains it.*

Also, we can compute the cohomology of the loop spaces and classifying spaces of these Lie groups, or their quotients, and so on.

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