CATEGORIES OF SPECTRA AND INFINITE LOOP SPACES

by

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At the Seattle conference, I presented a calculation of $H_*(F;Z_p)$ as an algebra, for odd primes p, where $F = \varinjlim F(n)$ and $F(n)$ is the topological monoid of homotopy equivalences of an n-sphere. This computation was meant as a preliminary step towards the computation of $H^*(BF;Z_p)$. Since then, I have calculated $H^*(BF;Z_p)$, for all primes p, as a Hopf algebra over the Steenrod and Dyer-Lashof algebras. The calculation, while not difficult, is somewhat lengthy, and I was not able to write up a coherent presentation in time for inclusion in these proceedings. The computation required a systematic study of homology operations on n-fold and infinite loop spaces. As a result of this study, I have also been able to compute $H_*(\Omega^n S^n X;Z_p)$, as a Hopf algebra over the Steenrod algebra, for all connected spaces X and prime numbers p. This result, which generalizes those of Dyer and Lashof [3] and Milgram [8], yields explicit descriptions of both $H_*(\Omega^n S^n X;Z_p)$ and $H_*(QX;Z_p)$, $QX = \varinjlim \Omega^n S^n X$, as functors of $H_*(X;Z_p)$.

An essential first step towards these results was a systematic categorical analysis of the notions of n-fold and infinite loop spaces. The results of this analysis will be presented here. These include certain adjoint functor relationships that provide the conceptual reason that $H_*(\Omega^n S^n X; Z_p)$ and $H_*(QX; Z_p)$ are functors of $H_*(X; Z_p)$ and that precisely relate maps between spaces to maps between spectra. These categorical considerations motivate the introduction of certain non-standard categories, $\mathcal{I}$ and $\mathcal{L}$, of (bounded) spectra and Ω-spectra, and the main purpose of this paper is to propagandize these categories. It is clear from their definitions that these categories are considerably easier to work with topologically than are the usual ones, but it is not clear that they are sufficiently large to be of interest. We shall remedy this by showing that, in a sense to be made precise, these categories are equivalent for the purposes of homotopy theory to the standard categories of (bounded) spectra and Ω-spectra. We extend the theory to unbounded spectra in the last section.

The material here is quite simple, both as category theory and as topology, but it turns out nevertheless to have useful concrete applications. We shall indicate two of these at the end of the paper. In the first, we observe that there is a natural epimorphism, realized by a map of spaces, from the stable homotopy groups of an infinite loop space to its ordinary homotopy groups. In the second, by coupling our results with other information, we shall construct a collection of interesting topological spaces and maps; the other information by itself gives no hint of the possibility of performing this construction.

1 THE CATEGORIES $\mathcal{L}_n$ AND HOMOLOGY

In order to sensibly study the homology of iterated loop spaces, it is necessary to have a precise categorical framework in which to work. It is the purpose of this section to present such a framework.

We let $\mathcal{T}$ denote the category of topological spaces with base-point and base-point preserving maps, and we let

$$\mu\colon \mathrm{Hom}_{\mathcal{T}}(X,\Omega Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(SX,Y) \qquad (1.1)$$

denote the standard adjunction homeomorphism relating the loop and suspension functors.

We define the category of n-fold loop sequences, $\mathcal{L}_n$, to have objects $B = \{B_i | 0 \le i \le n\}$ such that $B_i = \Omega B_{i+1} \in \mathcal{T}$ and maps $g = \{g_i | 0 \le i \le n\}$ such that $g_i = \Omega g_{i+1} \in \mathcal{T}$; clearly $B_0 = \Omega^i B_i$ and $g_0 = \Omega^i g_i$ for $0 \le i \le n$. We define $\mathcal{L} = \mathcal{L}_\infty$ to be the category with objects $B = \{B_i | i \ge 0\}$ such that $B_i = \Omega B_{i+1} \in \mathcal{T}$ and maps $g = \{g_i | i \ge 0\}$ such that $g_i = \Omega g_{i+1} \in \mathcal{T}$; clearly $B_0 = \Omega^i B_i$ and $g_0 = \Omega^i g_i$ for all $i \ge 0$. We call $\mathcal{L}_\infty$ the category of perfect Ω-spectra (or of infinite loop sequences). For all n, we define forgetful functors $U_n\colon \mathcal{L}_n \longrightarrow \mathcal{T}$ by $U_n B = B_0$ and $U_n g = g_0$. Of course, if $n < \infty$, $U_n B$ and $U_n g$ are n-fold loop spaces and maps. We say that a space $X \in T$ is a perfect infinite loop space if $X = U_\infty B$ for some object $B \in L_\infty$ and we say that a map $f \in T$ is a perfect infinite loop map if $f = U_\infty g$ for some map $g \in L_\infty$.

We seek adjoints $Q_n: T \longrightarrow L_n$, $1 \le n \le \infty$, to the functors U_n. For $n < \infty$, define $Q_n X = \{\Omega^{n-i} S^n X \mid 0 \le i \le n\}$ and $Q_n f = \{\Omega^{n-i} S^n f \mid 0 \le i \le n\}$. Clearly, $Q_n X$ and $Q_n f$ are objects and maps in L_n. For the case $n = \infty$, we first define a functor $Q: T \longrightarrow T$ by letting $QX = \varinjlim \Omega^n S^n X$, where the limit is taken with respect to the inclusions

$$\Omega^n \mu^{-1}(1_{S^{n+1}X}): \Omega^n S^n X \longrightarrow \Omega^{n+1} S^{n+1} X$$

For $f: X \longrightarrow Y$, we define $Qf = \varinjlim \Omega^n S^n f: QX \longrightarrow QY$. It is clear that $QX = \Omega QSX$ and $Qf = \Omega QSf$. We can therefore define a functor $Q_\infty: T \longrightarrow L_\infty$ by $Q_\infty X = \{QS^i X \mid i \ge 0\}$ and $Q_\infty f = \{QS^i f \mid i \ge 0\}$.

Proposition 1

For each n, $1 \le n \le \infty$, there is an adjunction

$$\phi_n: \mathrm{Hom}_T(X, U_n B) \longrightarrow \mathrm{Hom}_{L_n}(Q_n X, B).$$

Proof. Observe first that the following two composites are the identity.

$$S^n X \xrightarrow{S^n \mu^{-n}(1_{S^n X})} S^n \Omega^n S^n X \xrightarrow{\mu^n(1_{\Omega^n S^n X})} S^n X, \; X \in T \tag{1.2}$$

$$\Omega^n X \xrightarrow{\mu^{-n}(1_{S^n \Omega^n X})} \Omega^n S^n \Omega^n X \xrightarrow{\Omega^n \mu^n(1_{\Omega^n X})} \Omega^n X, \; X \in T \tag{1.3}$$

In fact, since $\mu(f) = \mu(1_{\Omega Z}) \cdot Sf$ for any map $f: Y \longrightarrow \Omega Z$ in $\mathcal{T}$, $\mu^n(1_{\Omega^n S^n X}) \cdot S^n\mu^{-n}(1_{S^n X}) = \mu^n\mu^{-n}(1_{S^n X}) = 1_{S^n X}$; this proves (1.2) and the proof of (1.3) is similar. Now define natural transformations $\Phi_n: Q_n U_n \longrightarrow 1_{\mathcal{L}_n}$ and $\Psi_n: 1_{\mathcal{T}} \longrightarrow U_n Q_n$ by

$$\Phi_n(B) = \{\Omega^{n-i}\mu^n(1_{B_0})\,|\,0 \le i \le n\}: Q_n U_n B \longrightarrow B \quad \text{if} \quad n < \infty; \tag{1.4}$$

$$\Phi_\infty(B) = \{\varinjlim \Omega^j\mu^{i+j}(1_{B_0})\,|\,i \ge 0\}: Q_\infty U_\infty B \longrightarrow B \quad \text{if} \quad n = \infty;$$

$$\Psi_n(X) = \mu^{-n}(1_{S^n X}): X \longrightarrow U_n Q_n X = \Omega^n S^n X \quad \text{if} \quad n < \infty; \tag{1.5}$$

$$\Psi_\infty(X) = \varinjlim \mu^{-j}(1_{S^j X}): X \longrightarrow U_\infty Q_\infty X = QX \quad \text{if} \quad n = \infty.$$

We claim that (1.2) and (1.3) imply that the following two composites are the identity for all n.

$$Q_n X \xrightarrow{Q_n\Psi_n(X)} Q_n U_n Q_n X \xrightarrow{\Phi_n(Q_n X)} Q_n X,\ X \in \mathcal{T} \tag{1.6}$$

$$U_n B \xrightarrow{\Psi_n(U_n B)} U_n Q_n U_n B \xrightarrow{U_n\Phi_n(B)} U_n B,\ B \in \mathcal{L}_n \tag{1.7}$$

For $n < \infty$, (1.6) follows from (1.2) by application of Ω^{n-i} for $0 \le i \le n$ and (1.7) is just (1.3) applied to $X = B_n$, since $B_0 = U_n B = \Omega^n B_n$. For $n = \infty$, observe that $\Psi_\infty(X)$ factors as the composite

$$X \xrightarrow{\mu^{-1}(1_{SX})} \Omega SX \xrightarrow{\Omega\Psi_\infty(SX)} \Omega QSX = QX \quad .$$

It follows that $\Psi_\infty(X) = \mu^{-i}\Psi_\infty(S^i X)$ for all $i \ge 0$ since $\mu^{-i}\Psi_\infty(S^i X) = \mu^{-i}(\Omega\Psi_\infty(S^{i+1}X) \cdot \mu^{-1}(1_{S^{i+1}X})) = \mu^{-(i+1)}\Psi_\infty(S^{i+1}X)$.
Observe also that

$$\Omega^j\Psi_\infty(S^{i+j}X): \Omega^j S^{i+j}X \longrightarrow \Omega^j QS^{i+j}X = QS^i X$$

is just the natural inclusion obtained from the definition of QS^iX as $\varinjlim \Omega^j S^{i+j}X$. We therefore have that:

$$\Phi_\infty(Q_\infty X)_i \cdot Q_\infty \Psi_\infty(X)_i$$

$$= \varinjlim \Omega^j \mu^{i+j}(1_{QX}) \cdot \varinjlim \Omega^k S^{i+k}\mu^{-(i+k)}\Psi_\infty(S^{i+k}X)$$

$$= \varinjlim \Omega^j \mu^{i+j}(1_{QX}) \cdot \Omega^j S^{i+j}\mu^{-(i+j)}\Psi_\infty(S^{i+j}X)$$

$$= \varinjlim \Omega^j \Psi_\infty(S^{i+j}X) = 1_{QS^iX} ;$$

$$U_\infty\Phi_\infty(B) \cdot \Psi_\infty(U_\infty B) = \varinjlim \Omega^j\mu^j(1_{B_0}) \cdot \varinjlim \mu^{-k}(1_{S^kB_0})$$

$$= \varinjlim \Omega^j\mu^j(1_{B_0}) \cdot \mu^{-j}(1_{S^jB_0}) = \varinjlim 1_{B_0} = 1_{B_0} .$$

In both calculations, the second equality is an observation about the limit topology. The third equalities follow from formulas (1.2) and (1.3) respectively. Finally, define

$$\phi_n(f) = \Phi_n(B) \cdot Q_n f \quad \text{if } f\colon X \longrightarrow U_nB \text{ is a map in } \mathcal{T} \qquad (1.8)$$

$$\psi_n(g) = U_n g \cdot \Psi_n(X) \quad \text{if } g\colon Q_nX \longrightarrow B \text{ is a map in } \mathcal{L}_n \qquad (1.9)$$

It is a standard fact that ϕ_n is an adjunction with inverse ψ_n since the composites (1.6) and (1.7) are each the identity.

If $B \in \mathcal{L}_n$, we define $H_*(B) = H_*(U_nB)$, where homology is taken with coefficients in any Abelian group Π. We regard H_* as a functor defined on $\mathcal{L}_n$, but we deliberately do not specify a range category. Indeed, the problem of determining the homology operations on n-fold and (perfect) infinite loop spaces may be stated as that of obtaining an appropriate algebraic description of the range category. It follows easily from (1.2) and (1.5) of the proof above that $\Psi_n(X)_*: H_*(X) \longrightarrow H_*(U_n Q_n X)$ is a monomorphism. Since Q_n is adjoint to U_n, the objects $Q_n X$ are, in a well-defined sense, free objects in the category $\mathcal{L}_n$. It is therefore natural to expect $H_*(Q_n X)$ to be a functor of $H_*(X)$, with values in the appropriate range category. I have proven that this is the case if $\pi = Z_p$ and have computed the functor. By the previous proposition, if $B \in \mathcal{L}_n$ then any map $f: X \longrightarrow U_n B$ in $\mathcal{T}$ induces a map $\phi_n(f): Q_n X \longrightarrow B$ in $\mathcal{L}_n$, and the functor describing $H_*(Q_n X)$ is geometrically free in the sense that $\phi_n(f)_*: H_*(Q_n X) \longrightarrow H_*(B)$ is determined by $f_* = U_n \phi_n(f)_* \Psi_n(X)_*: H_*(X) \longrightarrow H_*(U_n B)$ in terms of the homology operations that go into the definition of the functor. In this sense, we can geometrically realize enough free objects since $\phi_n(B)_*: H_*(Q_n U_n B) \longrightarrow H_*(B)$ is an epimorphism. All of these statements are analogs of well-known facts about the cohomology of spaces. The category of unstable algebras over the Steenrod algebra is the appropriate range category for cohomology with Z_p-coefficients. Products of $K(Z_p,n)$'s play the role analogous to that of the $Q_n X$ and their fundamental classes play the role analogous to that of $H_*(X) \subset H_*(Q_n X)$.

By use of Proposition 1, we can show the applicability of the method of acyclic models to the homology of iterated loop spaces. The applications envisaged are to natural transformations defined for iterated loop spaces but not for arbitrary spaces. The argument needed is purely categorical. Let $\mathcal{T}$ temporarily denote any category, let $\mathcal{A}$ denote the category of modules over a commutative ring Λ, and let $\mathcal{M}$ be a set of model objects in $\mathcal{T}$. Let $F: \mathcal{S} \longrightarrow \mathcal{A}$ be the free Λ-module functor, where $\mathcal{S}$ is the category of sets. If $R: \mathcal{T} \longrightarrow \mathcal{A}$ is any functor, define a functor $\tilde{R}: \mathcal{T} \longrightarrow \mathcal{A}$ by $\tilde{R}(X) = F[\bigcup_{M \in \mathcal{M}} \mathrm{Hom}_{\mathcal{T}}(M,X) \times R(M)]$ on objects and $\tilde{R}(f)(\nu,r) = (f \cdot \nu, r)$ on morphisms, where if $f: X \longrightarrow Y$, then $\nu \in \mathrm{Hom}_{\mathcal{T}}(M,X)$ and $r \in R(M)$. Define a natural transformation $\lambda: \tilde{R} \longrightarrow R$ by $\lambda(X)(\nu,r) = R(\nu)(r)$. Recall that R is said to be representable by $\mathcal{M}$ if there exists a natural transformation $\xi: R \longrightarrow \tilde{R}$ such that $\lambda \cdot \xi: R \longrightarrow R$ is the identity natural transformation. With these notations, we have the following lemma.

<u>Lemma 2</u>

Let $\phi: \mathrm{Hom}_{\mathcal{T}}(X,UB) \longrightarrow \mathrm{Hom}_{\mathcal{L}}(QX,B)$ be an adjunction and let $R: \mathcal{T} \longrightarrow \mathcal{A}$ be a functor representable by $\mathcal{M}$. Define $Q\mathcal{M} = \{QM | M \in \mathcal{M}\}$ and let $S = R \cdot U: \mathcal{L} \longrightarrow \mathcal{A}$. Then S is representable by $Q\mathcal{M}$.

<u>Proof</u>. Define a natural transformation $\eta: \tilde{R} \cdot U \longrightarrow \tilde{S}$ by $\eta(B)(\nu,r) = (\phi(\nu), R\phi^{-1}(1_{QM})(r))$ for $\nu: M \longrightarrow UB$, $r \in R(M)$. Write λ' for the natural transformation $\tilde{S} \longrightarrow S$ defined as above for $\tilde{R}$. We have $\lambda'\eta = \lambda U: \tilde{R}U \longrightarrow RU = S$ since $\lambda'\eta(B)(\nu,r)$ $= S\phi(\nu)[R\phi^{-1}(1_{QM})(r)] = R[U\phi(\nu) \cdot \phi^{-1}(1_{QM})](r) = R(\nu)(r)$.

Therefore, if $\xi\colon R \longrightarrow \tilde{R}$ satisfies $\lambda\xi = 1\colon R \longrightarrow R$, then $\lambda'(\eta\xi U) = \lambda U \cdot \xi U = 1\colon S \longrightarrow S$, and this proves the result.

Of course, if ϕ is an adjunction as in the lemma and if T^j denotes the product of j factors T, then $\phi^j\colon \mathrm{Hom}_{T^j}(X, U^j B) \longrightarrow \mathrm{Hom}_{L^j}(Q^j X, B)$ is also an adjunction $(X \in T^j,\ B \in L^j)$. Thus the lemma applies to functors $R\colon T^j \longrightarrow A$ and $RU^j\colon L^j \longrightarrow A$.

Returning to topology, let $C_*\colon T \longrightarrow A$ be the singular chain complex functor, with coefficients in Λ. The lemma applies to $C_* U_n\colon L_n \longrightarrow A$ for $1 \le n \le \infty$ and, by the remark above, to the usual related functors on L_n^j (tensor and Cartesian products of singular chain complexes). With $M = \{\Delta_m\}$, the standard set of models in T, we have $U_n Q_n \Delta_m = \Omega^n S^n \Delta_m$ if $n < \infty$ and $U_\infty Q_\infty \Delta_m = Q\Delta_m$; these spaces are contractible and the model objects $\{Q_n \Delta_m\} \subset L_n$ are therefore acyclic. We conclude that the method of acyclic models [4] is applicable to the study of the homology of n-fold and perfect infinite loop spaces.

2 COMPARISONS OF CATEGORIES OF SPECTRA

The work of the previous section shows that the category L is a reasonable object of study conceptually, but it is not obvious that L is large enough to be of topological interest. For example, it is not clear that the infinite classical groups are homotopy equivalent to perfect infinite loop spaces. We shall show that, from the point of view of homotopy theory, $\mathcal{L}$ is in fact equivalent to the usual category of (bounded) Ω-spectra. To do this, we shall have to proceed by stages through a sequence of successively more restrictive categories of spectra.

By a spectrum, we shall mean a sequence $B = \{B_i, f_i \mid i \geq 0\}$, where B_i is a space and $f_i \colon B_i \longrightarrow \Omega B_{i+1}$ is a map. By a map $g \colon B \longrightarrow B'$ of spectra we shall mean a sequence of maps $g_i \colon B_i \longrightarrow B_i'$ such that the following diagrams are homotopy commutative, $i \geq 0$.

$$\begin{array}{ccc} B_i & \xrightarrow{\;g_i\;} & B_i' \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_i'} \\ \Omega B_{i+1} & \xrightarrow{\;\Omega g_{i+1}\;} & \Omega B_{i+1}' \end{array} \quad . \qquad (2.1)$$

We call the resulting category $\mathcal{S}$. We say that $B \in \mathcal{S}$ is an inclusion spectrum if each f_i is an inclusion. We obtain the category $\mathcal{I}$ of inclusion spectra by letting a map in $\mathcal{I}$ be a map in $\mathcal{S}$ such that the diagrams (2.1) actually commute on the nose for each $i \geq 0$. (Thus, $\mathcal{I}$ is not a full subcategory of $\mathcal{S}$.) We say that $B \in \mathcal{S}$ is an Ω-spectrum if each f_i is a homotopy equivalence. We let $\Omega\mathcal{S}$ be the full subcategory of $\mathcal{S}$ whose objects are the Ω-spectra, and we let $\Omega\mathcal{I} = \mathcal{I} \cap \Omega\mathcal{S}$ be the full subcategory of $\mathcal{I}$ whose objects are the inclusion Ω-spectra. A spectrum $B \in \Omega\mathcal{I}$ will be said to be a retraction spectrum if B_i is a deformation retract of ΩB_{i+1} for all i. We let $\mathcal{R}$ denote the full subcategory of ΩI whose objects are the retraction spectra. Clearly, L is a full subcategory of R, since if $B \in L$ we may take $f_i = 1$ and then any map in R between objects of L will be a map in L by the commutativity of the diagrams (2.1). Thus we have the following categories and inclusions

$$L \subset R \subset \Omega I \subset \Omega S \quad \text{and} \quad I \subset S . \tag{2.2}$$

For each of these categories C, if $g,g': B \longrightarrow B'$ are maps in C, then we say that g is homotopic to g' if g_i is homotopic to g_i' in T for each i. We say that g is a (weak) homotopy equivalence if each g_i is a (weak) homotopy equivalence. Now each C has a homotopy category HC and a quotient functor $H: C \longrightarrow HC$. The objects of HC are the same as those of C and the maps of HC are homotopy equivalence classes of maps in C. Note that each of the inclusions of (2.2) is homotopy preserving in the sense that if $C \subset D$ and $g \simeq g'$ in C, then $g \simeq g'$ in D. We therefore have induced functors $HC \longrightarrow HD$ and these are still inclusions since if $g,g' \in C$ and $g \simeq g'$ in D, then $g \simeq g'$ in C.

The following definitions, due to Swan [11], will be needed in order to obtain precise comparisons of our various categories of spectra.

<u>Definitions 3</u>

(i) A category C is an H-category if there is an equivalence relation $\simeq$, called homotopy, on its hom sets such that $f \simeq f'$ and $g \simeq g'$ implies $fg \simeq f'g'$ whenever fg is defined. We then have a quotient category $H\mathcal{C}$ and a quotient functor $H: \mathcal{C} \longrightarrow H\mathcal{C}$.

(ii) Let $\mathcal{C}$ be any category and $\mathcal{D}$ an H-category. A prefunctor $T: \mathcal{C} \longrightarrow \mathcal{D}$ is a function, on objects and maps, such that $HT: \mathcal{C} \longrightarrow H\mathcal{D}$ is a functor. This amounts to requiring $T(1_C) \simeq 1_{T(C)}$ for each $C \in \mathcal{C}$ and $T(fg) \simeq T(f)T(g)$ whenever fg is defined in $\mathcal{C}$. If $\mathcal{C}$ is also an H-category, we say that a prefunctor $T: \mathcal{C} \longrightarrow \mathcal{D}$ is homotopy preserving if $f \simeq g$ in $\mathcal{C}$ implies $T(f) \simeq T(g)$ in $\mathcal{D}$. Clearly, T is homotopy preserving if and only if T determines a functor $T_*: H\mathcal{C} \longrightarrow H\mathcal{D}$ such that $HT = T_*H$.

(iii) Let $S,T: \mathcal{C} \longrightarrow \mathcal{D}$ be prefunctors. A natural transformation of prefunctors $\eta: S \longrightarrow T$ is a collection of maps $\eta(C): S(C) \longrightarrow T(C)$, $C \in \mathcal{C}$, such that $T(f)\eta(C) \simeq \eta(C')S(f)$ in $\mathcal{D}$ for each map $f: C \longrightarrow C'$ in $\mathcal{C}$. η is said to be a natural equivalence of prefunctors if there exists a natural transformation of prefunctors $\xi: T \longrightarrow S$ such that $\eta(C)\xi(C) \simeq 1_{T(C)}$ and $\xi(C)\eta(C) \simeq 1_{S(C)}$ for each $C \in \mathcal{C}$. A natural transformation of prefunctors $\eta: S \longrightarrow T$ determines a natural transformation of functors $H\eta: HS \longrightarrow HT$ and, if S and T are homotopy preserving, a natural transformation of functors $\eta_*: S_* \longrightarrow T_*$ such that $\eta_*H = H\eta$; if η is a natural equivalence of prefunctors, then $H\eta$ and, if defined, η_* are natural equivalences of functors.

(iv) If $S: \mathcal{D} \longrightarrow \mathcal{C}$ and $T: \mathcal{C} \longrightarrow \mathcal{D}$ are homotopy preserving prefunctors between H-categories, we say that T is adjoint to S if there exist natural transformations of prefunctors $\Phi: TS \longrightarrow 1_{\mathcal{D}}$ and $\Psi: 1_{\mathcal{C}} \longrightarrow ST$ such that for each $D \in \mathcal{D}$ the composite $S\Phi(D)\Psi(SD): SD \longrightarrow SD$ is homotopic in $\mathcal{C}$ to the identity map of SD and for each $C \in \mathcal{C}$ the composite $\Phi(TC) \cdot T\Psi(C): TC \longrightarrow TC$ is homotopic in $\mathcal{D}$ to the identity map of TC. If S and T are adjoint prefunctors, then $S_*: H\mathcal{D} \longrightarrow H\mathcal{C}$ and $T_*: H\mathcal{D} \longrightarrow H\mathcal{C}$ are adjoint functors, with adjunction $\phi_* = \Phi_* T_*: \mathrm{Hom}_{H\mathcal{C}}(A, S_*B) \longrightarrow \mathrm{Hom}_{H\mathcal{D}}(T_*A, B)$.

We can now compare our various categories of spectra. The following theorem implies that $\mathcal{I}$ is equivalent to $\mathcal{S}$ for the purposes of homotopy theory in the sense that no homotopy invariant information is lost by restricting attention to spectra and maps of spectra in $\mathcal{I}$, and that $\Omega\mathcal{I}$ is equivalent to $\Omega\mathcal{S}$ in this sense. Under restrictions on the types of spaces considered, it similarly compares $\mathcal{R}$ to $\Omega\mathcal{S}$. To state the restrictions, let $\mathcal{C}$ denote the full subcategory of $\mathcal{S}$ whose objects are those spectra $\{B_i, f_i\}$ such that each B_i is a locally finite countable simplicial complex and each $\mu(f_i): SB_i \longrightarrow B_{i+1}$ is simplicial. Observe that if $\mathcal{W}$ is the full subcategory of $\mathcal{S}$ whose objects are those spectra B such that each B_i has the homotopy type of a countable CW-complex, then every object of $\mathcal{W}$ is homotopy equivalent (in $\mathcal{S}$) to an object of $\mathcal{C}$. In fact, if $\{B_i, f_i\} \in \mathcal{W}$, then each B_i is homotopy equivalent to a locally finite simplicial complex B_i' by [9, Theorem 1]; if f_i' is the composite $B_i' \longrightarrow B_i \xrightarrow{f_i} \Omega B_{i+1} \longrightarrow \Omega B_{i+1}'$ determined by chosen homotopy equivalences $B_i \rightleftarrows B_i'$ and if $\mu(f_i'')$ is a simplicial approximation to $\mu(f_i')$, then $\{B_i, f_i\}$ is homotopy equivalent to $\{B_i', f_i'\}$ and therefore to $\{B_i', f_i''\} \in C$.

<u>Theorem 4</u>

There is a homotopy preserving prefunctor $M: S \longrightarrow I$ such that

(i) There exists a natural equivalence of prefunctors $\eta: 1_S \longrightarrow JM$, with inverse $\xi: JM \longrightarrow 1_S$, where $J: I \longrightarrow S$ is the inclusion. Therefore J_*M_* is naturally equivalent to the identity functor of HS.

(ii) $MJ: I \longrightarrow I$ is a functor, $\xi(JB): JMJB \longrightarrow JB$ is a map in I if $B \in I$, and if $\zeta: MJ \longrightarrow 1_I$ is defined by $\zeta(B) = \xi(JB)$, then ζ is a natural transformation of functors.

(iii) η and ζ establish an adjoint prefunctor relationship between J and M. Therefore $\phi_*: \mathrm{Hom}_{HS}(A, J_*B) \longrightarrow \mathrm{Hom}_{HI}(M_*A, B)$ is an adjunction, where $\phi_*(f) = \zeta_*(B)M_*f$, $f: A \longrightarrow J_*B$, and $\phi_*^{-1}(g) = J_*g \cdot \eta_*(A)$, $g: M_*A \longrightarrow B$.

(iv) By restriction, M induces a homotopy preserving prefunctor $\Omega S \longrightarrow \Omega I$ which satisfies (i) through (iii) with respect to the inclusion $\Omega I \longrightarrow \Omega S$.

(v) By restriction, M induces a homotopy preserving prefunctor $\Omega S \cap C \longrightarrow R \cap C$ which satisfies (i) through (iii) with respect to the inclusion $R \cap C \longrightarrow \Omega S \cap C$.

<u>Proof</u>. We first construct M and prove (i) and (ii) simultaneously. Let $B = \{B_i, f_i\} \in S$. Define $MB = \{M_i B, M_i f\} \in I$ by induction on i as follows. Let $M_0 B = B_0$. Assume that $M_j B$, $j \leq i$, and $M_j f$, $j < i$, have been constructed. Let $\eta_0 = 1 = \xi_0$ and assume further that $\eta_j \colon B_j \longrightarrow M_j B$ and $\xi_j \colon M_j B \longrightarrow B_j$ have been constructed such that

(a) $\xi_j \eta_j = 1 \colon B_j \longrightarrow B_j$ and $\eta_j \xi_j \simeq 1 \colon M_j B \longrightarrow M_j B$;

(b) $\Omega\xi_j \cdot M_{j-1} f = f_{j-1} \cdot \xi_{j-1}$ and $\Omega\eta_j \cdot f_{j-1} \simeq M_{j-1} f \cdot \eta_{j-1}$.

Define $M_{i+1} B$ to be the mapping cylinder of the map $\mu(f_i) \cdot S\xi_i \colon SM_i B \longrightarrow B_{i+1}$, let $k_i \colon SM_i B \longrightarrow M_{i+1} B$ denote the standard inclusion, and define $M_i f = \mu^{-1}(k_i) \colon M_i B \longrightarrow \Omega M_{i+1} B$. Clearly $M_i f$ is then an inclusion. Consider the diagram

$$\begin{array}{ccc} SM_i B & \underset{S\eta_i}{\overset{S\xi_i}{\rightleftarrows}} & SB_i \\ {\scriptstyle k_i = \mu(M_i f)} \downarrow & & \downarrow {\scriptstyle \mu(f_i)} \\ M_{i+1} B & \underset{\eta_{i+1}}{\overset{\xi_{i+1}}{\rightleftarrows}} & B_{i+1} \end{array}$$

Here η_{i+1} and ξ_{i+1} are the inclusion and retraction obtained by the standard properties of mapping cylinders, hence (a) is satisfied for $j = i + 1$. It is standard that $\xi_{i+1} \cdot \mu(M_i f) = \mu(f_i) S\xi_i$, and $\Omega\xi_{i+1} \cdot M_i f = f_i \xi_i$ follows by application of μ^{-1}. Now $\Omega\eta_{i+1} \cdot f_i \simeq M_i f \cdot \eta_i$ is obtained by a simple chase of the diagram. This proves (b) for $j = i + 1$ and thus constructs M on objects and constructs maps $\eta(B): B \longrightarrow JMB$ and $\xi(B): JMB \longrightarrow B$ in S. If $B \in I$, then $\xi(JB)$ is a map in I by (b) and we can define $\zeta(B) = \xi(JB): MJB \longrightarrow B$. We next construct M on maps. Let $g: B \longrightarrow B'$ be a map in S. Define $M_0 g = g_0$ and assume that $M_j g$ have been found for $j \leq i$ such that (with $\eta' = \eta(B')$, etc.)

(c) $\eta_j' g_j = M_j g \cdot \eta_j$; $\xi_j' \cdot M_j g \simeq g_j \xi_j$ with equality if $g \in I$;

(d) $\Omega M_j g \cdot M_{j-1} f = M_{j-1} f' \cdot M_{j-1} g$.

Then, by (c) and the definition of maps in the categories S and I, $f_i' \xi_i' M_i g \simeq f_i' g_i \xi_i \simeq \Omega g_{i+1} f_i \xi_i : M_i B \longrightarrow \Omega B_{i+1}'$, with equalities if $g \in I$. Applying μ, we see that there exists a homotopy $h_i: SM_i B \times I \longrightarrow B_{i+1}'$ from $\mu(f_i') S\xi_i' SM_i g$ to $g_{i+1}\mu(f_i) S\xi_i$, and we agree to choose h_i to be the constant homotopy if $g \in I$. Write $[x,t]$ and $[y]$ for the images of $(x,t) \in SM_i B \times I$ and $y \in B_{i+1}$ in the mapping cylinder $M_{i+1} B$ of $\mu(f_i) S\xi_i$, and similarly for $M_{i+1} B'$. Define $M_{i+1} g: M_{i+1} B \longrightarrow M_{i+1} B'$ by

$$\text{(e)}\quad M_{i+1} g[x,t] = \begin{cases} [SM_i g(x), 2t], & 0 \leq t \leq 1/2 \\ [h_i(x, 2t-1)], & 1/2 \leq t \leq 1. \end{cases}$$

$$M_{i+1} g[y] = [g_{i+1}(y)] \ .$$

It is trivial to verify that $M_{i+1} g$ is well-defined and continuous. Now consider the following diagram:

$$\begin{array}{ccccc}
SM_iB & \xrightarrow{\mu(M_if)} & M_{i+1}B & \underset{\eta_{i+1}}{\overset{\xi_{i+1}}{\rightleftarrows}} & B_{i+1} \\
\downarrow{\scriptstyle SM_ig} & & \downarrow{\scriptstyle M_{i+1}g} & & \downarrow{\scriptstyle g_{i+1}} \\
SM_iB' & \xrightarrow{\mu(M_if')} & M_{i+1}B' & \underset{\eta'_{i+1}}{\overset{\xi'_{i+1}}{\rightleftarrows}} & B'_{i+1}
\end{array}$$

Since $\eta_{i+1}(y) = [y]$, $\eta'_{i+1} \cdot g_{i+1} = M_{i+1}g \cdot \eta_{i+1}$ is obvious, and $\xi'_{i+1} \cdot M_{i+1}g \simeq g_{i+1}\xi_{i+1}$ then follows from (a) and a simple chase of the right-hand square. If the map g is in I, then $\xi'_{i+1}M_{i+1}g = g_{i+1}\xi_{i+1}$ is easily verified by explicit computation since $h_i(x,t) = g_{i+1}\mu(f_i)S\xi_i(x)$ for all t. This proves (c) for $j = i + 1$. To prove (d) for $j = i + 1$, merely observe that the left-hand square clearly commutes, since $\mu(M_if)(x) = [x,0]$, and apply μ^{-1} to this square. Of course, (d) proves that M_g is a map in I, and (c) completes the proof of (ii) of the theorem since $MJ\colon I \longrightarrow I$ is clearly a functor. If $\ell\colon M_{i+1}B \longrightarrow M_{i+1}B'$ is any map whatever such that $\ell\eta_{i+1} \simeq \eta'_{i+1}g_{i+1}$, then

$$M_{i+1}g \simeq \eta'_{i+1}\xi'_{i+1}M_{i+1}g \simeq \eta'_{i+1}g_{i+1}\xi_{i+1} \simeq \ell\eta_{i+1}\xi_{i+1} \simeq \ell .$$

It follows that the homotopy class of $M_{i+1}g$ is independent of the choice of h_i, and from this it follows easily that $M\colon S \longrightarrow I$ is a prefunctor. M is homotopy preserving since if $g \simeq g'\colon B \longrightarrow B'$ in S, then

$$M_ig \simeq M_ig \cdot \eta_i\xi_i = \eta'_ig_i\xi_i \simeq \eta'_ig'_i\xi_i = M_ig' \cdot \eta_i\xi_i \simeq M_ig' ,$$

$i \geq 0$. Now (i) of the theorem follows immediately from (a), (b), and (c).

(iii) To prove (iii), we must show that the following two composites are homotopic to the identity map.

(f) $JB \xrightarrow{\eta(JB)} JMJB \xrightarrow{J\zeta(B)} JB$, $B \in I$

(g) $MB \xrightarrow{M\eta(B)} MJMB \xrightarrow{\zeta(MB)} MB$, $B \in S$.

By (a) and $\zeta(B) = \xi(JB)$, the composite (f) is the identity map. For (g), note that $\xi(JMB)\eta(JMB) = 1 \simeq \eta(B)\xi(B)\colon JMB \longrightarrow JMB$. By the uniqueness proof above for the homotopy class of $M_{i+1}g$ applied to the case $g = \xi(B)$, we have $M\xi(B) \simeq \xi(JMB) = \zeta(MB)$. Since $M\xi(B)M\eta(B) \simeq 1$ by the fact that M is a prefunctor, this proves that the composite (g) is homotopic to the identity.

(iv) Since ΩS and ΩI are full subcategories of S and I, it suffices for (iv) to prove that $MB \in \Omega I$ if $B \in \Omega S$, and this follows from (a) and (b) which show that if $g_j\colon \Omega B_{j+1} \longrightarrow B_j$ is a homotopy inverse to f_j, then $\eta_j g_j \Omega\xi_{j+1}\colon \Omega M_{j+1}B \longrightarrow M_jB$ is a homotopy inverse to M_jf.

(v) Again, it suffices to show that $MB \in R \cap C$ if $B \in \Omega S \cap C$. By induction on i, starting with $M_0B = B$ and $\eta_0 = 1 = \xi_0$, we see that each M_iB is a locally finite countable simplicial complex and that each map $\mu(M_{i-1}f)$, η_i, and ξ_i is simplicial, since $M_{i+1}B$ is the mapping cylinder of the simplicial map $\mu(f_i)S\xi_i\colon SM_iB \longrightarrow B_{i+1}$ [10, p. 151]. By Hanner [5, Corollary 3.5], every countable locally finite simplicial complex is an absolute neighborhood retract (ANR) and, by Kuratowski [7, p. 284], the loop space of an ANR is an ANR. Since the image of M_if is a closed subspace of the

ANR $\Omega M_{i+1}B$, $M_i f$ has the homotopy extension property with respect to the ANR $M_i B$ [6, p. 86], and therefore $M_i B$ is a deformation retract of $\Omega M_{i+1}B$ [10, p. 31]. This proves that $MB \in \mathcal{R} \cap \mathcal{C}$, as was to be shown.

The category $\mathcal{I}$ is not only large and convenient. It is also conceptually satisfactory in view of the following observation relating maps in $\mathcal{T}$ to maps in $\mathcal{I}$. We can define a functor $\Sigma\colon \mathcal{T} \longrightarrow \mathcal{I}$ by letting ΣX be the suspension spectrum of X, $\Sigma_i X = S^i X$ and $f_i = \mu^{-1}(1_{S^{i+1}X})$. If $g\colon X \longrightarrow Y$ is a map in $\mathcal{T}$, define $\Sigma_i g = S^i g$; it is clear that Σg is in fact a map in $\mathcal{I}$. Let $U = U_{\mathcal{I}}\colon \mathcal{I} \longrightarrow \mathcal{T}$ be the forgetful functor, $UB = B_0$ and $Ug = g_0$. Observe that $U\Sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$ is the identity functor. With these notations, we have the following proposition.

<u>Proposition 5</u>

$$U\colon \mathrm{Hom}_{\mathcal{I}}(\Sigma X, B) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, UB) \text{ is an adjunction.}$$

<u>Proof</u>. If $B = \{B_i, f_i\} \in \mathcal{I}$, define $f^i\colon B_0 \longrightarrow \Omega^i B_i$ inductively by $f^0 = 1$, $f^1 = f_0$, and $f^{i+1} = \Omega^i f_i \cdot f^i$ if $i > 0$. Define a natural transformation $\Phi\colon \Sigma U \longrightarrow 1_{\mathcal{I}}$ by $\Phi(B) = \{\mu^i(f^i)\}\colon \Sigma UB \longrightarrow B$. Since $\Omega\mu^{i+1}(f^{i+1}) \cdot \mu^{-1}(1_{S^{i+1}B_0})$ $= \mu^i(f^{i+1}) = \mu^i(\Omega^i f_i \cdot f^i) = f_i\mu^i(f^i)$, $\Phi(B)$ is a map in $\mathcal{I}$. For $g\colon X \longrightarrow UB$, define $\phi(g) = \Phi(B)\Sigma g$. Clearly $U\phi(g) = \mu^0(f^0)\Sigma_0 g$ $= g$. Now f^i for ΣX is easily verified to be $\mu^{-i}(1_{S^i X})\colon X \longrightarrow \Omega^i S^i X$. Therefore $\Phi(\Sigma X) = 1\colon \Sigma X \longrightarrow \Sigma X$; since

we obviously have $\Sigma U(1_{\Sigma X}) = 1\colon \Sigma X \longrightarrow \Sigma U \Sigma X = \Sigma X$, this implies that $\phi U = 1$.

Finally, we compare $\mathcal{L}$ to the categories $\mathcal{I}$, $\Omega\mathcal{I}$, and $\mathcal{R}$. The following theorem shows that $\mathcal{L}$ is nicely related conceptually to $\mathcal{I}$ and is equivalent for the purposes of weak homotopy theory to $\Omega\mathcal{I}$ in the sense that no weak homotopy invariant information is lost by restricting attention to spectra and maps of spectra in $\mathcal{L}$; coupled with the remarks preceding Theorem 4, it also shows that $\mathcal{L} \cap \mathcal{W}$ is equivalent to $\mathcal{R} \cap \mathcal{W}$ for the purposes of homotopy theory.

Theorem 6

There is a functor $L\colon \mathcal{I} \longrightarrow \mathcal{L}$ and a natural transformation of functors $\eta\colon 1_{\mathcal{I}} \longrightarrow KL$, where $K\colon L \longrightarrow \mathcal{I}$ is the inclusion, such that

(i) $LK\colon \mathcal{L} \longrightarrow \mathcal{L}$ is the identity functor and

$$L\colon \mathrm{Hom}_{\mathcal{I}}(A,KB) \longrightarrow \mathrm{Hom}_{\mathcal{L}}(LA,B)$$

is an adjunction with $L^{-1}(g) = Kg \cdot \eta(A)$ for $g\colon LA \longrightarrow B$.

(ii) If $g \simeq g'$ in $\mathcal{I}$, then Lg is weakly homotopic to Lg' in $\mathcal{L}$, and if $B \in \Omega\mathcal{I}$, then $\eta(B)\colon B \longrightarrow KLB$ is a weak homotopy equivalence.

(iii) Let $B \in \mathcal{R} \cap \mathcal{C}$; then $\eta(B)\colon B \longrightarrow KLB$ is a homotopy equivalence and if $g \simeq g'\colon B \longrightarrow B'$ in $\mathcal{I}$, then $Lg \simeq Lg'\colon LB \longrightarrow LB'$ in $\mathcal{L}$.

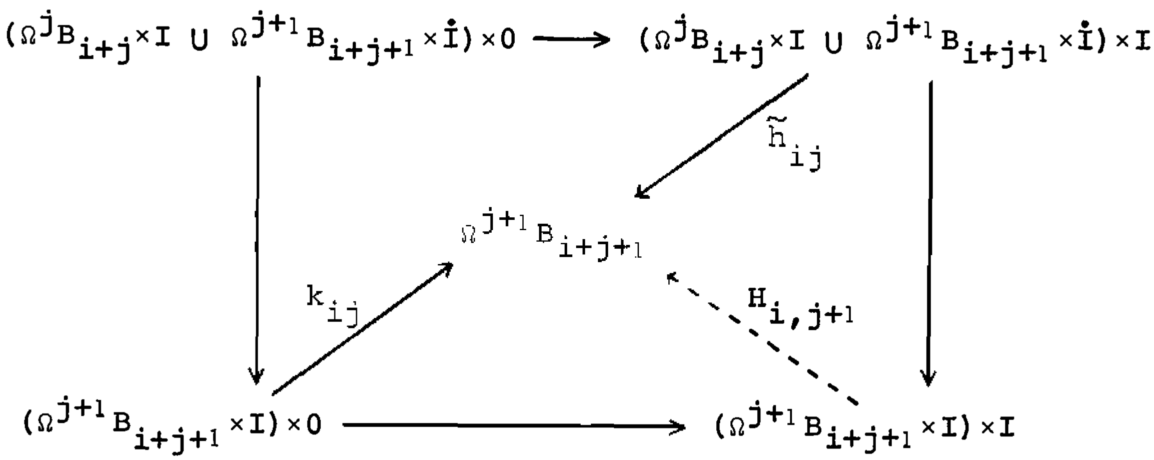
Proof. Let $B = \{B_i, f_i\} \in \mathcal{I}$. Since each f_i is an inclusion, we can define $L_i B = \varinjlim \Omega^i B_{i+j}$, where the limit is taken with respect to the inclusions $\Omega^j f_{i+j}\colon \Omega^j B_{i+j} \longrightarrow \Omega^{j+1} B_{i+j+1}$. Clearly $\Omega L_{i+1} B = L_i B$, hence $LB \in L$. If $g\colon B \longrightarrow B'$ is a map in I, define $L_i g = \varinjlim \Omega^j g_{i+j}\colon L_i B \longrightarrow L_i B'$; the limit makes sense since $\Omega^j f'_{i+j} \Omega^j g_{i+j} = \Omega^{j+1} g_{i+j+1} \Omega^j f_{i+j}$ by the definition of maps in I. Clearly $\Omega L_{i+1} g = L_i g$, hence $Lg \in L$. Define $\eta\colon 1_I \longrightarrow KL$ by letting $\eta_i(B)\colon B_i \longrightarrow L_i B$ be the natural inclusion; $\eta(B)$ is obviously a map in I since $\Omega\eta_{i+1}(B) \cdot f_i = \eta_i(B)$. Now (ii) of the theorem is a standard consequence of the definition of the limit topology. The fact that LK is the identity functor of L is evident, and $\eta K\colon K \longrightarrow KLK$ and $L\eta\colon L \longrightarrow LKL$ are easily verified to be the identity natural transformations. This implies (i) and it remains to prove (iii). If $B \in R$, with retractions $r_i\colon \Omega B_{i+1} \longrightarrow B_i$, define maps $r^{ij}\colon \Omega^j B_{i+j} \longrightarrow B_i$ inductively by $r^{i0} = 1$, $r^{i1} = r_i$, and $r^{i,j+1} = r^{ij} \Omega^j r_{i+j}$ if $j > 0$. Since $r_{i+j} f_{i+j} = 1$, we have $r^{i,j+1} \Omega^j f_{i+j} = r^{ij}$. We can therefore define maps

$\xi_i = \varinjlim r^{ij}\colon L_i B \longrightarrow B_i$. Obviously $\xi_i \eta_i\colon B_i \longrightarrow B_i$ is the identity map. Suppose further that $B \in C$. Then we claim that $\eta_i \xi_i \simeq 1\colon L_i B \longrightarrow L_i B$. As in the proof of (v) of Theorem 4, each $\Omega^j B_{i+j}$ is now an ANR. Let us identify $\Omega^j B_{i+j}$ with its image under $\Omega^j f_{i+j}$ in $\Omega^{j+1} B_{i+j+1}$ for all i and j and omit the inclusion maps $\Omega^j f_{i+j}$ from the notation. Then the inclusion

$$\Omega^j B_{i+j} \times I \cup \Omega^{j+1} B_{i+j+1} \times \dot{I} \subset \Omega^{j+1} B_{i+j+1} \times I$$

is that of a closed subset in an ANR, and it therefore has the homotopy extension property with respect to the ANR $\Omega^{j+1} B_{i+j+1}$.

In particular, by [10, p. 31], each B_i is a strong deformation retract of ΩB_{i+1}, and we assume given homotopies $k_i\colon \Omega B_{i+1} \times I \longrightarrow \Omega B_{i+1}$, $k_i\colon 1 \simeq r_i$ rel B_i. The k_i induce homotopies: $k_{ij}\colon \Omega^{j+1}B_{i+j+1} \times I \longrightarrow \Omega^{j+1}B_{i+j+1}$, $k_{ij}\colon 1 \simeq \Omega^j r_{i+j}$ rel $\Omega^j B_{i+j}$, in the obvious fashion ($k_{ij,t} = \Omega^j k_{i+j,t}$). We claim that, by induction on j, we can choose homotopies $h_{ij}\colon \Omega^j B_{i+j} \times I \longrightarrow \Omega^j B_{i+j}$, $h_{ij}\colon 1 \simeq r^{ij}$ rel B_i, such that $h_{i,j+1} = h_{ij}$ on $\Omega^j B_{i+j} \times I$. To see this, let h_{i0} be the constant homotopy, let $h_{i1} = k_i = k_{i0}$, and suppose given h_{ij} for some $j > 0$. Consider the following diagram:



The unlabeled arrows are inclusions, and $\tilde{h}_{ij}$ is defined by $\tilde{h}_{ij}(x,s,t) = h_{ij}(x,st)$ if $x \in \Omega^j B_{i+j}$; and $\tilde{h}_{ij}(y,0,t) = y$, $\tilde{h}(y,1,t) = h_{ij}(\Omega^j r_{i+j}(y),t)$ if $y \in \Omega^{j+1}B_{i+j+1}$. It is easily verified that $\tilde{h}_{ij}$ is well-defined and continuous and that $\tilde{h}_{ij} = k_{ij}$ on the common parts of their domains. We can therefore obtain $H_{i,j+1}$ such that the diagram commutes. Define $h_{i,j+1}(x,s) = H_{i,j+1}(x,s,1)$. It is trivial to verify that

$h_{i,j+1}$ has the desired properties. Now $\varinjlim h_{ij}: L_iB \times I \longrightarrow L_iB$ is defined and is clearly a homotopy from 1 to $\eta_i\xi_i$. Finally, if $g \simeq g': B \longrightarrow B'$ in $\mathcal{I}$ and $B \in \mathcal{R} \cap \mathcal{C}$, then

$$L_ig \simeq L_ig\eta_i\xi_i = \eta_i^! g_i\xi_i \simeq \eta' g_i^! \xi_i = L_ig'\eta_i\xi_i \simeq L_ig' \text{ , } i \geq 0 \text{ .}$$

This completes the proof of (iii) and of the theorem.

We remark that the categorical relationships of Propositions 1 and 5 and of the theorem are closely related. In fact, the composite functor $L\Sigma: \mathcal{T} \longrightarrow \mathcal{L}$ is precisely Q_∞, and the adjunction

$$\phi_\infty: \mathrm{Hom}_{\mathcal{T}}(X, U_\infty B) \longrightarrow \mathrm{Hom}_{\mathcal{L}}(Q_\infty X, B)$$

of Proposition 1 factors as the composite $(U_\infty = U)$

$$\mathrm{Hom}_{\mathcal{T}}(X, UKB) \xrightarrow{U^{-1}} \mathrm{Hom}_{\mathcal{I}}(\Sigma X, KB) \xrightarrow{L} \mathrm{Hom}_{\mathcal{L}}(Q_\infty X, B) .$$

The verification of these statements requires only a glance at the definitions.

3 INFINITE LOOP SPACES

We shall here summarize the implications of the work of the previous section for infinite loop spaces and give the promised applications. We then make a few remarks about the extension of our results to unbounded spectra and point out an interesting collection of connective cohomology theories.

It is customary to say that $X \in \mathcal{T}$ is an infinite loop space if X is the initial space B_0 of an Ω-spectrum B. If X is given as an H-space, it is required that its product be homotopic to the product induced from the homotopy equivalence $X \longrightarrow \Omega B_1$. Similarly, a map $f \in \mathcal{T}$ is said to be an infinite loop map if f is the initial map g_0 of a map of Ω-spectra g. The functor $M: \Omega\mathcal{S} \longrightarrow \Omega\mathcal{I}$ of Theorem 4 satisfies $M_0 B = B_0$ and $M_0 g = g_0$. We therefore see that the identical infinite loop spaces and maps are obtained if we restrict attention to inclusion Ω-spectra and maps in $\mathcal{I}$. If $f: X \longrightarrow X'$ is any infinite loop map, then Theorem 6 implies the existence of a commutative diagram of infinite loop maps

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \xrightarrow{g'} & Y' \end{array} \qquad (3.1)$$

such that f' is a perfect infinite loop map between perfect infinite loop spaces and g and g' are weak homotopy equivalences.

If X is an infinite loop space of the homotopy type of a countable CW-complex, then it follows from arguments of Boardman and Vogt [1, p. 15] that there is an infinite loop map $g: X \longrightarrow Y$ such that g is a homotopy equivalence and Y is the initial space of a spectrum in $\Omega\mathcal{S} \cap \mathcal{W}$. Combining this fact with (v) of Theorem 4, the remarks preceding that theorem, and (iii) of Theorem 6, we see that if $f: X \longrightarrow X'$ is any infinite loop map between spaces of the homotopy type of countable CW-complexes, then there is a homotopy commutative diagram of infinite loop maps, of the form given in (1), such that f' is a perfect infinite loop map and g and g' are homotopy equivalences.

Therefore nothing is lost for the purposes of weak homotopy theory if the notions of infinite loop spaces and maps are replaced by those of perfect infinite loop spaces and maps, and similarly for homotopy theory provided that we restrict attention to spaces of the homotopy type of countable CW-complexes.

The promised comparison of stable and unstable homotopy groups of infinite loop spaces is now an easy consequence of Proposition 1. In fact, if Y is an infinite loop space, say $Y = B_0$ where $B \in \Omega\mathcal{S}$, then that proposition gives a map $\Phi_\infty(LMB): Q_\infty L_0 MB \longrightarrow LMB$ in $\mathcal{L}$, and Theorem 6 gives a map $\eta(MB): MB \longrightarrow LB$ in $\mathcal{I}$. Define maps

$$QY \xrightarrow{\alpha} QL_0MB \xrightarrow{\beta} L_0MB \xleftarrow{\gamma} Y$$

by $\alpha = Q\eta_0(MB)$, $\beta = \Phi_{\infty,0}(LMB)$, and $\gamma = \eta_0(MB)$. γ is clearly a weak homotopy equivalence, and therefore so is α since $Q: \mathcal{T} \longrightarrow \mathcal{T}$ is easily verified to preserve weak homotopy equivalences. Since $\Phi_{\infty,0}(LMB) \cdot \Psi_\infty(L_0MB)$ is the identity map of L_0MB, β_* is an epimorphism on homotopy. If $X \in \mathcal{T}$, then $\Pi_n(QX) = \Pi_n^S(X)$, the $n^{\underline{th}}$ stable homotopy group of X. Therefore $\rho(Y) = \gamma_*^{-1}\beta_*\alpha_*: \Pi_*(QY) \longrightarrow \Pi_*(Y)$ gives an epimorphism $\Pi_*^S(Y) \longrightarrow \Pi_*(Y)$. It is clear that if $f: Y \longrightarrow Y'$ is any infinite loop map, then

$$\rho(Y')(Qf)_* = f_*\rho(Y): \Pi_n^S(Y) \longrightarrow \Pi_n(Y') .$$

It should be observed that the notions of infinite loop spaces and maps are not very useful from a categorical point of view since the composite of infinite loop maps need not be an infinite loop map. In fact, given infinite loop maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$, there need be no spectrum B with $B_0 = Y$ which is simultaneously the range of a map of spectra giving f and the domain of a map of spectra giving g. One can get around this by requiring infinite loop spaces to be topological monoids and using a classifying space argument to allow composition of maps, but this is awkward. These conditions motivate the use of $\mathcal{L}$ in the definition of homology in section 1.

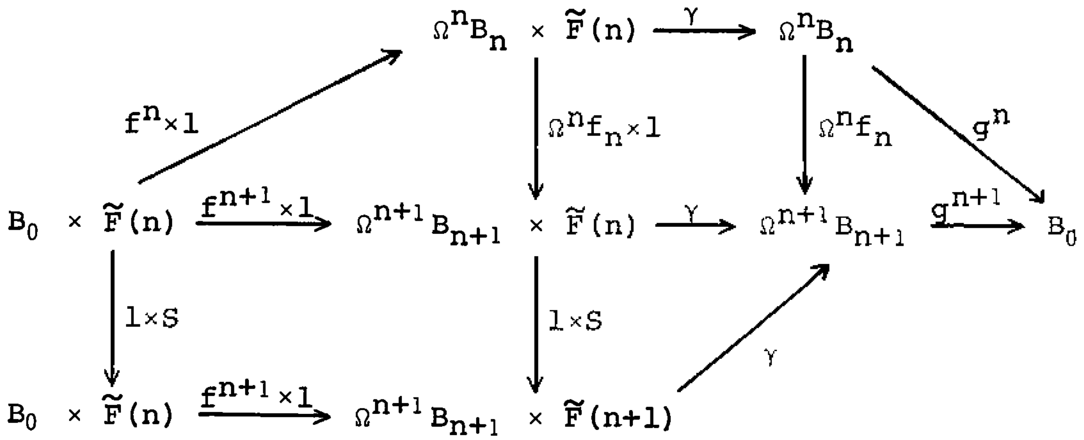
The following application of our results, which will be used in the computation of $H^*(BF)$, illustrates the technical convenience of the category $\mathcal{L}$. Let $\tilde{F}(n) = \mathrm{Hom}_{\mathcal{T}}(S^n, S^n)$ and let $\tilde{F} = \varinjlim \tilde{F}(n)$, where the limit is taken with respect to suspension of maps $S\colon \tilde{F}(n) \longrightarrow \tilde{F}(n+1)$. $\tilde{F}(n)$ and $\tilde{F}$ are topological monoids under composition of maps. If $X \in \mathcal{T}$, define $\gamma\colon \Omega^n X \times \tilde{F}(n) \longrightarrow \Omega^n X$ by $\gamma(x,f) = \mu^{-n}(\mu^n(x) \cdot f)$, that is, with $\Omega^n X$ identified with $\mathrm{Hom}_{\mathcal{T}}(S^0, \Omega^n X)$, by the composite

$$\Omega^n X \times \tilde{F}(n) \xrightarrow{\mu^n \times 1} \mathrm{Hom}_{\mathcal{T}}(S^n, X) \times \tilde{F}(n) \xrightarrow{\text{composition}} \mathrm{Hom}_{\mathcal{T}}(S^n, X) \xrightarrow{\mu^{-n}} \Omega^n X.$$

This defines an operation of $\tilde{F}(n)$ on $\Omega^n X$. Now let $B = \{B_i, f_i\} \in \Omega\mathcal{S}$, and let $g_i\colon \Omega B_{i+1} \longrightarrow B_i$ be a homotopy inverse to f_i. Define homotopy equivalences $f^n\colon B_0 \longrightarrow \Omega^n B_n$ and $g^n\colon \Omega^n B_n \longrightarrow B_0$ in the obvious inductive manner and define

$$\gamma_n = g^n\gamma(f^n \times 1): B_0 \times \tilde{F}(n) \longrightarrow B_0 .$$

Observe that γ_n fails to define an operation of $\tilde{F}(n)$ on B_0 since the associativity condition $(xf)g = x(fg)$ is lost. Of course, γ_n coincides with γ on $\Omega^n B_n$ if $B \in L$, and associativity is then retained. Now consider the following diagram:



The left-hand triangle and square commute trivially. Clearly γ is natural on n-fold loop maps, hence $\Omega^n f_n \gamma = \gamma(\Omega^n f \times 1)$. $\gamma(1 \times S) = \gamma$ since

$$\mu^{-n}(\mu^n(x)f) = \mu^{-(n+1)}\mu(\mu^n(x)f) = \mu^{-(n+1)}(\mu^{n+1}(x) \cdot Sf) .$$

g^n is homotopic to $g^{n+1}\Omega^n f_n$, and if $B \in R$ and the g_i are chosen retractions, then $g^n = g^{n+1}\Omega^n f_n$. Thus if $B \in R$ we have $\gamma_n = \gamma_{n+1}(1 \times S)$ and we can define $\gamma = \varinjlim \gamma_n: B_0 \times \tilde{F} \longrightarrow B_0$. Since the right-hand triangle is not transformed naturally by maps in R, the map γ is not natural on R. For $B \in L$, the f's and g's are the identity maps, and the diagram trivializes. Therefore, for each $B \in L$, we have an operation $\gamma: B_0 \times \tilde{F} \longrightarrow B_0$ and if $h: B \longrightarrow B'$ is a map in L, then $h_0(xf) = h_0(x)f$ for $x \in B_0$ and $f \in \tilde{F}$.

Stasheff [unpublished] has generalized work of Dold and Lashof [2] to show that if a topological monoid M operates on a space X, then there is a natural way to form an associated quasifibration $X \longrightarrow Xx_M EM \longrightarrow BM$ to the classifying principal quasifibration $M \longrightarrow EM \longrightarrow BM$. As usual, let $F \subset \tilde{F}$ consist of the homotopy equivalences of spheres. By restriction, if $B \in \mathcal{L}$ and $Y = B_0$, we have an operation of F on Y and we can therefore form $Yx_F EF$. Of course, this construction is natural on $\mathcal{L}$.

Boardman and Vogt [1] have proven that the standard inclusions $U \subset O \subset PL \subset Top \subset F$ are all infinite loop maps between infinite loop spaces with respect to the H-space structures given by Whitney sum (on F, this structure is weakly homotopic to the composition product used above). We now know that we can pass to $\mathcal{L}$ and obtain natural operations of F on (spaces homotopy equivalent to) each of these sub H-spaces G of F. The same is true for their classifying spaces BG. Observe that the resulting operation of F on F is not equivalent to its product. (In fact, if $\phi \in B_0$ is the identity under the loop product of ΩB_1, where $B \in \mathcal{L}$, then $\phi f = \phi$ for all $f \in \tilde{F}$ since composing any map with the trivial map gives the trivial map.) It would be of interest to understand the geometric significance of these operations by F on its various sub H-spaces and of the spaces $Gx_F EF$ and $BGx_F EF$.

I shall show elsewhere that, with mod p coefficients, $\gamma_*: H_*(B) \otimes H_*(\tilde{F}) \longrightarrow H_*(B)$ gives $H_*(B)$ a structure of Hopf algebra over $H_*(\widetilde{F})$ for $B \in \mathcal{L}$ (and, a fortiori, for $B \in \Omega\mathcal{S}$), where $H_*(B) = H_*(B_0)$ as in section 1. $H_*(B)$ is also a Hopf algebra over the opposite algebra of the Steenrod algebra and over the Dyer-Lashof algebra, which is defined in terms of the homology operations introduced by Dyer and Lashof in [3]. These operations are all natural on $\mathcal{L}$. The appropriate range category for $H_*\colon \mathcal{L} \longrightarrow$? is determined by specifying how these three types of homology operations commute, and, coupled with known information, these commutation formulas are all that is required to compute $H^*(BF)$.

Finally, we observe that there is a natural way to extend our results of section 2 to unbounded spectra. Let $\overline{\mathcal{S}}$ denote the category whose objects are sequences $\{B_i, f_i \mid i \in Z\}$ such that $\{B_i, f_i \mid i \geq 0\} \in \mathcal{S}$ and $B_i = \Omega^{-i}B_0$ and $f_i\colon B_i \longrightarrow \Omega B_{i+1}$ is the identity map for $i < 0$. The maps in $\overline{\mathcal{S}}$ are sequences $g = \{g_i \mid i \in Z\}$ such that $\{g_i \mid i \geq 0\} \in \mathcal{S}$ and $g_i = \Omega g_{i+1}$ if $i < 0$. We have an obvious completion functor $C\colon \mathcal{S} \longrightarrow \overline{\mathcal{S}}$ defined on objects by $C_iB = B_i$ if $i \geq 0$ and $C_iB = \Omega^{-i}B_0$ if $i < o$, with $C_if = f_i$ for $i \geq 0$ and $C_if = 1$ for $i < 0$, and defined similarly on maps. C is an isomorphism of categories with inverse the evident forgetful functor $\overline{\mathcal{S}} \longrightarrow \mathcal{S}$. For each of our previously defined subcategories $\mathcal{D}$ of $\mathcal{S}$ define $\overline{\mathcal{D}}$ to be the image of $\mathcal{D}$ under C in $\overline{\mathcal{S}}$. $\overline{\mathcal{T}}$ is of particular interest. Its objects and maps are sequences $\{B_i \mid i \in Z\}$ and $\{g_i \mid i \in Z\}$ such that $B_i = \Omega B_{i+1}$ for all i and $g_i = \Omega g_{i+1}$ for all i. Clearly all of the results of section 2 remain valid for the completed categories.

Our results show that any reasonable cohomology theory, by which we mean any cohomology theory determined by a spectrum $B \in \Omega\mathcal{S} \cap \mathcal{W}$, is isomorphic to a cohomology theory determined by a spectrum in $\mathcal{L} \cap \mathcal{W}$ and that any transformation of such theories determined by a map $g: B \longrightarrow B'$ in $\Omega\mathcal{S} \cap \mathcal{W}$ is naturally equivalent to a transformation determined by a map in $\mathcal{L} \cap \mathcal{W}$. Recall that

$$H^n(X,A;B) = \mathrm{Hom}_{H\mathcal{T}}(X/A,B_n)$$

defines the cohomology theory determined by $B \in \Omega\mathcal{S}$ on CW pairs (X,A). Call such a theory connective if $H^n(P;B) = 0$ for $n > 0$, where P is a point. Of course, $H^{-n}(P;B) = \Pi_0(\Omega^n B_0) = \Pi_n(B_0)$. Any infinite loop space Y determines a connective (additive) cohomology theory since, by a classifying space argument, we can obtain $CB \in \Omega\mathcal{S}$ such that B_0 is homotopy equivalent to Y and $\Pi_0(B_n) = 0$ for $n > 0$; according to Boardman and Vogt [1], any such cohomology theory is so obtainable and determines Y up to homotopy equivalence of infinite loop spaces. If $X \in \mathcal{T}$, then $CQ_\infty X$ determines a connective cohomology theory, since $C_n Q_\infty X = QS^n X$ for $n > 0$, and $H^{-n}(P;CQ_\infty X) = \Pi_n(QX) = \Pi_n^S(X)$ if $n \geq 0$. In view of Proposition 1, these theories play a privileged role among all connective cohomology theories, and an analysis of their properties might prove to be of interest. Observe that if $B \in \mathcal{L}$, then $C\Phi_\infty B: CQ_\infty B_0 \longrightarrow CB$ determines a natural transformation of cohomology theories $H^*(X,A;CQ_\infty B_0) \longrightarrow H^*(X,A;CB)$ and, if the theory determined by CB is connective, this transformation is epimorphic on the cohomology of a point.

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